Certain classes of meromorphic functions by using the linear operator

Sultan Abdullah Alzahrani*
King Faisal Air Academy
Scientific Studies Squadron
Adress Details: 2782, AL Fairouz 7847
Irqah Riyadh, 12534
Kingdom Of Saudi Arabia
sultan.aa@hotmail.com

Maslina Darus
School of Mathematical Sciences Faculty of Science and Technology
University Kebangsaan Malaysia
Bangi 43600 Selangor D. Ehsan
Malaysia
maslina@ukm.edu.my

Abstract. In this paper, we introduce a new certain differential operator $A_\lambda^n f(z)$ with subclass $S_p^*(\alpha, \lambda, n, \beta)$ for functions of the form $f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_k z^k$. For functions in $S_p^*(\alpha, \lambda, n, \beta)$, we give coefficient inequalities, distortion theorem, radii of starlikeness and convexity.
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1. Introduction and preliminaries

Let $\mathcal{A}$ denote the class of functions f of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. As usual, we denote by S the subclass of $\mathcal{A}$, consisting of functions which are also univalent in $\mathbb{U}$. We recall here the definitions of the well-known classes of starlike functions and convex functions:

$$S^* = \left\{ f \in \mathcal{A} : \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \right\} \quad (z \in \mathbb{U}),$$

$$S^c = \left\{ f \in \mathcal{A} : \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \right\} \quad (z \in \mathbb{U})$$

∗. Corresponding author

Later Acu and Owa [2] studied the classes extensively. The class S_w^* is defined by geometric property that the image of any circular arc centered at w is starlike with respect to $f(w)$ and the corresponding class S_w^c is defined by the property that the image of any circular arc centered at w is convex. We observe that the definitions are somewhat similar to the ones introduced by Amourah in [3] and [4] for starlike and convex functions.

Let S denoted the subclass of $\mathcal{A}(p)$ consisting of the function of the form:

$$f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_{k+p-1} z^{k+p-1}, \tag{2}$$
$$(a_{k+p-1} > 0,\ z \in \mathbb{U}^* = \{z : z \in \mathbb{C} \ \text{ and } \ 0 < |z| < 1\}).$$

The function $f(z)$ in S is said to be starlike functions of order α if and only if

$$\operatorname{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in \mathbb{U}^*), \tag{3}$$

for some $\alpha(0 \leq \alpha < 1)$. We denote by $S^*(\alpha)$ the class of all starlike functions of order α. Similarly, a function f in S is said to be convex of order α if and only if

$$\operatorname{Re}\left\{-1 - \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (z \in \mathbb{U}^*), \tag{4}$$

for some $\alpha(0 \leq \alpha < 1)$. We denote by $VR(\alpha)$ the class of all convex functions of order α. We note that the class $S^*(\alpha)$ and various other subclasses have been studied rather extensively by Nehari and Netanyahu [5], Acu and Owa [2], Amourah ([6],[7],[10],[11],[13]), Aouf [12], Miller [8] and Royster [9].

For the function $f \in \mathcal{A}(p)$, the definition of linear operator $A_\lambda^n f(z)$ introduced by [1] to define the linear operator $A_\lambda^n f(z)$ as the following:

$$A_\lambda^0 f(z) = f(z),$$
$$A_\lambda^1 f(z) = (1 + p\lambda)\, A_\lambda^0 f(z) + \lambda z \left(A_\lambda^0 f(z)\right)',$$

and for $n = 1, 2, 3, \cdots$

$$A_\lambda^n f(z) = A(A_\lambda^{n-1} f(z)), \tag{5}$$
$$= \frac{1}{z^p} + \sum_{k=1}^{\infty} [1 + 2p\lambda + k\lambda - \lambda]^n\, a_{k+p-1} z^{k+p-1}, \tag{6}$$

for $\lambda \geq 0$, $z \in \mathbb{U}^*$, $p \in \mathbb{N}$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Then, we can observe easily that for

$$\lambda z \left(A_\lambda^n f(z)\right)' = A_\lambda^{n+1} f(z) - (1 + p\lambda) A_\lambda^n f(z), \quad (p \in \mathbb{N},\ n \in \mathbb{N}_0)$$

Definition 1.1. *A function $f(z) \in S$ is said to be in $S_p(\alpha,\lambda,n,\beta)$ if and only if*

$$\left|\frac{z(A_\lambda^n f(z))'}{pA_\lambda^n f(z)}+\alpha+\alpha\beta\right| \le \operatorname{Re}\left\{-\frac{z(A_\lambda^n f(z))'}{pA_\lambda^n f(z)}\right\}+\alpha-\alpha\beta, \tag{7}$$

for some $0\le\beta<1$, $\alpha\ge\frac{1}{2+\beta}$, $p\in\mathbb{N}$ and $n\in\mathbb{N}_0$ and for all $z\in\mathbb{U}^*$.

Let $\mathcal{A}^*(p)$ denote the subclass of $\mathcal{A}(p)$ consisting of functions of the form:

$$f(z)=\frac{1}{z^p}+\sum_{k=1}^{\infty}a_k z^k, \qquad (a_k\ge 0). \tag{8}$$

Further, we define the class $S_p(\alpha,\lambda,n,\beta)$ by

$$S_p^*(\alpha,\lambda,n,\beta)=S_p(\alpha,\lambda,n,\beta)\cap\mathcal{A}^*(p). \tag{9}$$

In this paper, coefficient inequalities, growth and distortion theorem, radii of starlikeness and convexity.

2. Coefficient inequalities

In this section, the result provides a sufficient condition for a function, regular in $\mathbb{U}^*$, to be in $S_p^*(\alpha,\lambda,n,\beta)$.

Theorem 2.1. *Let the function $f(z)$ be given by (8). If*

$$\sum_{k=1}^{\infty}\left[p(\alpha\beta+1)+k-1\right]\gamma_n a_{k+p-1}\le p(1-\alpha\beta), \quad (z\in\mathbb{U}^*) \tag{10}$$

where $\gamma_n=(1+2p\lambda+k\lambda-\lambda)^n$, $0\le\beta<1, \alpha\ge\frac{1}{2+\beta}$, $p\in\mathbb{N}$ and $n\in\mathbb{N}_0$.

Proof. Suppose that $f\in S_p^*(\alpha,\lambda,n,\beta)$. Then, by the inequality (7), we get that

$$\left|\frac{z(A_\lambda^n f(z))'}{pA_\lambda^n f(z)}+\alpha+\alpha\beta\right| \le \operatorname{Re}\left\{-\frac{z(A_\lambda^n f(z))'}{pA_\lambda^n f(z)}\right\}+\alpha-\alpha\beta.$$

That is,

$$\operatorname{Re}\left\{\frac{z(A_\lambda^n f(z))'}{pA_\lambda^n f(z)}+\alpha+\alpha\beta\right\}\le\left|\frac{z(A_\lambda^n f(z))'}{pA_\lambda^n f(z)}+\alpha+\alpha\beta\right|$$
$$\le\operatorname{Re}\left\{-\frac{z(A_\lambda^n f(z))'}{pA_\lambda^n f(z)}\right\}+\alpha-\alpha\beta.$$

That is,

$$\operatorname{Re}\left\{\frac{2z(A_\lambda^n f(z))'}{pA_\lambda^n f(z)}2\alpha\beta\right\}\le 0.$$

Hence, by the inequalities (7) and (8)

$$\operatorname{Re}\left\{\frac{-2p\left(1-\alpha\beta\right)+\sum_{k=1}^{\infty}2\left[p\left(\alpha\beta+1\right)+k-1\right]\gamma_n a_{k+p-1}z^{k+2p-1}}{p+\sum_{k=1}^{\infty}p\gamma_n a_{k+p-1}z^{k+2p-1}}\right\}\leq 0. \tag{11}$$

Taking z to be real and putting $z\to 1^-$ through real values, then the inequality (11) yields

$$\frac{-2p\left(1-\alpha\beta\right)+\sum_{k=1}^{\infty}2\left[p\left(\alpha\beta+1\right)+k-1\right]\gamma_n a_{k+p-1}}{p+\sum_{k=1}^{\infty}p\gamma_n a_{k+p-1}}\leq 0$$

Hence,

$$\sum_{k=1}^{\infty}\left[p\left(\alpha\beta+1\right)+k-1\right]\gamma_n a_{k+p-1}\leq p\left(1-\alpha\beta\right)$$

This completes the proof of Theorem 2.1. □

Corollary 2.1. *Let the function* $f(z)$ *be defined by* (8). *If* $f\in S_p^*\left(\alpha,\lambda,n,\beta\right)$, *then*

$$a_{k+p-1}\leq\frac{p\left(1-\alpha\beta\right)}{\left[p\left(\alpha\beta+1\right)+k-1\right]\left(1+2p\lambda+k\lambda-\lambda\right)^n},\quad (k\in\mathbb{N}). \tag{12}$$

The result (12) *is sharp for functions of the form:*

$$f(z)=\frac{1}{z^p}+\frac{p\left(1-\alpha\beta\right)}{\left[p\left(\alpha\beta+1\right)+k-1\right]\left(1+2p\lambda+k\lambda-\lambda\right)^n}z^{k+p-1},\ (k\in\mathbb{N}). \tag{13}$$

where $0\leq\beta<1,\alpha\geq\frac{1}{2+\beta}$, $p\in\mathbb{N}$ *and* $n\in\mathbb{N}_0$.

Proof. Since $f\in S_p^*\left(\alpha,\lambda,n,\beta\right)$, then from Theorem 2.1 above, we get that

$$\sum_{k=1}^{\infty}\left[p\left(\alpha\beta+1\right)+k-1\right]\left(1+2p\lambda+k\lambda-\lambda\right)^n a_{k+p-1}\leq p\left(1-\alpha\beta\right).$$

Next, note that

$$\begin{aligned}&\left[p\left(\alpha\beta+1\right)+k-1\right]\left(1+2p\lambda+k\lambda-\lambda\right)^n a_{k+p-1}\\&\leq\sum_{k=1}^{\infty}\left[p\left(\alpha\beta+1\right)+k-1\right]\left(1+2p\lambda+k\lambda-\lambda\right)^n a_{k+p-1}\leq p\left(1-\alpha\beta\right).\end{aligned}$$

Hence,

$$a_{k+p-1}\leq\frac{p\left(1-\alpha\beta\right)}{\left[p\left(\alpha\beta+1\right)+k-1\right]\left(1+2p\lambda+k\lambda-\lambda\right)^n}$$

Thus, the equality (12) is attained for the function f given by

$$f(z)=\frac{1}{z^p}+\frac{p\left(1-\alpha\beta\right)}{\left[p\left(\alpha\beta+1\right)+k-1\right]\left(1+2p\lambda+k\lambda-\lambda\right)^n}z^{k+p-1}.$$ □

3. Growth and distortion theorem

In this section we will prove the following growth and distortion theorems for the class $S_p^*(\alpha,\lambda,n,\beta)$.

Theorem 3.1. *Let the function $f(z)$ given by (8) be in the class $S_p^*(\alpha,\lambda,n,\beta)$, where $0\le\beta<1,\alpha\ge\frac{1}{2+\beta}$, $p\in\mathbb{N}$, $p>m$, $0<|z|=r<1$ and $n\in\mathbb{N}_0$. Then, we have*

$$\begin{aligned}&\left\{\frac{(p+m-1)!}{(p-1)!}-\frac{(1-\alpha\beta)}{(1+\alpha\beta)(1+2p\lambda)^n}\cdot\frac{p!}{(2p-m-1)!}r^{3p-1}\right\}r^{-(p+m)}\\ &\le\left|f^{(m)}(z)\right|\\ &\le\left\{\frac{(p+m-1)!}{(p-1)!}-\frac{(1-\alpha\beta)}{(1+\alpha\beta)(1+2p\lambda)^n}\cdot\frac{p!}{(2p-m-1)!}r^{3p-1}\right\}r^{-(p+m)}.\end{aligned}\tag{14}$$

The result is sharp for the function f given by

$$f(z)=\frac{1}{z^p}+\sum_{k=1}^{\infty}\frac{(1-\alpha\beta)}{(1+\alpha\beta)(1+2p\lambda)^n}z^{k+p-1}.\tag{15}$$

Proof. Since $f\in S_p^*(\alpha,\lambda,n,\beta)$, from Theorem 2.1 readily yields the inequality

$$\frac{(1+\alpha\beta)(1+2p\lambda)^n}{(p-1)!}\sum_{k=1}^{\infty}(k+p-1)!a_{k+p-1},\tag{16}$$

$$\le[p(\alpha\beta+1)+k-1](1+2p\lambda+k\lambda-\lambda)^n a_{k+p-1}\le p(1-\alpha\beta),\tag{17}$$

that is,

$$\begin{aligned}\sum_{k=1}^{\infty}(k+p-1)!a_{k+p-1}&\le\frac{p(1-\alpha\beta)(p-1)!}{(1+\alpha\beta)(1+2p\lambda)^n}\\ &=\frac{(1-\alpha\beta)p!}{(1+\alpha\beta)(1+2p\lambda)^n}.\end{aligned}\tag{18}$$

By differentiating the function f in the form m times with respect to z, we get that

$$\begin{aligned}f^{(m)}(z)&=(-1)^m\frac{(p+m-1)!}{(p-1)!}z^{-(p+m)}\\ &+\sum_{k=1}^{\infty}\frac{(k+p-1)!}{(k+p-m-1)!}a_{k+p-1}z^{k+p-m-1}.\end{aligned}\tag{19}$$

From (18) and (19), we get that

$$\left|f^{(m)}(z)\right| \leq \frac{(p+m-1)!}{(p-1)!}r^{-(p+m)} + \sum_{k=1}^{\infty}\frac{(k+p-1)!}{(k+p-m-1)!}a_{k+p-1}r^{k+p-m-1}$$

$$\leq \left\{\frac{(p+m-1)!}{(p-1)!} + \sum_{k=1}^{\infty}\frac{(k+p-1)!}{(2p-m-1)!}a_{k+p-1}r^{3p-1}\right\}r^{-(p+m)} \tag{20}$$

$$\leq \left\{\frac{(p+m-1)!}{(p-1)!} + \frac{(1-\alpha\beta)}{(1+\alpha\beta)(1+2p\lambda)^n}\frac{p!}{(2p-m-1)!}r^{3p-1}\right\}r^{-(p+m)}, \tag{21}$$

and

$$\left|f^{(m)}(z)\right| \geq \frac{(p+m-1)!}{(p-1)!}r^{-(p+m)}$$

$$- \sum_{k=1}^{\infty}\frac{(k+p-1)!}{(k+p-m-1)!}a_{k+p-1}r^{k+p-m-1} \tag{22}$$

$$\geq \left\{\frac{(p+m-1)!}{(p-1)!} - \sum_{k=1}^{\infty}\frac{(k+p-1)!}{(2p-m-1)!}a_{k+p-1}r^{3p-1}\right\}r^{-(p+m)}$$

$$\geq \left\{\frac{(p+m-1)!}{(p-1)!} - \frac{(1-\alpha\beta)}{(1+\alpha\beta)(1+2p\lambda)^n}\frac{p!}{(2p-m-1)!}r^{3p-1}\right\}r^{-(p+m)}.$$

We can easily prove that the bounds of (14) are attained for the function f given by the form (15).

This completes the proof of Theorem 3.1. □

4. Radii of starlikeness and convexity

The radii of starlikeness and convexity for the class $S_p^*(\alpha,\lambda,n,\beta)$ is given by the following theorems.

Theorem 4.1. *If the function $f(z)$ given by (8) is in the class $S_p^*(\alpha,\lambda,n,\beta)$, where $0<\beta\leq 1$ and $n\in\mathbb{N}_0$, then $f(z)$ is starlike of order $\mu(\ 0\leq\mu<p)$ in $|z|<r_1$, that is*

$$\operatorname{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \mu, \tag{23}$$

where

$$r_1 = \inf_{k\geq 1}\left\{\frac{(p-\mu)\left[p(\alpha\beta+1)+k-1\right](1+2p\lambda+k\lambda-\lambda)^n}{p(k+2\mu-1)(1-\alpha\beta)}\right\}^{\frac{1}{k+2p-1}}. \tag{24}$$

Proof. It suffices to prove that

$$\left|\frac{\frac{zf'(z)}{f(z)}+p}{\frac{zf'(z)}{f(z)}-p+2\mu}\right| = \left|\frac{\sum_{k=1}^{\infty}(k+2p-1)\,a_{k+p-1}z^{k+2p-1}}{2(p-\mu)-\sum_{k=1}^{\infty}(k+2\mu-1)\,a_{k+p-1}z^{k+2p-1}}\right| \leq \frac{\sum_{k=1}^{\infty}(k+2p-1)\,a_{k+p-1}\,|z|^{k+2p-1}}{2(p-\mu)-\sum_{k=1}^{\infty}(k+2\mu-1)\,a_{k+p-1}\,|z|^{k+2p-1}}. \tag{25}$$

Then, the following

$$\left|\frac{\frac{zf'(z)}{f(z)}+p}{\frac{zf'(z)}{f(z)}-p+2\mu}\right| \leq 1, \ \ (0\leq\mu<p,\ p\in\mathbb{N}) \tag{26}$$

will hold if

$$\sum_{k=1}^{\infty}\frac{k+2\mu-1}{p-\mu}a_{k+p-1}\,|z|^{k+2p-1}\leq 1. \tag{27}$$

Then, by Corollary 2.1 the inequality (27) will be true if

$$\frac{k+2\mu-1}{(p-\mu)}\,|z|^{k+2p-1}\leq\frac{[p\,(\alpha\beta+1)+k-1]\,(1+2p\lambda+k\lambda-\lambda)^n}{p\,(1-\alpha\beta)}$$

that is,

$$|z|^{k+2p-1}\leq\frac{(p-\mu)\,[p\,(\alpha\beta+1)+k-1]\,(1+2p\lambda+k\lambda-\lambda)^n}{p\,(k+2\mu-1)\,(1-\alpha\beta)}. \tag{28}$$

This completes the proof of Theorem 4.1. □

Theorem 4.2. *If the function* $f(z)$ *given by* (8) *is in the class* $S_p^*(\alpha,\lambda,n,\beta)$, *where* $0<\beta\leq 1$ *and* $n\in\mathbb{N}_0$, *then* $f(z)$ *is convex of order* $\mu(\ 0\leq\mu<p)$ *in* $|z|<r_2$, *that is,*

$$\operatorname{Re}\left\{-1-\frac{zf''(z)}{f'(z)}\right\}>\mu,$$

where

$$r_2=\inf_{k\geq 1}\left\{\frac{(p-\mu)\,[p\,(\alpha\beta+1)+k-1]\,(1+2p\lambda+k\lambda-\lambda)^n}{(k+\mu-1)(k+2\mu-1)\,(1-\alpha\beta)}\right\}^{\frac{1}{k+2p-1}},$$

$$(k\geq 1). \tag{29}$$

Proof. By using the same technique employed in the proof of Theorem 4.1, we can show that

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}+p}{\frac{zf''(z)}{f'(z)}-p+2\mu}\right| = \left|\frac{\sum_{k=1}^{\infty}(k+p-1)(k+2p-1)a_{k+p-1}z^{k+2p-1}}{2p(p-\mu)z^{-p}-\sum_{k=1}^{\infty}(k+p-1)(k+2\mu-1)a_{k+p-1}z^{k+2p-1}}\right| \leq \frac{\sum_{k=1}^{\infty}(k+p-1)(k+2p-1)a_{k+p-1}\,|z|^{k+2p-1}}{2p(p-\mu)-\sum_{k=1}^{\infty}(k+p-1)(k+2\mu-1)a_{k+p-1}\,|z|^{k+2p-1}}.$$

Then, the following

$$\left| \frac{1 + \frac{zf''(z)}{f'(z)} + p}{\frac{zf''(z)}{f'(z)} - p + 2\mu} \right| \leq 1 \tag{30}$$

will hold if

$$\sum_{k=1}^{\infty} \frac{(k + \mu - 1)(k + 2\mu - 1)}{p(p - \mu)} a_{k+p-1} |z|^{k+2p-1} \leq 1. \tag{31}$$

Then, by Corollary 2.1 the inequality (31) will be true if

$$\frac{(k + \mu - 1)(k + 2\mu - 1)}{p(p - \mu)} |z|^{k+2p-1} \leq \frac{[p(\alpha\beta + 1) + k - 1](1 + 2p\lambda + k\lambda - \lambda)^n}{p(1 - \alpha\beta)}$$

that is,

$$|z|^{k+2p-1} \leq \frac{(p - \mu)[p(\alpha\beta + 1) + k - 1](1 + 2p\lambda + k\lambda - \lambda)^n}{(k + \mu - 1)(k + 2\mu - 1)(1 - \alpha\beta)}. \tag{32}$$

Therefore, the inequality (32) leads us to the disk $|z| < r_2$, where r_2 is given by the form (29). □

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