The total graph of a commutative ring with respect to multiplication

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Abstract. Let R be a commutative ring with $1 \neq 0$, $Z(R)$ be the set of zero-divisors of R, and $Reg(R)$ be the set of regular elements of R. In this paper, we introduce and investigate the *dot total graph of* R and denote by $T_{Z(R)}(\Gamma(R))$. It is the (undirected) simple graph with all elements of R as vertices, and any two distinct vertices $x, y \in R$ are adjacent if and only if $xy \in Z(R)$. The graph $T_{Z(R)}(\Gamma(R))$ is shown to be connected and has a small diameter of at most two. Furthermore, $T_{Z(R)}(\Gamma(R))$ divides into two distinct subsets of R, i.e., $Z(R)$ and $Reg(R)$. Following that, the connectivity, clique number, and girth of the graph $T_{Z(R)}(\Gamma(R))$ were investigated. Finally, the traversability of the graph $T_{Z(R)}(\Gamma(R))$ is investigated.
Keywords: commutative rings, zero-divisor graph, regular elements, zero-divisors.

1. Introduction

Throughout this paper, let R be a commutative ring with unity $1 \neq 0$. In 1988, Beck [10] considered $\Gamma(R)$ as a simple graph, whose vertices are the elements of R and any two different elements x and y are adjacent if and only if $xy = 0$, but he was mainly interested in colorings. In 1993, Anderson and Naseer [6] continued this study by giving a counterexample, where R is a finite local ring.

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In 1999, Anderson and Livingston [3], associated a (simple) graph $\Gamma(R)$ to R with vertices $Z(R)^* = Z(R) \setminus \{0\}$, the set of nonzero zero-divisors of R, and for distinct $x, y \in Z(R)^*$, the vertices x and y are adjacent if and only if $xy = 0$ and they were interested to study the interplay of ring-theoretic properties of R with graph-theoretic properties of $\Gamma(R)$. In 2008, Anderson and Badawi [4] introduced the total graph of R, denoted by $T(\Gamma(R))$, as the (undirected) graph with all elements of R as vertices and for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x + y \in Z(R)$. In 2012, Abbasi and Habibi [2] introduced and studied the total graph of a commutative ring R with respect to proper ideal I, denoted by $T(\Gamma_I(R))$. In addition, some fundamental graphs with vector spaces can be identified in [7, 8].

Let G be a graph. We say that G is connected if there is a path between any two distinct vertices of G. For distinct vertices x and y of G, we define $d(x, y)$ to be the length of the shortest path from x to y ($d(x, y) = \infty$ if there is no such path). The diameter of G is $diam(G) = sup\{d(x, y) \mid x$ and y are distinct vertices of $G\}$. The girth of G, denoted by $gr(G)$, is defined as the length of the shortest cycle in G ($gr(G) = \infty$ if G contains no cycle). Note that if G contains a cycle, then $gr(G) \leq 2\ diam(G) + 1$. The complement $\overline{G}$ of a graph G is that graph whose vertex set is $V(G)$ and such that for each pair u, v of distinct vertices of G, uv is an edge of $\overline{G}$ if and only if uv is not an edge of G. The degree of vertex v, written $deg_G(v)$ or $deg(v)$, is the number of edges incident to v, (or the degree of the vertex v is the number of vertices adjacent to v). In a connected graph G, a vertex v is said to be a cut-vertex of G if and only if $G \setminus \{v\}$ is disconnected. Let $V(G)$ be a vertex set of G. Then the subset $U \subseteq V(G)$ is called as vertex-cut if $G \setminus U$ is disconnected. The connectivity of a graph G denoted by $k(G)$ and is defined as the cardinality of a minimum vertex-cut of G, also the same concepts we have for the edges. In a connected graph G, an edge e is said to be a bridge of G if and only if $G \setminus \{e\}$ is disconnected. Let $E(G)$ be an edge set of G. Then the subset $X \subseteq E(G)$ is called an edge-cut if $G \setminus X$ is disconnected. The edge-connectivity of a graph G denoted by $\lambda(G)$ and is defined as the cardinality of a minimum edge-cut of G. A complete subgraph of a graph G is called a clique. The clique number denoted by $\omega(G)$, is the greatest integer $n \geqslant 1$ such that $K_n \subseteq G$, and $\omega(G) = \infty$ if $K_n \subseteq G$ for all $n \geqslant 1$. A nontrivial connected graph G is Eulerian if and only if every vertex of G has even degree. Also, G contains an Eulerian trail if and only if exactly two vertices of G have odd degree. In addition, let G be a graph of order $n \geq 3$. If $deg(u) + deg(v) \geq n$ for each pair u, v of nonadjacent vertices of G, then G is Hamiltonian. The present paper is organise as follows:

In Section 2, we introduce the definition of the total graph of R with respect to multiplication. We give some examples, and show that $T_{Z(R)}(\Gamma(R))$ is always connected with $diam(T_{Z(R)}(\Gamma(R))) \leqslant 2$ and $gr(T_{Z(R)}(\Gamma(R))) \leqslant 5$, and we establish if the graph $T_{Z(R)}(\Gamma(R))$ is a complete graph or a star graph based on the type of ring and we observe that if R is not trivial then $T_{Z(R)}(\Gamma(R))$ is not null graph. Also, we find the degree of each vertex of $T_{Z(R)}(\Gamma(R))$. Further, in Section 3, we study the connectivity of $\overline{K}_n \vee K_m$ when $T_{Z(R)}(\Gamma(R))$ has no cut-vertex and $T_{Z(R)}(\Gamma(R))$ has a bridge. We also, find the $k(T_{Z(R)}(\Gamma(R)))$. Furthermore, in Section 4, we study the clique number of the graph $\overline{K}_n \vee K_m$. Also, we find the girth of $T_{Z(R)}(\Gamma(R))$ i.e., $gr(\overline{K}_n \vee K_m)$. Finally, in Section 5, we study the traversability of the graph $T_{Z(R)}(\Gamma(R))$ when the graph $T_{Z(R)}(\Gamma(R))$ have an Eulerian trail and $T_{Z(R)}(\Gamma(R))$ is Hamiltonian. Further, we generalized the definition of the graph $T_{Z(R)}(\Gamma(R))$ and denoted by $T_A(\Gamma(B))$. Also, we investigate some properties viz complement graph, spaning subgraph, induced subgraph of $T_A(\Gamma(B))$.

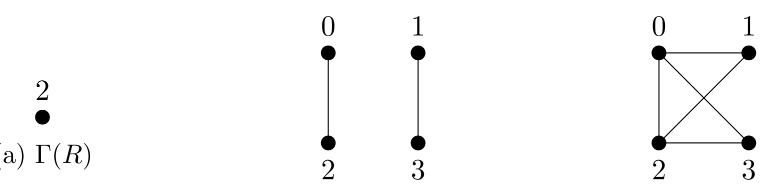
2. Definition and properties of $T_{Z(R)}(\Gamma(R))$

We begin this section by define dot total graph of a commutative ring and denoted by $T_{Z(R)}(\Gamma(R))$. We demonstrate that $T_{Z(R)}(\Gamma(R))$ is always connected and has small diameter which is less than or equal to two and girth which is less than or equal to five. We start with some examples which motivate later results and we associate some examples from zero-divisor graph of a commutative ring, total graph and compare them with $T_{Z(R)}(\Gamma(R))$.

Definition 2.1. *Let R be a commutative ring with $1 \neq 0$ and $Z(R)$ be the set of zero-divisors of R, and $Reg(R)$ be the set of regular elements of R. We define an undirected simple graph $T_{Z(R)}(\Gamma(R))$, whose vertices are all the elements of R and any two distinct vertices x and y of $T_{Z(R)}(\Gamma(R))$ are adjacent if and only if $xy \in Z(R)$.*

Example 2.1. We have several rings with its set of zero-divisor $Z(R)$ and its set of regular elements $Reg(R)$ and comparisons $\Gamma(R)$, $T(\Gamma(R))$ and $T_{Z(R)}(\Gamma(R))$:

(i) $R = \mathbb{Z}_4, Z(R) = \{0, 2\}$ and $Reg(R) = \{1, 3\}$ (see Fig.1)



(a) $\Gamma(R)$ (b) $T(\Gamma(R))$ (c) $T_{Z(R)}(\Gamma(R))$

Figure 1: (a)$\Gamma(R)$, (b)$T(\Gamma(R))$ and (c)$T_{Z(R)}(\Gamma(R))$, when $R = \mathbb{Z}_4$

(ii) $R = \mathbb{Z}_2[x]/(x^2) = \{0, 1, x, 1 + x\}, Z(R) = \{0, x\}$ and $Reg(R) = \{1, 1 + x\}$ (see Fig. 2)

(iii) $R = \mathbb{Z}_9, Z(R) = \{0, 3, 6\}$ and $Reg(R) = \{1, 2, 4, 5, 7, 8\}$ (see Fig. 3)

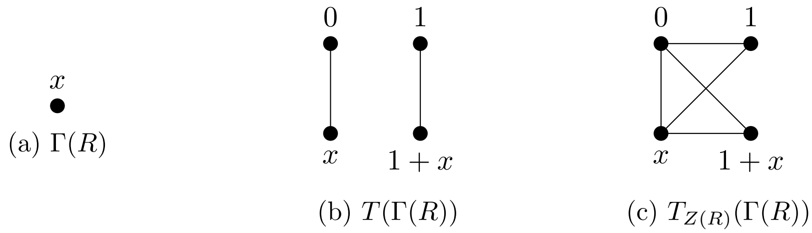


Figure 2: (a)$\Gamma(R)$, (b)$T(\Gamma(R))$ and (c)$T_{Z(R)}(\Gamma(R))$, when $R = \mathbb{Z}_2[x]/(x^2)$

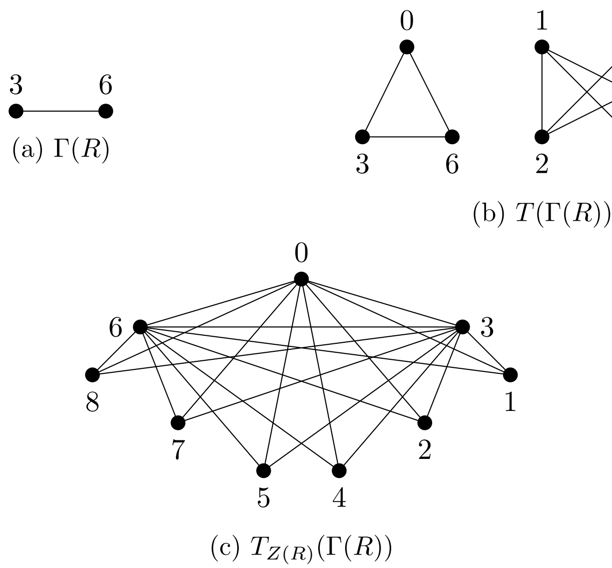


Figure 3: (a)$\Gamma(R)$, (b)$T(\Gamma(R))$ and (c)$T_{Z(R)}(\Gamma(R))$, when $R = \mathbb{Z}_9$

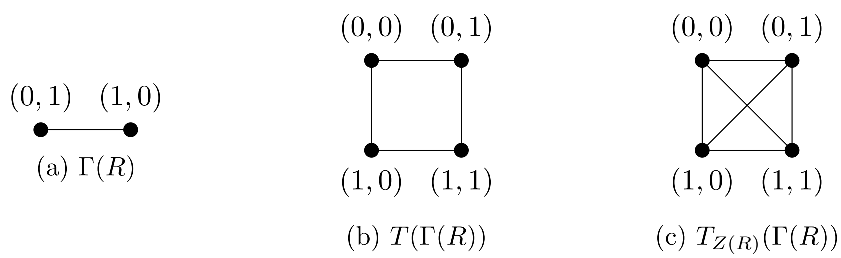


Figure 4: (a)$\Gamma(R)$, (b)$T(\Gamma(R))$ and (c)$T_{Z(R)}(\Gamma(R))$, when $R = \mathbb{Z}_2 \times \mathbb{Z}_2$

(iv) $R = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0),(0,1),(1,0),(1,1)\}, Z(R) = \{(0,0),(0,1),(1,0)\}$ and $Reg(R) = \{(1,1)\}$ (see Fig. 4)

(v) $R = \mathbb{Z}_3[x]/(x^2) = \{0,1,2,x,2x,1+x,2+x,1+2x,2+2x\}$, $Z(R) = \{0,x,2x\}$ and $Reg(R) = \{1,2,1+x,2+x,1+2x,2+2x\}$ (see Fig. 5)

(vi) $R = \mathbb{Z}_6, Z(R) = \{0,2,3,4\}$ and $Reg(R) = \{1,5\}$ (see Fig. 6)

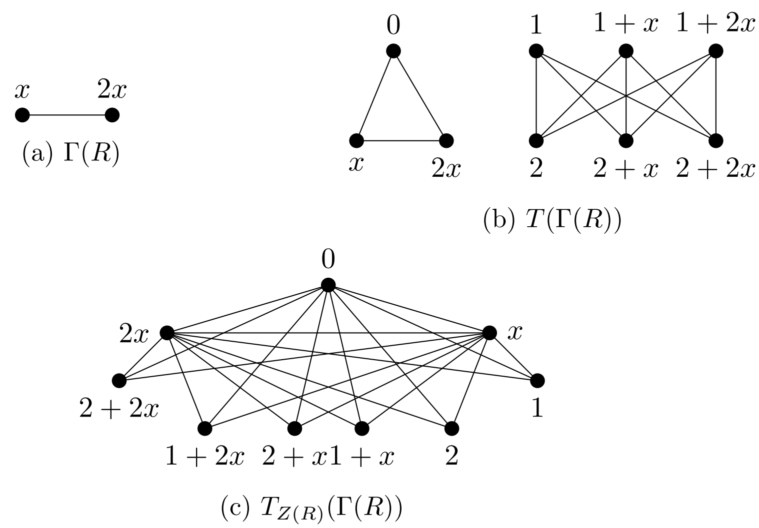


Figure 5: (a)$\Gamma(R)$, (b)$T(\Gamma(R))$ and (c)$T_{Z(R)}(\Gamma(R))$, when $R = \mathbb{Z}_3[x]/(x^2)$

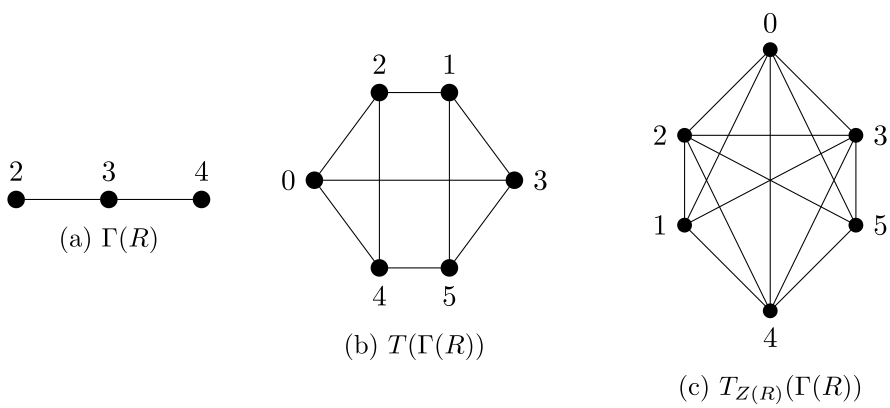


Figure 6: (a)$\Gamma(R)$, (b)$T(\Gamma(R))$ and (c)$T_{Z(R)}(\Gamma(R))$, when $R = \mathbb{Z}_6$

(vii) $R = \mathbb{Z}_8, Z(R) = \{0, 2, 4, 6\}$ and $Reg(R) = \{1, 3, 5, 7\}$ (see Fig. 7)

$(viii)$ $R = \mathbb{Z}_7, Z(R) = \{0\}$ and $Reg(R) = \{1, 2, 3, 4, 5, 6\}$ (see Fig 8)

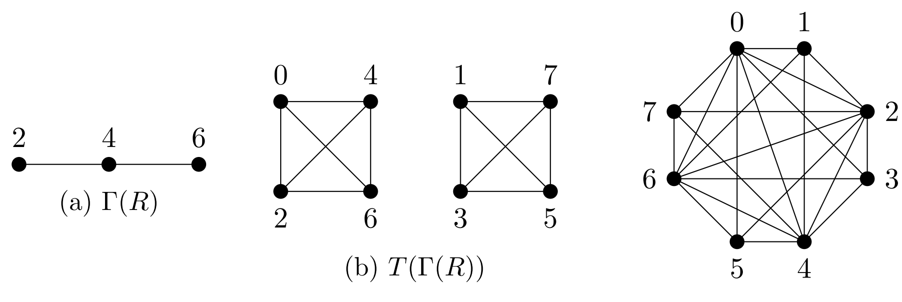
Remark 2.1.

(1) Note that these examples show that non isomorphic rings may have the same zero-divisor graph, but in dot total graph the non isomorphic rings R_1 and R_2 have the following:

  (a) If $|R_1| \neq |R_2|$, then $T_{Z(R_1)}(\Gamma(R_1)) \ncong T_{Z(R_2)}(\Gamma(R_2))$.

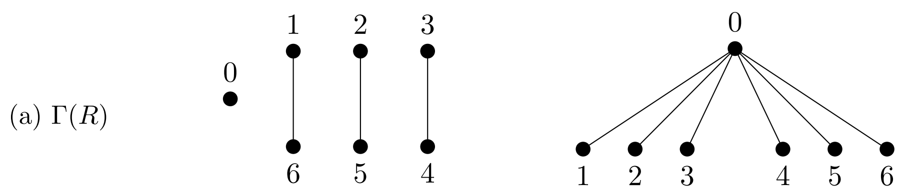
  (b) If $|R_1| = |R_2|$, then they may have the same dot total graph.

(2) For any integral domain R, we know that $\Gamma(R) = \emptyset$ (null graph), but here $T_{Z(R)}(\Gamma(R))$ is complete bipartite graph of the form $K_{1,n}$ is called



(a) $\Gamma(R)$ (b) $T(\Gamma(R))$ (c) $T_{Z(R)}(\Gamma(R))$

Figure 7: (a)$\Gamma(R)$, (b)$T(\Gamma(R))$ and (c)$T_{Z(R)}(\Gamma(R))$, when $R = \mathbb{Z}_8$



(a) $\Gamma(R)$ (b) $T(\Gamma(R))$ (c) $T_{Z(R)}(\Gamma(R))$

Figure 8: (a)$\Gamma(R)$, (b)$T(\Gamma(R))$ and (c)$T_{Z(R)}(\Gamma(R))$, when $R = \mathbb{Z}_7$

a star graph and $n = |R| - 1$, if R is finite (previous Example $(viii)$) otherwise $n = \infty$, if R is infinite.

(3) Let R be a commutative ring. Then the following statements hold:

(i) If $x \in Z(R)$, then x is adjacent to each vertex $y \in R$.

(ii) If $x \in Reg(R)$, then x is adjacent to $y \in Z(R)$, only.

(iii) Any two distinct verties of $Reg(R)$ are not adjacent in $T_{Z(R)}(\Gamma(R))$.

(4) $T(\Gamma(R))$ may be connected and may not. That is, if R is a finite commutative ring and $Z(R)$ is not an ideal of R, then $T(\Gamma(R))$ is connected [4], but $T_{Z(R)}(\Gamma(R))$ is connected as we prove in next theorem.

We next show that the all dot total graphs of R are connected and study the diameter and girth.

Theorem 2.1. *$T_{Z(R)}(\Gamma(R))$ is connected and $diam(T_{Z(R)}(\Gamma(R))) \leq 2$. Moreover, if $T_{Z(R)}(\Gamma(R))$ contains a cycle, then $gr(T_{Z(R)}(\Gamma(R))) \leq 5$.*

Proof. Let x and y be distinct vertices of $T_{Z(R)}(\Gamma(R))$.

$Case(i)$ If $x, y \in Z(R)$, then $x - y$ is a path in $T_{Z(R)}(\Gamma(R))$.

$Case(ii)$ If $x, y \in Reg(R)$, then there is some $z \in Z(R)$ such that $xz \in Z(R)$ and $yz \in Z(R)$. Thus $x - z - y$ is a path.

Case(iii) If $x \in Z(R)$ and $y \in Reg(R)$, then $x - y$ is a path.

Thus $T_{Z(R)}(\Gamma(R))$ is connected and $diam(T_{Z(R)}(\Gamma(R))) \leq 2$. Since for any undirected graph H, $gr(H) \leq 2\, diam(H) + 1$, H contains a cycle (for reference see [12]). Thus $gr(T_{Z(R)}(\Gamma(R))) \leq 5$. □

Remark 2.2. For any commutative ring R with $1 \neq 0$, we know that $\Gamma(R)$ is connected and has $diam(\Gamma(R)) \leq 3$ and if $\Gamma(R)$ contains a cycle, then $gr(\Gamma(R)) \leq 7$ (for reference see [3]). Also, the same results hold for $\Gamma_I(R)$ (for reference see [15]). In addition, if $T(\Gamma(R))$ is connected, then $diam(T(\Gamma(R))) = d(0,1)$ (for reference see [4]). But for $T_{Z(R)}(\Gamma(R))$, we get a connected graph which has $diam(T_{Z(R)}(\Gamma(R))) \leq 2$ and if $T_{Z(R)}(\Gamma(R))$ contains a cycle, then $gr(T_{Z(R)}(\Gamma(R))) \leq 5$.

The graph $T_{Z(R)}(\Gamma(R))$ has a very special form. In fact, if $|Z(R)| = m$ and $|Reg(R)| = n$ then $T_{Z(R)}(\Gamma(R)) \cong \overline{K}_n \vee K_m$, where $\vee$ is used for the join of two graphs.

Theorem 2.2. *The graph $\overline{K}_n \vee K_m$ is complete iff $n = 1$.*

Proof. Suppose $\overline{K}_n \vee K_m$ is complete. Then each distinct vertices in R are adjacent. If $n > 1$, then there is at least two vertices x and y in $\overline{K}_n$ which are non adjacent, which is a contradiction. Hence $n = 1$.

Conversely, suppose that $n = 1$. Then it is clear that $\overline{K}_n \vee K_m$ is complete graph. □

Corollary 2.1. $T_{Z(R)}(\Gamma(R))$ *is not complete if and only if* $|Reg(R)| \geqslant 2$.

Corollary 2.2. $\overline{T_{Z(R)}(\Gamma(R))}$ *is K_n with vertices of regular elements of R, where $n = |Reg(R)|$ and other vertices are isolated (elements of $Z(R)$).*

Remark 2.3. Let R be a finite commutative ring. Then the following statements hold:

(i) If $Z(R)$ is an ideal, then $T(\Gamma(R))$ is not connected [11, 13] and for any element $x \in R$, there are two possibilities:

- (a) If $2 \in Z(R)$, then $deg(x) = |Z(R)| - 1$ for each $x \in R$.
- (b) If $2 \notin Z(R)$, then $deg(x) = |Z(R)| - 1$ for each $x \in Z(R)$ and $deg(x) = |Z(R)|$ for each $x \in Reg(R)$.

(ii) If $Z(R)$ is not an ideal, then $T(\Gamma(R))$ is connected and $deg(x) = |Z(R)| - 1$ for each $x \in R$.

In the next theorem, we find the degree of each vertex of $T_{Z(R)}(\Gamma(R)) \cong \overline{K}_n \vee K_m$.

Theorem 2.3. *The degree of vertices in the graph $\overline{K}_n \vee K_m$ are m or $m+n-1$.*

Proof. Since vertices in the graph $\overline{K}_n \vee K_m$ are belong to either K_m or $\overline{K}_n$, we have the following two cases:

Case(i) If $x \in K_m$, then x is adjacent to each vertex in $\overline{K}_n \vee K_m$ except x, that is, x is adjacent to $m+n-1$ vertices and hence degree of x is $m+n-1$.

Case(ii) If $x \in \overline{K}_n$, then x is adjacent to the vertices, which belongs to K_m, that is, x is adjacent to m vertices and hence $deg(x) = m$. □

Corollary 2.3. *The graph $\overline{K}_n \vee K_m$ is regular graph iff $n = 1$.*

Remark 2.4. For any graph G, $\delta(G)$ is the minimum degree of G and $\Delta(G)$ is the maximum degree of G. Here for $G = T_{Z(R)}(\Gamma(R))$, $\delta(G) = |Z(R)|$ and $\Delta(G) = |R| - 1$.

3. Connectivity of $\overline{K}_n \vee K_m$

In this section, we study the connectivity of $\overline{K}_n \vee K_m$.

Theorem 3.1. *The graph $\overline{K}_n \vee K_m$ has a cut vertex iff $m = 1$. i.e., R is an integral domain.*

Proof. Assume that the vertex x of $\overline{K}_n \vee K_m$ is a cut-vertex. Then there exist $u, w \in \overline{K}_n \vee K_m$ such that x lies on every path from u to w. Thus we have the following two cases:

Case(i) If u is adjacent to w, then we get a contradiction.

Case(ii) If u is not adjacent to w, then $u, w \in \overline{K}_n$ and $x \in K_m$. Now, if $m > 1$, then K_m have more than one vertices. i.e., $x \neq y \in K_m$. Therefore, there is at least one path from u to w and x does not lie on it, which is a contradiction. Hence $m = 1$.

Conversely, assume that $m = 1$. Then it is clear that $\overline{K}_n \vee K_m$ has a cut vertex. □

Theorem 3.2. *The graph $\overline{K}_n \vee K_m$ has a bridge iff $m = 1$. i.e., R is an integral domain.*

Proof. Suppose that $\overline{K}_n \vee K_m$ has a bridge. Now we have the following cases:

Case(i) If $|R| = 2$, then it is clear that $m = 1$.

Case(ii) If $|R| \geqslant 3$, then either $V(\overline{K}_n \vee K_m) \subseteq V(\overline{K}_n)$ or $V(\overline{K}_n \vee K_m) \subseteq V(K_m)$ and we know that there is no edge between any two elements of $\overline{K}_n$, and we have an edge either between each $x, y \in V(K_m)$ or each $x \in V(K_m)$ with all $y \in R$. Therefor we have the following subcases:

Subcase(a) If $x, y \in V(K_m)$ and $|R| \geqslant 3$, then there exists $z \in R$ such that x and y are adjacent to z. We note that $x - y - z - x$ is a cycle, and there is no bridge between them, we get a contradiction.

Subcase(b) If $x \in V(K_m)$, $y \in V(\overline{K}_n)$ and $|R| \geqslant 3$, then there exists at least one element $z \in R \setminus \{x, y\}$. There are two possibilities:

If $z \in V(K_m)$, then z is adjacent to x and y. Thus $x - z - y - x$ is a cycle and there is no bridge between them. This is a contradiction.

If $z \in V(\overline{K}_n)$ and only $x \in V(K_m)$ (here $x = 0$, additive identity), then x is adjacent to each $z \in V(\overline{K}_n)$ and there is no adjacency between any two elements of $\overline{K}_n$. Thus there are more than one vertex adjacent to x and $0 = x \in V(K_m)$ only, otherwise, there is a cycle. Thus all edges are bridge. Hence $m = 1$.

Converse of the proof is trivial. □

Remark 3.1. If the ring $R \cong \mathbb{Z}_2$ or R is an integral domain, then $T_{Z(R)}(\Gamma(R))$ has a bridge and vice versa.

Theorem 3.3. *$k(\overline{K}_n \vee K_m) = m$.*

Proof. We know that, for any graph G, $k(G) \leqslant \lambda(G) \leqslant \delta(G)$ and by Remark 2.4, $\delta(\overline{K}_n \vee K_m) = |Z(R)| = m$. Therefore,

$$k(\overline{K}_n \vee K_m) \leqslant m.$$

Now if $x \in V(K_m)$, then x is adjacent to each vertex $y \in R$. Hence the minimum vertex-cut is the set of all those vertices in $V(K_m)$, otherwise, $\overline{K}_n \vee K_m$ is connected. Hence $k(\overline{K}_n \vee K_m)) = m$. □

Remark 3.2. For any commutative ring R with $1 \neq 0$, $Z(R)$ is the minimum vertex-cut of $T_{Z(R)}(\Gamma(R))$.

4. Clique number of $\overline{K}_n \vee K_m$

In this section, we study the clique number of $\overline{K}_n \vee K_m$.

Theorem 4.1. *$\omega(\overline{K}_n \vee K_m) = m + 1$.*

Proof. We know that each pair of elements in K_m are adjacent. In general, they are adjacent to all elements of $\overline{K}_n \vee K_m$. Thus each element is adjacent at least to one element in $\overline{K}_n$. Since $|K_m| = m$, we find that $m + 1$ elements are adjacent. This completes the proof. □

Corollary 4.1. *If $m \geqslant 2$, then $gr(\overline{K}_n \vee K_m) = 3$. If $m = 1$,i.e., R is an integral domain, then $gr(\overline{K}_n \vee K_m) = \infty$.*

Proof. Suppose that $m \geqslant 2$. Then by the same arguments as used in the above theorem, and given that $|K_m| = m \geqslant 2$, we find that at least two elements are in K_m. Let $u, v \in K_m$. Also, $\overline{K}_n \vee K_m$ has at least one element $w \in \overline{K}_n$. Then $u - w - v - u$ is a cycle of length 3, which is the smallest cycle in $\overline{K}_n \vee K_m$. Hence $gr(\overline{K}_n \vee K_m) = 3$.

Suppose that $m = 1$. Then there is no cycle in $\overline{K}_n \vee K_m$. Hence $gr(\overline{K}_n \vee K_m) = \infty$. □

5. Traversability of $T_{Z(R)}(\Gamma(R))$

In this section, we show that $T_{Z(R)}(\Gamma(R))$ can not be an Eulerian graph. Also, we discover the types of rings that make the graph $T_{Z(R)}(\Gamma(R))$ have an Eulerian trail. Further, we find out when the graph $T_{Z(R)}(\Gamma(R))$ is Hamiltonian graph.

Theorem 5.1. *$T_{Z(R)}(\Gamma(R))$ can not be an Eulerian graph.*

Proof. First of all, we prove that $T_{Z(R)}(\Gamma(R))$ is an Eulerian if and only if $|R|$ is odd and $|Z(R)|$ is even. Moreover, $|Reg(R)|$ is odd. Suppose that $T_{Z(R)}(\Gamma(R))$ is an Eulerian. Then every vertex of $T_{Z(R)}(\Gamma(R))$ has even degree. Since the degree of each vertex of $T_{Z(R)}(\Gamma(R))$ either $(|R| - 1)$ or $|Z(R)|$ (Theorem 2.3), we have the following cases:

$Case(i)$ If $x \in Z(R)$, then $deg(x) = |R| - 1$, which is even, and we get $|R|$ is odd.

$Case(ii)$ If $x \in Reg(R)$, then $deg(x) = |Z(R)|$, which is even. Thus $|Z(R)|$ is even.

Hence $|R|$ is odd and $|Z(R)|$ is even. Moreover, $|Reg(R)|$ is odd.

Conversely, suppose that $|R|$ is odd and $|Z(R)|$ is even. Then $|R| - 1$ is even and $|Z(R)|$ is also even. Since the degree of each vertex of $T_{Z(R)}(\Gamma(R))$ is either $|R| - 1$ or $|Z(R)|$, degree of each vertex of $T_{Z(R)}(\Gamma(R))$ is even. Hence $T_{Z(R)}(\Gamma(R))$ is an Eulerian.

Second, we show that there is no ring R such that $T_{Z(R)}(\Gamma(R))$ be an Eulerian graph. If $u \in Reg(R) = U(R)$ then $u^n = 1$ where $n = |U(R)|$. So, if n is an odd number, then $-1 = (-1)^n = 1$. Hence $2 = 0$ and $Char(R) = 2$. Thus $|R| = 2^k$. So, there is no ring R such that $T_{Z(R)}(\Gamma(R))$ be an Eulerian graph. □

Theorem 5.2. *$T_{Z(R)}(\Gamma(R))$ has an Eulerian trail iff $R \cong \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \frac{\mathbb{Z}_2[x]}{(x^2)}$.*

Proof. Suppose that $|Z(R)| = |Reg(R)| = 1$. Then $T_{Z(R)}(\Gamma(R))$ has an Eulerian trail and $R \cong \mathbb{Z}_2$. Now suppose that $|Z(R)| > 1$ or $|Reg(R)| > 1$. Then we prove that $T_{Z(R)}(\Gamma(R))$ has an Eulerian trail if and only if either $|Z(R)| = 2$ and $|Reg(R)|$ is even or $|Reg(R)| = 2$ and $|Z(R)|$ is odd. Suppose that $T_{Z(R)}(\Gamma(R))$ has an Eulerian trail. Then exactly two vertices of $T_{Z(R)}(\Gamma(R))$ have odd degree. Let u and v be the two vertices of odd degree and let $x_1, x_2, ..., x_n$ be the vertices of even degree. Then we have the following cases:

Case(i) If $u, v \in Z(R)$ and $x_i \in Reg(R)$ for all $1 \leq i \leq n$, then $deg(u) = deg(v)$ is odd and $deg(x_i)$ for all $1 \leq i \leq n$ is even, therefor $|R| - 1$ is odd and $|Z(R)| = 2$ is even, thus $|R|$ is even and $|Z(R)| = 2$. Hence $|Z(R)| = 2$ and $|Reg(R)|$ is even. Moreover, $|R|$ is even.

Case(ii) If $u, v \in Z(R)$ and there exists at least one $x_j \in Z(R)$, then $deg(u) = deg(v) = deg(x_j)$ is odd. Hence there are more than two odd vertices in $T_{Z(R)}(\Gamma(R))$, we get a contradiction.

Case(iii) If $u, v \in Reg(R)$ and $x_i \in Z(R)$ for all $1 \leq i \leq n$, then $deg(u) = deg(v)$ is odd and $deg(x_i)$ for all $1 \leq i \leq n$ is even. Note that $|Z(R)|$ is odd and $|R| - 1$ is even. We get $|Z(R)|$ is odd and $|R|$ is odd. Since $u, v \in Reg(R)$ only, we have $|Reg(R)| = 2$. Hence $|Reg(R)| = 2$ and $|Z(R)|$ is odd. Moreover, $|R|$ is odd.

Case(iv) If $u, v \in Reg(R)$ and there exists at least one $x_j \in Reg(R)$, then $deg(u) = deg(v) = deg(x_j)$ is odd. Thus there are more than two odd vertices in $T_{Z(R)}(\Gamma(R))$, we get a contradiction.

Case(v) If $u \in Z(R)$ and $v \in Reg(R)$, then $deg(u) = deg(v) = deg(x_i)$ for all $1 \leq i \leq n$ is odd. Thus all the vertices of $T_{Z(R)}(\Gamma(R))$ have odd degree, we get a contradiction.

Therefore in all the cases, we get that either $|Z(R)| = 2$ and $|Reg(R)|$ is even or $|Reg(R)| = 2$ and $|Z(R)|$ is odd.

Conversely, suppose that either $|Z(R)| = 2$ and $|Reg(R)|$ is even or $|Reg(R)| = 2$ and $|Z(R)|$ is odd. Now we assume that $|Z(R)| = 2$ and $|Reg(R)|$ is even, let x be any vertex of $T_{Z(R)}(\Gamma(R))$, then we have the following cases:

Case(i) If $x \in Z(R)$, then $deg(x) = |R| - 1$, which is odd. Since $|Z(R)| = 2$ and $|Reg(R)|$ is even, there are only two vertices in $Z(R)$ have odd degree and each other vertices in $Reg(R)$ have even degree. Hence $T_{Z(R)}(\Gamma(R))$ contains an Eulerian trail.

Case(ii) If $x \in Reg(R)$, then $deg(x) = |Z(R)| = 2$, which is even, by the same argument, there are only two vertices $x_1, x_2 \in Z(R)$ such that x_1 and x_2 are adjacent to each vertices in $Reg(R)$ and x_1 adjacent to x_2 and $deg(x_1) = deg(x_2) = |Reg(R)| + 1$ which is odd. Therefor, there are only two vertices in $Z(R)$ have odd degree and each other vertices in $Reg(R)$ have even degree. Hence $T_{Z(R)}(\Gamma(R))$ contains an Eulerian trail.

After that, we assume that $|Reg(R)| = 2$ and $|Z(R)|$ is odd. Then $|R|$ is odd, and let x be any vertex of $T_{Z(R)}(\Gamma(R))$. Then we have the following cases:

Case(i) If $x \in Z(R)$, then $deg(x) = |R| - 1$, which is even. Since $|Reg(R)| = 2$ and $|Z(R)|$ is odd, there are only two vertices in $Reg(R)$ have odd degree and each other vertices in $Z(R)$ have even degree. Hence $T_{Z(R)}(\Gamma(R))$ contains an Eulerian trail.

Case(ii) If $x \in Reg(R)$, then $deg(x) = |Z(R)|$, which is odd, thus $|R|$ is odd. By the same argument, there are only two vertices in $Reg(R)$ have odd degree and each other vertices in $Z(R)$ have degree $|R|-1$, which is even. Hence $T_{Z(R)}(\Gamma(R))$ contains an Eulerian trail.

From all the above cases we conclude that $T_{Z(R)}(\Gamma(R))$ contains an Eulerian trail. Hence if either $|Z(R)| = 2$ and $|Reg(R)|$ is even or $|Reg(R)| = 2$ and $|Z(R)|$ is odd, then $T_{Z(R)}(\Gamma(R))$ contains an Eulerian trail.

Second, we show that $T_{Z(R)}(\Gamma(R))$ has an Eulerian trail iff $R \cong \mathbb{Z}_3, \mathbb{Z}_4, \frac{\mathbb{Z}_2[x]}{(x^2)}$.

(i) Assume $|Z(R)| = 2$. Let $0 \neq x \in Z(R)$. Since $ann(x), Rx \subseteq Z(R)$ we conclude $ann(x) = Rx = Z(R)$. So, the isomorphism $\frac{R}{ann(x)} \cong Rx$ implies $|R| = 4$. Hence $R \cong \mathbb{Z}_4$ or $R \cong \frac{\mathbb{Z}_2[x]}{(x^2)}$.

(ii) It is well known that every commutative artinian ring is isomorphic to direct product of finitely many local rings. If R is a finite local ring with the unique maximal ideal M, then $|R| = p^r, m = |Z(R)| = |M| = p^s$ and $n = |Reg(R)| = |U(R)| = p^r - p^s$. In particular, $m|n$. So, if $n = 1$, then $R \cong \mathbb{Z}_2$. If $n = 2$, then $m = 1$ or $m = 2$. If $m = 1$, then $R \cong \mathbb{Z}_3$. If $m = 2$, then $|R| = 4$ and $R \cong \mathbb{Z}_4$. So, the only odd order ring with $|Reg(R)| = 2$ is $\mathbb{Z}_3$. □

Remark 5.1. In the above theorem, if $|Z(R)| = 2$, then Eulerian trail of $T_{Z(R)}(\Gamma(R))$ begins at one of these two elements of $Z(R)$ and ends at other. Also, if $|Reg(R)| = 2$, then Eulerian trail of $T_{Z(R)}(\Gamma(R))$ begins at one of these two elements of $Reg(R)$ and ends at other.

Theorem 5.3. *Let R has a maximal ideal of index 2 and $|R| > 2$, then $T_{Z(R)}(\Gamma(R))$ is Hamiltonian.*

Proof. The graph $\overline{K}_n \vee K_m$ is Hamiltonian iff $m \geq max\{n, 2\}$. So, $T_{Z(R)}(\Gamma(R))$ is Hamiltonian iff $|R| > 2$ and $|Z(R)| \geq \frac{|R|}{2}$ iff $|R| > 2$ and $\frac{|U(R)|}{|R|} \leq \dfrac{1}{2}$. Since $\frac{|U(R)|}{|R|} = \frac{|U(\frac{R}{J(R)})|}{|\frac{R}{J(R)}|}$ ($J(R)$ is the Jacobson radial of R). So, $T_{Z(R)}(\Gamma(R))$ is Hamiltonian if $T_{Z(\frac{R}{J(R)})}(\Gamma(\frac{R}{J(R)}))$ is Hamiltonian. Also If $T_{Z(R)}(\Gamma(R))$ is Hamiltonian and $|R/J(R)| > 2$, then $T_{Z(\frac{R}{J(R)})}(\Gamma(\frac{R}{J(R)}))$ is Hamiltonian. Since $\frac{R}{J(R)} \cong \prod_{M_i \in Max(R)} \dfrac{R}{M_i} \cong \prod F_{q_i}$ ($\frac{R}{M_i} \cong F_{q_i}$ is a field). So, $\frac{|U(\frac{R}{J(R)})|}{|\frac{R}{J(R)}|} = \prod \frac{q_i - 1}{q_i}$. In particular, if R has a maximal ideal of index 2 and $|R| > 2$, then $T_{Z(R)}(\Gamma(R))$ is Hamiltonian. Also $T_{Z(R)}(\Gamma(R))$ is Hamiltonian for a local ring (R, M) iff $|R/M| = 2$ and $|R| > 2$. □

Corollary 5.1. *Let R be a local ring and has k maximal ideal. If $T_{Z(R)}(\Gamma(R))$ is Hamiltonian, then $R/J(R) \cong F$, i.e., $k = 1$ and $J(R)$ is maximal ideal of R.*

Corollary 5.2. *Let R be a finite commutative ring with $1 \neq 0$, such that $|R| = n \geq 3$. If $|Z(R)| \geq \frac{n}{2}$ for each pair u, v of $Reg(R)$, then $T_{Z(R)}(\Gamma(R)) + uv$ is Hamiltonian if and only if $T_{Z(R)}(\Gamma(R))$ is Hamiltonian.*

Let $A, B \subseteq R$. Define $T_A(\Gamma(B))$ be a graph whose vertex set is B and two distinct vertices x, y are adjacent if $xy \in A$.

Theorem 5.4. *The graph $T_A(\Gamma(B))$ is the complement graph of $T_{A^c}(\Gamma(B))$ where $A^c = R \setminus A$.*

Proof. Let u and v be two distinct vertices of B. Then $T_A(\Gamma(B))$ and $T_{A^c}(\Gamma(B))$ have the same set of vertices. Since $uv \in A$ if and only if $uv \notin A^c$, we get that uv is an edge of $T_A(\Gamma(B))$ if and only if uv is not an edge of $T_{A^c}(\Gamma(B))$. $\square$

Theorem 5.5. *The graph $T_A(\Gamma(B))$ is a spaning subgraph of $T_C(\Gamma(B))$ if $A \subseteq C$.*

Proof. Let $A \subseteq C$. Since $T_A(\Gamma(B))$ and $T_C(\Gamma(B))$ have the same set of vertices depending on B, we have to prove that the edge set of $T_A(\Gamma(B))$ contains in the edge set of $T_C(\Gamma(B))$. To complete the prove, assume, on contrary, that the edge set of $T_A(\Gamma(B))$ contains the edge set of $T_C(\Gamma(B))$. Then for every two distinct vertices $u, v \in B$ that adjacent in $T_C(\Gamma(B))$ should be adjacent in $T_A(\Gamma(B))$. By definitions of $T_C(\Gamma(B))$ and $T_A(\Gamma(B))$, we get that $C \subseteq A$, which is a contradiction. Hence $T_A(\Gamma(B))$ is the spanning subgraph of $T_C(\Gamma(B))$. $\square$

Corollary 5.3. *The graph $T_A(\Gamma(B))$ is an induced subgraph of $T_A(\Gamma(C))$ if $B \subseteq C$.*

Corollary 5.4. *If A is multiplicatively closed subset of R and $B \subseteq A$, then $T_A(\Gamma(B))$ is a complete graph.*

Corollary 5.5. *If A and B are two disjoint multiplicatively closed subsets of R, then $T_A(\Gamma(B))$ is the empty graph.*

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