e-semicommutative modules

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Abstract. e-semicommutative modules and their related properties. Also, we investigate some extensions of rings and modules in terms of e-semicommutativity.
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1. Introduction

Throughout this paper, all rings are associative with unity. R denotes an associative ring with unity, M_R is a unitary right R-module, $\mathrm{Id}\,(R)$ denotes the set of all idempotent elements of R, $\mathrm{N}\,(R)$ denotes the set of all nilpotent elements of R, $\mathrm{C}(R)$ denotes the center of R, $\mathrm{S_r}(R) = \{e \in \mathrm{Id}\,(R) : e\,R\,e = e\,R\}$ denotes the set of all right semicentral idempotent elements of R, $\mathrm{S}_\ell(R) = \{e \in \mathrm{Id}\,(R) : e\,R\,e = R\,e\}$ denotes the set of all left semicentral idempotent elements of R, and $\mathrm{r}_R(M) = \{a \in R : Ma = 0\}$ denotes the right annihilator of M in R.

A ring R is said to be abelian if $\mathrm{Id}\,(R) \subseteq \mathrm{C}(R)$. A ring R is called reduced if $\mathrm{N}\,(R) = 0$. This concept of reduced rings was extended to modules [9] as follows: a right R-module M_R is reduced if, for any $m \in M$ and any $a \in R$, $ma = 0$ implies $mR \cap Ma = 0$. Recall from [10], R is a right e-reduced ring, where $e \in \mathrm{Id}(R)$, if $\mathrm{N}\,(R)\,e = 0$. A ring R is called symmetric [8] if whenever $a, b, c \in R$ such that $abc = 0$, we have $acb = 0$. Recall from Refs. [8] and [11], a right R-module M_R is called *symmetric* if whenever $a, b \in R$ and $m \in M$ such that $mab = 0$ implies $mba = 0$. Following [10], a ring R is called *e-symmetric*, for $e \in \mathrm{Id}(R)$, if whenever $a, b, c \in R$ such that $abc = 0$, we have $acbe = 0$.

Introduce of these properties via idempotents, inspires us to extend the notions of e-reduced and e-symmetric to modules as follows:

Definition 1.1 ([1]). *A right R-module M_R is called e-reduced, where $e \in \mathrm{Id}(R)$, if whenever $a \in R$ and $m \in M$ such that $ma = 0$ implies $mR \cap Mae = 0$.*

Definition 1.2 ([1]). *A right R-module M_R is called e-symmetric, where $e \in \mathrm{Id}(R)$, if whenever $a, b \in R$ and $m \in M$ such that $mab = 0$ implies $mbae = 0$.*

A ring R is an e-reduced (e-symmetric) ring if and only if R_R is an e-reduced (e-symmetric) module.

According to [3] a ring R is called semicommutative, if whenever $a, b \in R$ satisfy $ab = 0$, then $aRb = 0$. A right R-module M_R is called semicommutative [5], if whenever $a \in R$ and $m \in M$ satisfy $ma = 0$, then $mRa = 0$. Recall from [7], a ring R is called e-semicommutative, for $e \in \mathrm{Id}(R)$, if whenever $a, b \in R$ such that $ab = 0$, we have $aRbe = 0$.

So it is natural to motivate us to extend the condition of e-semicommutativity to Module Theory.

2. Modules with e-semicommutative condition

In this section, we extend the notion of e-semicommutative rings to modules as follows:

Definition 2.1. *A right R-module M_R is called e-semicommutative, where $e \in \mathrm{Id}(R)$, if whenever $a \in R$ and $m \in M$ such that $ma = 0$ implies $mRae = 0$.*

Obviously, R is an e-semicommutative ring if and only if R_R is an e-semicommutative module.

Clearly, any semicommutative module is an e-semicommutative module, for any $e \in \mathrm{Id}(R)$, and every an e-reduced (e-symmetric) module is e-semicommutative. The following examples demonstrate rather strikingly that the class of e-semicommutative modules is properly contains the class of semicommutative modules.

Example 2.1. Let S be a semicommutative ring and $R = \begin{pmatrix} S & S \\ 0 & S \end{pmatrix}$. Consider a right R-module $M_R = R[x]_R$. Assume that $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in R$. We see that $(Ax + A)B = 0$ but $(Ax + A)CB \neq 0$. Then, M_R is not semicommutative. Now for the idempotent $E = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in R$, we can show that M_R is E-semicommutative. Let $f(x) = \sum_{i=0}^{n} A_i x^i \in M$, where $A_i = \begin{pmatrix} a_i & b_i \\ 0 & c_i \end{pmatrix} \in R$ for every $i = 0, 1, ..., n$, and $B = \begin{pmatrix} w & u \\ 0 & v \end{pmatrix} \in R$ such that $f(x)B = 0$. Then, $0 = A_iB = \begin{pmatrix} a_iw & a_iu + b_iv \\ 0 & c_iv \end{pmatrix}$ for every $i = 0, 1, ..., n$. Hence, $a_iw = 0$, $c_iv = 0$ and $a_iu + b_iv = 0$. For any element $C = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in R$, we have $f(x)CBE = \sum_{i=0}^{n} (A_iCBE)\, x^i = 0$. Therefore, M_R is E-semicommutative.

Example 2.2. Let S be a semicommutative ring and $R = \begin{pmatrix} S & 0 & 0 \\ S & S & S \\ 0 & 0 & S \end{pmatrix}$. Consider R_R as a right R-module. Assume that $m = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in R$. We see that $ma = 0$ but $mba \neq 0$. Then, R_R is not semicommutative. Now for the idempotent $e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, we can show that R_R is e-semicommutative. Let $m = \begin{pmatrix} x_1 & 0 & 0 \\ y_1 & z_1 & w_1 \\ 0 & 0 & v_1 \end{pmatrix}$, $a = \begin{pmatrix} x_2 & 0 & 0 \\ y_2 & z_2 & w_2 \\ 0 & 0 & v_2 \end{pmatrix} \in R$ such that $ma = 0$. Hence, $x_1x_2 = z_1z_2 = v_1v_2 = z_1w_2 = w_1v_2 = 0$ and $y_1x_2 + z_1y_2 = 0$. For any element $r = \begin{pmatrix} x & 0 & 0 \\ y & z & w \\ 0 & 0 & v \end{pmatrix} \in R$, we have $mrae = \begin{pmatrix} 0 & 0 & 0 \\ 0 & z_1zz_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$, since S_S is semicommutative. Therefore, R_R is e-semicommutative.

Proposition 2.1. *The class of e-semicommutative modules is closed under submodules, direct products and so direct sums.*

Proof. The proof is immediate from the definitions and algebraic structures. □

Proposition 2.2. *Let R be a ring, $e \in \mathrm{Id}(R)$ and M_R a right R-module. M_R is e-semicommutative if and only if every cyclic submodule of M_R is e-semicommutative.*

Proof. Assume that every cyclic submodule of M_R is e-semicommutative. Let $a \in R$ and $m \in M$ such that $ma = 0$ in M. Consider the cyclic submodule mR, we have $ma = 0$ in mR. Since mR is e-semicommutative, we get $mRae = 0$. Hence, M_R is e-semicommutative. □

Proposition 2.3. *Let R be a ring, $e \in \mathrm{Id}(R)$ and M_R a right R-module. Then, the following two conditions are equivalent:*

1) M_R is an e-semicommutative module.

2) $NA = 0$ implies $NRAe = 0$ for any nonempty subset N in M and A in R.

Proof. "(1) $\Longrightarrow$ (2)" Assume that M_R is e-semicommutative and N is a subset of M and A is a subset of R such that $NA = 0$. Then, for any $n \in N$ and $a \in A$, we have $na = 0$. Thus, $nRae = 0$. Then, $\sum_{n\in N, a\in A} nRae = 0$. Hence, $NRAe = 0$.

"(2) $\Longrightarrow$ (1)" Assume that $a \in R$ and $m \in M$ such that $ma = 0$. Then, M_R is e-semicommutative follows directly if we set $N = \{m\}$ and $A = \{a\}$. □

Proposition 2.4. *Let R be a ring with every right ideal is two sided and $e \in \mathrm{Id}(R)$. Then, every right R-module is e-semicommutative.*

Proof. Suppose that M_R is a right R-module. Let $a \in R$ and $m \in M$ such that $ma = 0$. From our assumption, the right ideal $ae\,R$ is two sided. Then, we have $R\,ae \subseteq ae\,R$. So, we get $m\,R\,ae \subseteq mae\,R = 0$. Therefore, M_R is e-semicommutative. □

Proposition 2.5. *Let R, S be rings, $e \in \mathrm{Id}(R)$ and $\varphi : R \to S$ be a ring homomorphism. If M_S is a right S-module, then M is a right R-module via $mr = m\varphi(r)$ for all $r \in R$ and $m \in M$. Then, we get:*

(1) If M_S is a $\varphi(e)$-semicommutative module, then M_R is an e-semicommutative module.

(2) If φ is onto and M_R is an e-semicommutative module, then M_S is a $\varphi(e)$-semicommutative module.

Proof. (1) Suppose that M_S is a $\varphi(e)$-semicommutative module. Let $a \in R$ and $m \in M$ such that $ma = 0$. Then, $m\varphi(a) = 0$. Since M_S is $\varphi(e)$-semicommutative, we have $ms\varphi(a)\varphi(e) = 0$ for all $s \in S$. Hence, for any $r \in R$, we have $mrae = m\varphi(rae) = m\varphi(r)\varphi(a)\varphi(e) = 0$. Therefore, M_R is an e-semicommutative module.

(2) Suppose that φ is onto and M_R is an e-semicommutative module. Let $x \in S$ and $m \in M$ such that $mx = 0$. Since φ is onto, there exists $a \in R$ such that $x = \varphi(a)$. Then, $0 = mx = m\varphi(a) = ma$. Since M_R is e-semicommutative, implies $mRae = 0$. Hence, $0 = m\varphi(R)\varphi(a)\varphi(e) = mSx\varphi(e)$. Thus M_S is a $\varphi(e)$-semicommutative module. □

Corollary 2.1. *Let R be a ring, $e \in \mathrm{Id}(R)$, M_R a right R-module and $\overline{R} = R/\mathrm{r}_R(M)$. M_R is an e-semicommutative module if and only if $M_{\overline{R}}$ is an $\overline{e}$-semicommutative module.*

Proof. This is a consequence of Proposition 2.5, if we consider the canonical epimorphism $\varphi : R \to \overline{R}$ defined by $\varphi(r) = \overline{r} = r + \mathrm{r}_R(M)$, for all $r \in R$. □

Proposition 2.6. *Let R be a ring, $e \in \mathrm{C}(R)$ and M_R a right R-module. Then, M_R is an e-semicommutative module if and only if M_{Re} is a semicommutative module.*

Proof. "$\Longrightarrow$" Assume that M_R is an e-semicommutative module. Let $a \in R\,e \subseteq R$ and $m \in M$ such that $ma = 0$. Then, we get $mRae = 0$. Since $e \in \mathrm{C}(R)$, we have $m\,R\,ea = 0$. Hence, M_{Re} is a semicommutative module.

"$\Longleftarrow$" Assume that M_{Re} is a semicommutative module. Let $a \in R$ and $m \in M$ such that $ma = 0$. Then, we get $m\,R\,ea = 0$. Since $e \in \mathrm{C}(R)$, we have $m\,R\,ae = 0$. Thus M_R is an e-semicommutative module. □

Corollary 2.2. *Let R be a ring, $e \in \mathrm{C}(R)$ and M_R a right R-module. If M_{Re} and $M_{R(1-e)}$ are semicommutative modules, then M_R is a semicommutative module.*

Proof. We can easily check that $e \in \mathrm{C}(R)$ if and only if $(1-e) \in \mathrm{C}(R)$. From Proposition 2.6, we conclude that M_R is both e-semicommutative and $(1-e)$-semicommutative. Now let $a \in R$ and $m \in M$ such that $ma = 0$. Thus $mRae = 0$ and $mRa(1-e) = 0$, which implies that $mRa = 0$. Therefore, M_R is a semicommutative module. □

Proposition 2.7. *Let R be a ring, $e \in \mathrm{S}_\ell(R)$ and M_R a right R-module. Then, M_R is an e-semicommutative module if and only if M_{eRe} is a semicommutative module.*

Proof. " $\Longrightarrow$"Assume that M_R is an e-semicommutative module. Let $ere \in e\,R\,e$ and $m \in M$ such that $m\,(ere) = 0$. Then, we get $m\,R\,(ere) = 0$. Since $e \in \mathrm{S}_\ell(R)$, we have $0 = m\,(R\,e)\,(ere) = m\,(e\,R\,e)\,(ere)$. Hence, M_{eRe} is a semicommutative module.

" $\Longleftarrow$"Assume that M_{eRe} is a semicommutative module. Let $a \in R$ and $m \in M$ such that $ma = 0$. Then, we get $mae = 0$. Since $e \in \mathrm{S}_\ell(R)$, we have $meae = 0$. Hence, $0 = m\,(e\,R\,e)\,(eae) = m\,(R\,e)\,(eae) = mR\,(eae) = mRae$. Thus M_R is an e-semicommutative module. □

Recall from [4], that a right R-module M_R is called principally quasi-Baer (p.q.-Baer for short) if for any $m \in M$, $\mathrm{r}_R(mR) = gR$, where $g \in \mathrm{Id}(R)$.

Proposition 2.8. *Let R be an abelian ring, $e \in \mathrm{Id}(R)$ and M_R a p.q.-Baer right R-module. If M_R is e-semicommutative, then M_R is e-reduced.*

Proof. Assume that M_R is e-semicommutative. Let $a \in R$ and $m \in M$ such that $ma = 0$. Then, we get $mRae = 0$. Let $x \in mR \cap Mae$, so there exist $r \in R$ and $n \in M$ such that $x = mr$ and $x = nae$. Since $ae \in \mathrm{r}_R(mR) = gR$, where $g \in \mathrm{Id}(R)$, we get $ae = gae$. Thus $x = ngae = naeg = xg = mrg = mgr = 0$. Hence, $mR \cap Mae = 0$. Therefore, M_R is an e-reduced module. □

3. Matrix extensions

This section is devoted to characterize right e-semicommutative 2-by-2 generalized upper triangular matrix rings. Moreover, as a corollary we obtain that a ring R is a right e-semicommutative ring if and only if $T_n(R)$ is right E-semicommutative for all positive integers n.

Theorem 3.1. *Let $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ where R and S are rings, and ${}_RM_S$ an (R,S)-bimodule. If T is a right $\begin{pmatrix} e & k \\ 0 & g \end{pmatrix}$-semicommutative ring, where $\begin{pmatrix} e & k \\ 0 & g \end{pmatrix} \in \mathrm{Id}(T)$, then:*

(1) R is a right e-semicommutative ring;
(2) S is a right g-semicommutative ring;
(3) M_S is a right g-semicommutative S-module.

Proof of Theorem 3.1. Assume that T is a right $\begin{pmatrix} e & k \\ 0 & g \end{pmatrix}$-semicommutative ring, where $\begin{pmatrix} e & k \\ 0 & g \end{pmatrix} \in \mathrm{Id}(T)$. Then, by easy computations, we can check that $e \in \mathrm{Id}(R), g \in \mathrm{Id}(S)$ and $ek + kg = k$.

(1) Assume that $ab = 0$, for $a, b \in R$. Consider the following elements $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \in T$. We have $0 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}$. Since T is a right $\begin{pmatrix} e & k \\ 0 & g \end{pmatrix}$-semicommutative ring, we get for any $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in T$,

$$0 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} e & k \\ 0 & g \end{pmatrix}.$$

Hence, $axbe = 0$ in R, for any $x \in R$. Therefore, R is a right e-semicommutative ring.

(2) Assume that $\alpha\beta = 0$, for $\alpha, \beta \in S$. Consider the following elements $\begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix} \in T$. We have $0 = \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix}$. Since T is a right $\begin{pmatrix} e & k \\ 0 & g \end{pmatrix}$-semicommutative ring, we get for any $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in T$,

$$0 = \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} e & k \\ 0 & g \end{pmatrix}.$$

Hence, $\alpha z \beta g = 0$ in S, for any $z \in S$. Therefore, S is a right g-semicommutative ring.

(3) Let $a \in S$ and $m \in M$ such that $ma = 0$. Consider the following elements $\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \in T$. We have $0 = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$. Since T is a right $\begin{pmatrix} e & k \\ 0 & g \end{pmatrix}$-semicommutative ring, we get for any $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in T$,

$$0 = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} e & k \\ 0 & g \end{pmatrix}.$$

Hence, $mzag = 0$ in M_S, for any $z \in S$. Therefore, M_S is a right g-semicommutative S-module.

Theorem 3.2. *Let $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$, where R and S are rings, and ${}_RM_S$ an (R,S)-bimodule. If T is a left $\begin{pmatrix} e & k \\ 0 & g \end{pmatrix}$-semicommutative ring, where $\begin{pmatrix} e & k \\ 0 & g \end{pmatrix} \in$* $\mathrm{Id}(T)$, *then:*

(1) R is a left e-semicommutative ring,
(2) S is a left g-semicommutative ring, and
(3) $_RM$ is a left e-semicommutative R-module.

Proof of Theorem 3.2. The proof is similar to the proof of Theorem 3.1.

Theorem 3.3. *Let $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ where R and S are rings, and $_RM_S$ an (R,S)-bimodule. If R is a right e-semicommutative ring, where $e \in \mathrm{Id}(R)$, then T is a right $\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}$-semicommutative ring.*

Proof of Theorem 3.3. Assume that R is a right e-semicommutative ring, where $e \in \mathrm{Id}(R)$. Let $\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}, \begin{pmatrix} q & n \\ 0 & p \end{pmatrix} \in T$ such that

$$0 = \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \begin{pmatrix} q & n \\ 0 & p \end{pmatrix} = \begin{pmatrix} aq & an + mp \\ 0 & bp \end{pmatrix}.$$

Hence, $aq = 0$ in R. Since R is a right e-semicommutative ring, we have $auqe = 0$, for any $u \in R$. Thus, for any $\begin{pmatrix} u & t \\ 0 & v \end{pmatrix} \in T$, we have

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \begin{pmatrix} u & t \\ 0 & v \end{pmatrix} \begin{pmatrix} q & n \\ 0 & p \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

Therefore, T is a right $\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}$-semicommutative ring.

Corollary 3.1. *Let $T_n(R)$ be the n-by-n upper triangular matrix ring over a ring R, where $n \geq 1$. Then, the following are equivalent:*
(1) *R is a right e-semicommutative ring, where $e \in \mathrm{Id}(R)$.*
(2) *$T_2(R) = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$ is a right $\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}$-semicommutative ring.*
(3) *$T_n(R)$ is a right $\begin{pmatrix} e & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$-semicommutative ring for every positive integer n.*

Proof. "(3) $\Longrightarrow$ (1)" follows directly from the fact that $T_1(R) \cong R$.

"(1) $\Longrightarrow$ (2)" is clear from Theorem 3.3.

"(2) $\Longrightarrow$ (3)" Note that $T_{n+1}(R) \cong \begin{pmatrix} R & M \\ 0 & T_n(R) \end{pmatrix}$ where M is the 1-by-n row matrix with R in every entry and 0 is the n-by-1 column zero matrix. So, this implication is proved by using induction on n. □

4. Polynomial extensions

This section is intended to motivate our investigation of the behavior of right e-semicommutative modules with respect to polynomial extensions.

Recall the following extensions of a right R-module M_R :

$$M\left[x\right]=\left\{\varphi(x)=\sum_{i=0}^{n}m_ix^i : m_i\in M\right\}.$$

$M\left[x\right]$ is a right $R\left[x\right]$-module and $M\left[x\right]_{R[x]}$ is called *the usual polynomial extension of* M_R.

$$M[x,x^{-1}]=\left\{\varphi(x)=\sum_{i=-k}^{n}m_ix^i : m_i\in M\right\}.$$

$M[x,x^{-1}]$ is a right $R[x,x^{-1}]$-module and $M[x,x^{-1}]_{R[x,x^{-1}]}$ is called *the usual Laurent polynomial extension of* M_R.

We mean by a regular element of a ring R, a nonzero element which is not a zero divisor.

Theorem 4.1. *Let R be a ring, Δ be a multiplicatively closed subset of R consisting of central regular elements, $1\in\Delta$ and $e\in \mathrm{Id}(R)$. Then, M_R is e-semicommutative if and only if $\left(\Delta^{-1}M\right)_{(\Delta^{-1}R)}$ is $\left(1^{-1}e\right)$-semicommutative.*

Proof of Theorem 4.1. Suppose that M_R is e-semicommutative. Let $a\in R$, $m\in M$ and $u,w\in\Delta$ such that $\left(w^{-1}m\right)\left(u^{-1}a\right)=0$ in $\left(\Delta^{-1}M\right)_{(\Delta^{-1}R)}$. Since Δ is contained in the center of R, we have $0=\left(w^{-1}u^{-1}\right)(ma)=(wu)^{-1}(ma)$, and so $ma=0$. Hence, for any $r\in R$, we have $mrae=0$. So, in $\left(\Delta^{-1}M\right)_{(\Delta^{-1}R)}$, we have for any $v\in\Delta$, $0=(wvu)^{-1}(mrae)=\left(w^{-1}v^{-1}u^{-1}1^{-1}\right)(mrae)$. Thus

$$\left(w^{-1}m\right)\left(v^{-1}r\right)\left(u^{-1}a\right)\left(1^{-1}e\right)=0.$$

Hence, $\left(\Delta^{-1}M\right)_{(\Delta^{-1}R)}$ is $\left(1^{-1}e\right)$-semicommutative.

It is clear that if $\left(\Delta^{-1}M\right)_{(\Delta^{-1}R)}$ is $\left(1^{-1}e\right)$-semicommutative, then M_R is e-semicommutative.

Corollary 4.1. *Let R be a ring and $e\in \mathrm{Id}(R)$. Then, $M\left[x\right]_{R[x]}$ is e-semicommutative if and only if $M[x,x^{-1}]_{R[x,x^{-1}]}$ is e-semicommutative.*

Proof. Consider the multiplicatively closed set $\Delta=\left\{1,x,x^2,x^3,...\right\}$ which is clearly a subset of $R[x]$ consisting of central regular elements. Since $\Delta^{-1}R\left[x\right]=R[x,x^{-1}]$ and $\Delta^{-1}M\left[x\right]=M[x,x^{-1}]$, the result follows directly from Theorem 4.1. □

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