Fixed point theorems for monotone mappings on partial M^*-metric spaces

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Abstract. In this paper, we introduce the concept of partial M^*-metric on a nonempty set X, and we give some properties supported by some examples to illustrate

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our results. Furthermore, we establish some fixed points results for partial M^*-metric. Also, we extend our result for monotone mappings on partial M^*-metric spaces.
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1. Introduction

Bakhtin [2] and Czerwik [3] are defined a $b-$metric space and the idea of a $b-$metric space the triangle inequality axiom is weaker than for metric space. Also, many authors gives many fixed point theorems in a b-metric space (see [6−15]), Aydi et al. [8] gave some interesting theories for fixed point for set-valid quasi contraction in b-metric space.

In 2021 [37], Malkawi et al. introduced the notion of MR-metric space and MR-metric space is a generalization of a b-metric space [2, 3] and the tetrahedral inequality axiom is weaker than for a $D-metric$ space [1]. Also, there are many fixed point theorems in different type spaces for more information. I Refer to the reader to look at $[4-36]$.

Definition 1 ([37]). *Let X be a non empty set and $R \geq 1$ be a real number. $M : X \times X \times X \to [0, \infty)$ a function which is called an MR-metric, if it satisfies the following axioms for each $x, y, z \in X$.*

$(M1) : M(x, y, z) \geq 0.$

$(M2) : M(x, y, z) = 0$ *iff* $x = y = z.$

$(M3) : M(x, y, z) = M(p(x, y, z))$; *for any permutation* $p(x, y, z)$ *of* $x, y, z.$

$(M4) : M(x, y, z) \leq R\left[M(x, y, \ell) + M(x, \ell, z) + M(\ell, y, z)\right].$

A pair (X, M) is called an MR-metric space.

Also, Gharib et al. [38] introduced the concept of M^*-metric spaces, the importance of which lies in this property $M^*(x, x, y) = M^*(x, y, y)$. It is worth noting that these characteristics need not be satisfied in MR-metric space [37].

Definition 2 ([38]). *Let X be a non empty set and $R \geq 1$ be a real number. A function $M^* : X \times X \times X \to [0, \infty)$ is called M^*-metric, if the following properties are satisfied for each $x, y, z \in X$.*

$(M^*1) : M^*(x, y, z) \geq 0.$

$(M^*2) : M^*(x, y, z) = 0$ *iff* $x = y = z.$

$(M^*3) : M^*(x, y, z) = M^*(p(x, y, z))$; *for any permutation* $p(x, y, z)$ *of* $x, y, z.$

$(M^*4) : M^*(x, y, z) \leq RM^*(x, y, u) + M^*(u, z, z).$

A pair (X, M^*) is called an M^*-metric space.

The following are examples of M^*-metric space.

Example 1. a) Let (X, d) be a metric space then $M^*(x, y, z) = \frac{1}{R}\max\{d(x, y), d(y, z), d(z, x)\}$ and $M^*(x, y, z) = \frac{1}{R}[d(x, y), d(y, z), d(z, x)]$ are M^*-metric on X.

b) If $X = \mathbb{R}^n$, then

$$M^*(x, y, z) = \frac{1}{R}[\|x + y - 2z\| + \|y + z - 2x\| + \|z + x - 2y\|],$$

for every $x, y, z \in \mathbb{R}^n$ is an M^*-metric on X.

Example 2. Let $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ be a mapping defined as the following:

$$\psi(x, y) = 0 \text{ if } x = y, \psi(x, y) = \frac{1}{2} \text{ if } x > y, \psi(x, y) = \frac{1}{3} \text{ if } x < y.$$

Then, clearly ψ is not a metric, since $\psi(1, 2) \neq \psi(2, 1)$. Define $G : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ by

$$G(x, y, z) = \frac{1}{R} \max\{\psi(x, y), \psi(y, z), \psi(z, x)\}.$$

Then, G is an M^*-metric.

Example 3. Let $\psi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be a mapping defined as the following:

$\psi(x, y) = \max\{x, y\}$. Clearly it is not a metric. Define $G : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\psi(x, y) = \frac{1}{R}[\max\{x, y\} + \max\{y, x\} + \max\{z, x\}] - x - y - z,$$

for every $x, y, z \in \mathbb{R}^+$. Then G is an M^*-metric.

2. Partial M^*-metric space

The Authors defined $b-$metric space by replacing the triangular inequality axiom with a weaker one. Also, for some work on b-metric, we refer the reader to [40, 41, 42, 43, 44, 45, 46].

Now, we present the concept of a partial M^*-metric space and prove its properties.

Definition 3. *A partial M^*-metric on a nonempty set X is a function $M_p^* : X \times X \times X \to \mathbb{R}^+$ such that for all $x, y, z, a \in X$:*

(M_p^*1) $x = y = z \Leftrightarrow M_p^*(x, x, x) = M_p^*(x, y, z) = M_p^*(y, y, y) = M_p^*(z, z, z)$,

(M_p^*2) $M_p^*(x, x, x) \leq M_p^*(x, y, z)$,

(M_p^*3) $M_p^*(x, y, z) = M_p^*(p\{x, y, z\})$, *where p is a permutation function,*

(M_p^*4) $M_p^*(x, y, z) \leq RM_p^*(x, y, a) + M_p^*(a, z, z) - M_p^*(a, a, a)$.

(X, M_p^*) is a partial M^*-metric space on a nonempty set X and M_p^* is a partial M^*-metric on X. It is clear that, if $M_p^*(x, y, z) = 0$, then from (M_p^*1) and (M_p^*2) $x = y = z$. But if $x = y = z$, $M_p^*(x, y, z)$ may not be 0. The basic example of a partial M^*-metric space $(\mathbb{R}^+, M_p^*)$ is $M_p^*(x, y, z) = \frac{1}{R} \max\{x, y, z\}$ for all $x, y, z \in \mathbb{R}^+$.

It is obvious that every M^*-metric is a partial M^*-metric, but the converse need not be true. We will explain this in the following example.

Example 4. Let $M_p^* :: \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be a nonempty defined as follows:

$$M_p^*(x, y, z) = \frac{1}{R}[|x - y| + |y - z| + |x - z|] + \max\{x, y, z\},$$

such that $R \geq 1$. Then clearly it is a partial M^*-metric, but it is not an M^*-metric.

Example 5. Let (X, p) be a partial b-metric space and $M_p^* :: \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be a nonempty defined as:

$$M_p^*(x, y, z) = \frac{1}{R}[p(x, y) + p(x, z) + p(y, z)] - p(x, x) - p(y, y) - p(z, z).$$

Then, clearly M_p^* is a partial M^*-metric, but it is not an M^*-metric.

Remark 1. $M_p^*(x, x, y) = M_p^*(x, y, y)$

Proof.

$$\begin{aligned} M_p^*(x, x, y) &\leq RM_p^*(x, x, x) + M_p^*(x, y, y) - M_p^*(x, x, x) \\ &\leq RM_p^*(x, x, x) + M_p^*(x, y, y) - RM_p^*(x, x, x) \\ &\leq M_p^*(x, y, y). \end{aligned} \tag{2.1}$$

$$\begin{aligned} M_p^*(x, y, y) &\leq RM_p^*(y, y, y) + M_p^*(y, x, x) - M_p^*(y, y, y) \\ &\leq RM_p^*(y, y, y) + M_p^*(y, x, x) - RM_p^*(y, y, y) \\ &\leq M_p^*(y, x, x). \end{aligned} \tag{2.2}$$

From (2.1) and (2.2), we get $M_p^*(x, x, y) = M_p^*(x, y, y)$. □

Lemma 1. *Let (X, M_p^*) be a partial M^*-metric space. If we define $p(x, y) = M_p^*(x, y, y)$, then (X, p) is a partial b-metric space*

Proof. (M_p^*1) $x = y \Leftrightarrow M_p^*(x, x, x) = M_p^*(x, y, y) = p(y, y, y) \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,

(M_p^*2) $M_p^*(x, x, x) \leq M_p^*(x, y, y)$ implies that $p(x, x) \leq p(x, y)$,

(M_p^*3) $M_p^*(x, y, y) = M_p^*(y, x, x)$ implies that $p(x, y) = p(y, x)$,

(M_p^*4) $M_p^*(y, y, x) \leq RM_p^*(y, y, z) + M_p^*(z, x, x) - M_p^*(z, z, z)$ implies that

$$p(x, y) \leq R[p(y, z) + p(z, x)] - p(z, z).$$ □

Let (X, M_p^*) be a partial M^*-metric space. For $r > 0$ define

$$B_{M_p^*}(x, r) = \{y \in X : M_p^*(x, y, y) < M_p^*(x, x, x) + r\}.$$

Definition 4. *Let (X, M_p^*) be a partial M^*-metric space and $A \subset X$.*

(1) *If, for every $x \in A$ there exists $r > 0$ such that $B_{M_p^*}(x, r) \subset A$, then the subset A is called an open subset of X.*

(2) *$\{x_n\}$ is a sequence in a partial M^*-metric space (X, M_p^*) converges to x if and only if $M_p^*(x, x, x) = \lim_{n\to\infty} M_p^*(x_n, x_n, x)$. That is for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$M_p^*(x, x, x_n) < M_p^*(x, x, x) + \epsilon \ \forall n \geq n_0, \tag{1}$$

or equivalently, for each $\epsilon > 0$, *there exists* $n_0 \in \mathbb{N}$ *such that*

$$M_p^*(x, x_n, x_m) < M_p^*(x, x, x) + \epsilon \ \forall n, m \geq n_0. \tag{2}$$

Indeed, if (1) *holds then*

$$\begin{aligned} M_p^*(x, x_n, x_m) &= M_p^*(x_n, x, x_m) \\ &\leq RM_p^*(x_n, x, x) + M_p^*(x, x_m, x_m) - M_p^*(x, x, x) \\ &< R\epsilon + \epsilon + M_p^*(x, x, x). \end{aligned}$$

Conversely, set $m = n$ *in* (2) *we have* $M_p^*(x_n, x_n, x) < M_p^*(x, x, x) + \epsilon$.

(3) $\{x_n\}$ *is a sequence in a partial* M^*-*metric space* (X, M_p^*) *is called a Cauchy if* $\lim_{n\to\infty} M_p^*(x_n, x_m, x_m)$ *exists.*

Let $\tau_{M_p^*}$ *be the set of all open subsets* X, *then* $\tau_{M_p^*}$ *is a topolpgy on* X *(induced by the partial* M^*-*metric* M_p^**).*

A partial M^*-*metric space* (X, M_p^*) *is said to be complete if every Cauchy sequence* $\{x_n\}$ *in* X *converges to a point* $x \in X$ *with respect to* $\tau_{M_p^*}$.

If a sequence $\{x_n\}$ in a partial M^*-metric space (X, M_p^*) converges to x, then we have

$$\begin{aligned} M_p^*(x_n, x_n, x_m) &\leq RM_p^*(x_n, x_n, x) + M_p^*(x, x_m, x_m) - M_p^*(x, x, x) \\ &< R\epsilon + \epsilon + M_p^*(x, x, x). \end{aligned}$$

Lemma 2. *Let* (X, M_p^*) *be a partial* M^*-*metric space. If* $r > 0$, *then the ball* $B_{M_p^*}(x, r)$ *with center* $x \in X$ *and radius* r *is an open ball.*

Proof. Let $y \in B_{M_p^*}(x, r)$, then $M_p^*(x, y, y) < M_p^*(x, x, x) + r$. Let $RM_p^*(x, y, y) - M_p^*(x, x, x) = \delta$. Let $z \in B_{M_p^*}(y, r - \delta)$, by triangular inequality, we have

$$\begin{aligned} M_p^*(x, x, z) &\leq RM_p^*(x, y, y) + M_p^*(y, z, z) + M_p^*(y, y, y) \\ &= RM_p^*(x, y, y) - M_p^*(x, x, x) + M_p^*(z, z, y) \\ &\quad - M_p^*(y, y, y) + M_p^*(x, x, x) \\ &< \delta + r - \delta + M_p^*(x, x, x) \\ &= M_p^*(x, x, x) + r. \end{aligned}$$

Thus, $z \in B_{M_p^*}(x, r)$. Hence $B_{M_p^*}(y, r - \delta) \subseteq B_{M_p^*}(x, r)$. Therefore, the ball $B_{M_p^*}(x, r)$ is an open ball. □

Each partial M^*-metric M_p^* on X generates a topology $\tau_{M_p^*}$ on X which has as a base the family of open M_p^*-balls $\{B_{M_p^*}(x, \epsilon) : x \in X, \epsilon > 0\}$.

The following example shows that a convergent sequence $\{x_n\}$ in a partial M^*-metric space (X, M_p^*) need not be a Cauchy sequence. In particular, it shows that the limit of a convergent sequence is not necessarily unique, to explain that see the following example

Example 6. Let $X = [0, \infty)$ and $M_p^*(x, y, z) = \frac{1}{R} \max\{x, y, z\}$. Then, it is clear that (X, M_p^*) is a complete partial M^*-metric space. Let

$$x_n = \begin{cases} 1, & n = 2k \\ 2, & n = 2k + 1. \end{cases}$$

Then, clearly it is convergent sequence and for every $x \geq 2$ we have

$$\lim_{n \to \infty} M_p^*(x_n, x_n, x) = M_p^*(x, x, x),$$

therefore

$$L(x_n) = \{x : x_n \to x\} = [2, \infty).$$

But, $\lim_{n \to \infty} M_p^*(x_n, x_m, x_m)$ does not exist. Hence $\{x_n\}$ is not a Cauchy sequence.

The following Lemma plays an important role in this paper.

Lemma 3. *Let (X, p) be a partial $b-$metric space then there exists a partial M^*-metric M_p^* on X such that*

(a) $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the partial M^*-metric space (X, M_p^*),

(b) the partial $b-$metric space (X, p) is complete if and only if the partial M^*-metric space (X, M_p^*) is complete. Furthermore, $M_p^*(x, x, y) = p(x, y)$, for every $x, y \in X$.

Proof. Define

$$M_p^*(x, y, z) = \frac{1}{R} \max\{p(x, y), p(x, z), p(y, z)\}, \ \ \forall x, y, z \in X.$$

Then, it is easy to see that M_p^* is a partial M^*-metric and $M_p^*(x, x, y) = p(x, y)$, for every $x, y \in X$. $\square$

The following Lemma shows that under certain conditions the limit is unique.

Lemma 4. *Let $\{x_n\}$ be a convergent sequence in a partial M^*-metric space (X, M_p^*) such that $x_n \to x$ and $x_n \to y$. If*

$$\lim_{n \to \infty} M_p^*(x_n, x_n, x_n) = M_p^*(x, x, x) = M_p^*(y, y, y),$$

then $x = y$.

Proof. As

$$M_p^*(x, y, y) = M_p^*(x, x, y) \leq R M_p^*(x, x, x_n) + M_p^*(x_n, y, y) - M_p^*(x_n, x_n, x_n),$$

therefore

$$M_p^*(x_n, x_n, x_n) \leq R M_p^*(x, x, x_n) + M_p^*(x_n, y, y) - M_p^*(x, y, y).$$

By given assumptions, we have

$$\begin{aligned}
\lim_{n\to\infty} M_p^*(x_n, x_n, x) &= M_p^*(x, x, x),\\
\lim_{n\to\infty} M_p^*(x_n, x_n, y) &= M_p^*(y, y, y),\\
\lim_{n\to\infty} M_p^*(x_n, x_n, x_n) &= M_p^*(x, x, x).
\end{aligned}$$

Therefore

$$M_p^*(x, x, x) \leq RM_p^*(x, x, x) + M_p^*(y, y, y) - M_p^*(x, y, y),$$

which shows that $M_p^*(y, y, y) \leq (1 - R)M_p^*(x, x, x) + M_p^*(x, y, y) \leq M_p^*(y, y, y)$. So,

$$M_p^*(y, y, y) \leq M_p^*(x, y, y) \leq M_p^*(y, y, y).$$

Also,

$$M_p^*(x, y, y) = M_p^*(y, y, x) \leq RM_p^*(y, y, x_n) + M_p^*(x_n, x, x) - M_p^*(x_n, x_n, x_n),$$

implies that

$$M_p^*(x_n, x_n, x_n) \leq RM_p^*(y, y, x_n) + M_p^*(x_n, x, x) - M_p^*(x, y, y),$$

by on taking limit as $n \to \infty$ gives

$$M_p^*(y, y, y) \leq RM_p^*(y, y, y) + M_p^*(x, x, x) - M_p^*(x, y, y),$$

which shows that

$$M_p^*(x, x, x) \leq (1 - R)M_p^*(y, y, y) + M_p^*(x, y, y) \leq M_p^*(x, x, x).$$

So,

$$M_p^*(x, x, x) \leq M_p^*(x, y, y) \leq M_p^*(x, x, x).$$

Thus, $M_p^*(x, x, x) = M_p^*(x, y, y) = M_p^*(y, y, y)$. Therefore, $x = y$. □

Lemma 5. *Let $\{x_n\}$ and $\{y_n\}$ be two sequences in partial M^*-metric space (X, M_p^*) such that*

$$\lim_{n\to\infty} M_p^*(x_n, x, x) = \lim_{n\to\infty} M_p^*(x_n, x_n, x_n) = M_p^*(x, x, x),$$

and

$$\lim_{n\to\infty} M_p^*(y_n, y, y) = \lim_{n\to\infty} M_p^*(y_n, y_n, y_n) = M_p^*(y, y, y).$$

Then $\lim_{n\to\infty} M_p^*(x_n, y_n, y_n) = M_p^*(x, y, y)$. In particular, $\lim_{n\to\infty} M_p^*(x_n, y_n, z)$ $= M_p^*(x, y, z)$, for every $z \in X$.

Proof. As $\{x_n\}$ and $\{y_n\}$ converges to a $x \in X$ and $y \in X$ respectively, for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} M_p^*(x,x,x_n) &< M_p^*(x,x,x) + \frac{\epsilon}{2R}, \\ M_p^*(y,y,y_n) &< M_p^*(y,y,y) + \frac{\epsilon}{2R}, \\ M_p^*(x,x,x_n) &< M_p^*(x_n,x_n,x_n) + \frac{\epsilon}{2R} \end{aligned}$$

and

$$M_p^*(y,y,y_n) < M_p^*(y_n,y_n,y_n) + \frac{\epsilon}{2R}$$

for $n \geq n_0$. Now,

$$\begin{aligned} M_p^*(x_n,x_n,y_n) &\leq RM_p^*(x_n,x_n,x) + M_p^*(x,y_n,y_n) - M_p^*(x,x,x) \\ &\leq RM_p^*(x_n,x_n,x) + RM_p^*(y,y_n,y_n) + M_p^*(x,x,y) \\ &\quad -M_p^*(y,y,y) - M_p^*(x,x,x) \\ &< M_p^*(x,y,y) + \frac{R\epsilon}{2R} + \frac{R\epsilon}{2R} \\ &= M_p^*(x,y,y) + \epsilon, \end{aligned} \tag{1}$$

and so we have

$$M_p^*(x_n,y_n,y_n) - M_p^*(x,y,y) < \epsilon.$$

Also,

$$\begin{aligned} M_p^*(x,y,y) &\leq RM_p^*(x_n,y,y) + M_p^*(x,x,x_n) - M_p^*(x_n,x_n,x_n) \\ &\leq RM_p^*(x,x,x) + RM_p^*(x_n,x_n,y_n) + M_p^*(y_n,y,y) \\ &\quad -M_p^*(y_n,y_n,y_n) - M_p^*(x_n,x_n,x) \\ &< M_p^*(x_n,x_n,y) + \frac{R\epsilon}{2R} + \frac{R\epsilon}{2R} \\ &= M_p^*(x,y,y) + \epsilon. \end{aligned} \tag{2}$$

Thus,

$$M_p^*(x,x,y) - M_p^*(x_n,x_n,y_n) < \epsilon.$$

Hence, for all $n \geq n_0$, we have $\left|M_p^*(x_n,x_n,y_n) - M_p^*(x,x,y)\right| < \epsilon$. Hence, the result follows. □

Lemma 6. *If M_p^* is a partial M^*-metric on X, then the functions $M_{p^s}^*, M_{p^m}^* : X \times X \times X \to \mathbb{R}^+$ are given by:*

$$\begin{aligned} M_{p^s}^*(x,y,z) = {}& RM_p^*(x,x,y) + RM_p^*(y,y,z) + M_p^*(z,z,x) \\ & - M_p^*(x,x,x) - M_p^*(y,y,y) - M_p^*(z,z,z) \end{aligned}$$

and

$$M^*_{p^m}(x,y,z) = \max\left\{\begin{array}{c} 2RM^*_p(x,x,y) - M^*_p(x,x,x) - M^*_p(y,y,y), \\ 2RM^*_p(y,y,z) - M^*_p(y,y,y) - M^*_p(z,z,z), \\ 2RM^*_p(z,z,x) - M^*_p(z,z,z) - M^*_p(x,x,x) \end{array}\right\},$$

for every $x, y, z \in X$ are equivalent M^*-metrics on X.

Proof. It is easy to see that $M^*_{p^s}$ and $M^*_{p^m}$ are M^*-metrics on X. Let $x, y, z \in X$. It is obvious that

$$M^*_{p^m}(x,y,z) \leq 2M^*_{p^s}(x,y,z).$$

On the other hand, since $a+b+c \leq 3\max\{a,b,c\}$, it provides that

$$\begin{aligned} M^*_{p^s}(x,y,z) &= RM^*_p(x,x,y) + RM^*_p(y,y,z) + M^*_p(z,z,x) \\ &- M^*_p(x,x,x) - M^*_p(y,y,y) - M^*_p(z,z,z) \\ &\leq \frac{1}{2}\left[2RM^*_p(x,x,y) - M^*_p(x,x,x) - M^*_p(y,y,y)\right] \\ &+ \frac{1}{2}\left[2RM^*_p(y,y,z) - M^*_p(y,y,y) - M^*_p(z,z,z)\right] \\ &+ \frac{1}{2}\left[2RM^*_p(z,z,x) - M^*_p(z,z,z) - M^*_p(x,x,x)\right] \\ &\leq \frac{3}{2}\max\left\{\begin{array}{c} 2RM^*_p(x,x,y) - M^*_p(x,x,x) - M^*_p(y,y,y), \\ 2RM^*_p(y,y,z) - M^*_p(y,y,y) - M^*_p(z,z,z), \\ 2RM^*_p(z,z,x) - M^*_p(z,z,z) - M^*_p(x,x,x) \end{array}\right\} \\ &= \frac{3}{2}M^*_{p^m}(x,y,z). \end{aligned}$$

Thus, we have

$$\frac{1}{2}M^*_{p^m}(x,y,z) \leq M^*_{p^s}(x,y,z) \leq \frac{3}{2}M^*_{p^m}(x,y,z).$$

These inequalities implies that $M^*_{p^s}$ and $M^*_{p^m}$ are equivalent. □

Remark 2. Note that:

$$M^*_{p^s}(x,x,y) = 2RM^*_p(x,x,y) - M^*_p(x,x,x) - M^*_p(y,y,y) = M^*_{p^m}(x,x,y).$$

A mapping $F : X \to X$ is said to be continuous at $x_0 \in X$, if for every $\epsilon > 0$, there exists $\delta > 0$ such that $F(B_{M^*_p}(x_0,\delta)) \subseteq B_{M^*_p}(Fx_0,\epsilon)$.

The following lemma plays an important role to prove fixed point results on a partial M^*-metric space.

Lemma 7. *Let (X, M^*_p) be a partial M^*-metric space.*

(a) $\{x_n\}$ is a Cauchy sequence in (X, M^*_p) if and only if it is a Cauchy sequence in the M^*-metric space $(X, M^*_{p^s})$

(b) A partial M^*-metric space (X, M_p^*) is complete if and only if the M^*-metric space $(X, M_{p^s}^*)$ is complete. Furthermore,

$$\lim_{n\to\infty} M_{p^s}^*(x_n, x_n, x) = 0$$

if and only if

$$M_p^*(x, x, x) = \lim_{n\to\infty} M_p^*(x_n, x_n, x) = \lim_{n,m\to\infty} M_p^*(x_n, x_n, x_m).$$

Proof. Let $\{x_n\}$ be a Cauchy sequence in (X, M_p^*), we want to prove $\{x_n\}$ is a Cauchy sequence in the M^*-metric space $(X, M_{p^s}^*)$.

Since $\{x_n\}$ be a Cauchy sequence in (X, M_p^*), then there exists $\alpha \in \mathbb{R}$ and for every $\epsilon > 0$, there is $n_\epsilon \in \mathbb{N}$ such that $\left|M_p^*(x_n, x_n, x_m) - \alpha\right| < \frac{\epsilon}{4R}$ for all $n, m \geq n_\epsilon$. Hence

$$\begin{aligned} M_{p^s}^*(x_n, x_n, x_m) &\leq \big|2RM_p^*(x_n, x_n, x_m) - M_p^*(x_n, x_n, x_n) \\ &\quad - M_p^*(x_m, x_m, x_m) + 2\alpha - 2\alpha\big| \\ &\leq \big|2RM_p^*(x_n, x_n, x_m) - 2\alpha\big| + \big|M_p^*(x_n, x_n, x_n) - \alpha\big| \\ &\quad + \big|M_p^*(x_m, x_m, x_m) - \alpha\big| \leq 4R\frac{\epsilon}{4R} = \epsilon, \end{aligned}$$

for all $n, m \geq n_\epsilon$. Thus $\{x_n\}$ is a Cauchy sequence in $(X, M_{p^s}^*)$.

Now, we prove that completeness of $(X, M_{p^s}^*)$ implies completeness of (X, M_p^*).

Indeed, if $\{x_n\}$ be a Cauchy sequence in (X, M_p^*) then it is $\{x_n\}$ be a Cauchy sequence in $(X, M_{p^s}^*)$. Since the M^*-metric space $(X, M_{p^s}^*)$ is complete we deduce that there exists $y \in X$ such that $\lim_{n\to\infty} M_{p^s}^*(x_n, x_n, y) = 0$. Thus,

$$\begin{aligned} &\lim_{n\to\infty} \sup \big|M_p^*(x_n, x_n, y) - M_p^*(y, y, y)\big| \\ &\leq \lim_{n\to\infty} \big|2RM_p^*(x_n, x_n, y) - M_p^*(x_n, x_n, x_n) - M_p^*(y, y, y)\big| = 0. \end{aligned}$$

Hence, we follow that $\{x_n\}$ is a convergent sequence in (X, M_p^*). That is meaning

$$\lim_{n\to\infty} M_p^*(x_n, x_n, y) = M_p^*(y, y, y).$$

Now, we prove that every Cauchy sequence $\{x_n\}$ in $(X, M_{p^s}^*)$ is a Cauchy sequence in (X, M_p^*). Let $\epsilon = \frac{1}{2R}$, then there exists $n_0 \in \mathbb{N}$ such that $M_{p^s}^*(x_n, x_n, x_m) < \frac{1}{2R}$ for all $n, m \geq n_0$. Since

$$\begin{aligned} M_p^*(x_n, x_n, x_n) &\leq 4RM_p^*(x_{n_0}, x_{n_0}, x_n) - 3M_p^*(x_n, x_n, x_n) \\ &\quad - M_p^*(x_{n_0}, x_{n_0}, x_{n_0}) + M_p^*(x_n, x_n, x_n) \\ &\leq 2RM_{p^s}^*(x_n, x_n, x_{n_0}) + M_p^*(x_{n_0}, x_{n_0}, x_{n_0}). \end{aligned}$$

Thus, we have

$$\begin{aligned} M_p^*(x_n, x_n, x_n) &\leq 2RM_{p^s}^*(x_n, x_n, x_{n_0}) + M_p^*(x_{n_0}, x_{n_0}, x_{n_0}) \\ &\leq 1 + M_p^*(x_{n_0}, x_{n_0}, x_{n_0}). \end{aligned}$$

Consequently the sequence $\{M_p^*(x_n,x_n,x_n)\}$ is bounded in $\mathbb{R}$ and so there exists an $a\in\mathbb{R}$ such that a sub sequence $\{M_p^*(x_{n_k},x_{n_k},x_{n_k})\}$ is convergent to a, i.e. $\lim_{k\to\infty} M_p^*(x_{n_k},x_{n_k},x_{n_k})=0$.

It remains to prove that $\{M_p^*(x_n,x_n,x_n)\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\{x_n\}$ is a Cauchy sequence in $(X,M_{p^s}^*)$, for $\epsilon>0$, there exists n_ϵ such that $M_{p^s}^*(x_n,x_n,x_m)<\frac{\epsilon}{2R}$ for all $n,m\geq n_\epsilon$. Hence, for all $n,m\geq n_\epsilon$,

$$\begin{aligned}&\left|M_p^*(x_n,x_n,x_n)-M_p^*(x_m,x_m,x_m)\right|\leq 4RM_p^*(x_n,x_n,x_m)-3M_p^*(x_n,x_n,x_n)\\&-M_p^*(x_m,x_m,x_m)+M_p^*(x_n,x_n,x_n)-M_p^*(x_m,x_m,x_m)\\&\leq 2RM_{p^s}^*(x_n,x_n,x_m)<\epsilon.\end{aligned}$$

On the other hand,

$$\begin{aligned}\left|M_p^*(x_n,x_n,x_n)-a\right|&\leq\left|M_p^*(x_n,x_n,x_m)-M_p^*(x_{n_k},x_{n_k},x_{n_k})\right|\\&\quad+\left|M_p^*(x_{n_k},x_{n_k},x_{n_k})-a\right|<\epsilon+\epsilon=2\epsilon,\end{aligned}$$

for all $n,n_k\geq n_\epsilon$. Hence $\lim_{n\to\infty} M_p^*(x_n,x_n,x_n)=a$.

Now, we show that $\{x_n\}$ is a Cauchy sequence in (X,M_p^*). We have

$$\begin{aligned}&\left|2RM_p^*(x_n,x_n,x_m)-2a\right|\\&=\left|RM_{p^s}^*(x_n,x_n,x_m)+M_p^*(x_n,x_n,x_n)-a+M_p^*(x_m,x_m,x_m)-a\right|\\&\leq RM_{p^s}^*(x_n,x_n,x_m)+\left|M_p^*(x_n,x_n,x_n)-a\right|+\left|M_p^*(x_m,x_m,x_m)-a\right|\\&<\frac{\epsilon}{2R}+2\epsilon+2\epsilon=(\frac{1}{2R}+4)\epsilon.\end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence in (X,M_p^*).

We shall have established the lemma if we prove that $(X,M_{p^s}^*)$ is complete if so is (X,M_p^*). Let $\{x_n\}$ be a Cauchy sequence in $(X,M_{p^s}^*)$. Then $\{x_n\}$ is a Cauchy sequence in (X,M_p^*) and so it is convergent to point $y\in X$ with

$$\lim_{n,m\to\infty} M_p^*(x_n,x_n,x_m)=\lim_{m\to\infty} M_p^*(y,y,x_m)=M_p^*(y,y,y).$$

Thus, for $\epsilon>0$, there exists $n_\epsilon\in\mathbb{N}$ such that

$$\left|M_p^*(y,y,x_n)-M_p^*(y,y,y)\right|<\frac{\epsilon}{2R}$$

and

$$\left|M_p^*(y,y,y)-M_p^*(x_n,x_n,x_n)\right|<\frac{\epsilon}{2R}$$

whenever $n\geq n_\epsilon$. As a consequence, we have

$$\begin{aligned}M_{p^s}^*(y,y,x_n)&=2RM_p^*(y,y,x_n)-M_p^*(x_n,x_n,x_n)-M_p^*(y,y,y)\\&\leq\left|RM_p^*(y,y,x_n)-M_p^*(y,y,y)\right|+\left|RM_p^*(y,y,x_n)-M_p^*(x_n,x_n,x_n)\right|\\&<R\frac{\epsilon}{2R}+R\frac{\epsilon}{2R}=\epsilon,\end{aligned}$$

whenever $n \geq n_\epsilon$. Therefore $(X, M^*_{p^s})$ is complete.

Finally, it is easy to check that $\lim_{n\to\infty} M^*_{p^s}(a,a,x_n) = 0$ if and only if

$$M^*_p(a,a,a) = \lim_{n\to\infty} M^*_p(a,a,x_n) = \lim_{n,m\to\infty} M^*_p(x_n,x_n,x_m). \qquad \square$$

Definition 5. *Let (X, M^*_p) be a partial M^*-metric space, then M^*_p is said to first type if*

$$M^*_p(x,x,y) \leq M^*_p(x,y,z),$$

for all $x,y,z \in X$.

3. Fixed point result

We begin this section giving the concept of weakly increasing mappings.

Definition 6 ([39])**.** *Let $(X, \preceq)$ be a partially ordered set. Two mappings $S,T : X \to X$ are said to be $S-T$ weakly increasing if $Sx \preceq TSx$ for all $x \in X$.*

Remark 3 ([39])**.** (i) Two weakly increasing mappings need not be nondecreasing. for examples see [4].

(ii) $\mathcal{F}$ denote the set of all functions $F : [0,\infty) \to [0,\infty)$ such that F is nondecreasing and continuous, $F(0) = 0 < F(t)$, for every $t > 0$ and $F(x+y) \leq F(x) + F(y)$ for all $x,y \in [0,+\infty)$.

(iii) Ψ denote the set of all functions $\psi : [0,\infty) \to [0,\infty)$ where ψ is continuous, nondecreasing function such that $\sum_{n=0}^{\infty} \psi^n(t)$ is convergent for each $t > 0$. From the conditions on ψ, it is clear that $\lim_{n\to\infty} \psi^n(t) = 0$ and $\psi(t) < t$, for every $t > 0$.

Now, we begin the our main results is as follows:

Theorem 8. *Let $(X, \preceq)$ be a partially ordered set and suppose that the partial M^*-metric space M^*_p is a first type on X and (X, M^*_p) is a complete partial M^*-metric space. Let $S,T,G : X \to X$ be three self-mappings such that $S-T$, $T-G$ and $G-S$ are weakly increasing mappings such that*

$$F(M^*_p(Sx,Ty,Gz)) \leq \frac{1}{R}\psi(RF(\varphi(x,y,z)) \tag{3.1}$$

for all $x,y,z \in X$ with x,y,z are comparable with respect to partially order $\preceq$, where $F \in \mathcal{F}$, $\psi \in \Psi$ and

$$\varphi(x,y,z) = \max\left\{ \begin{array}{c} M^*_p(x,y,z), M^*_p(x,x,Sx), \\ M^*_p(y,y,Ty), M^*_p(z,z,Gz) \end{array} \right\}. \tag{3.2}$$

Further assume that if, for every increasing sequence $\{x_n\}$ convergent to $x \in X$, we have $x_n \preceq x$. Then S,T and G have a common fixed point.

Proof. Let x_0 be arbitrary point of X. We can define a sequence in X as follows $x_{3n+1} = Sx_{3n}$, $x_{3n+2} = Tx_{3n+1}$ and $x_{3n+3} = Gx_{3n+2}$ for $n = 0, 1, 2, ...$

Since $S - T$, $T - G$ and $G - S$ are weakly increasing mappings, we have $x_1 = Sx_0 \preceq TSx_0 = x_2 = Tx_1 \preceq GTx_1 = x_3 = Gx_2 \preceq SGx_2 = x_4$ and continuing this process, we have $x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots$

Case 1. Suppose there exists $n_0 \in \mathbb{N}$ such that $M_p^*(x_{3n_0}, x_{3n_0+1}, x_{3n_0+2}) = 0$. Now, we show that $M_p^*(x_{3n_0+1}, x_{3n_0+2}, x_{3n_0+3}) = 0$. Otherwise, from (3.1), we get

$$\begin{aligned}
F(M_p^*(x_{3n_0+2}, x_{3n_0+2}, x_{3n_0+3})) &\leq F(M_p^*(x_{3n_0+1}, x_{3n_0+2}, x_{3n_0+3})) \\
&= F(M_p^*(Sx_{3n_0}, Tx_{3n_0+1}, Gx_{3n_0+2})) \\
&\leq \frac{1}{R}\psi(RF(\varphi(x_{3n_0}, x_{3n_0+1}, x_{3n_0+2}))) \\
&= \frac{1}{R}\psi(RF(\varphi(x_{3n_0+2}, x_{3n_0+2}, x_{3n_0+3}))) \\
&< F(x_{3n_0+2}, x_{3n_0+2}, x_{3n_0+3}),
\end{aligned}$$

which is a contradiction. Hence $M_p^*(x_{3n_0}, x_{3n_0+1}, x_{3n_0+1}) = 0$. Therefore, $x_{3n_0} = x_{3n_0+1} = x_{3n_0+2} = x_{3n_0+3}$. Thus, $Sx_{3n_0} = Tx_{3n_0} = Gx_{3n_0} = x_{3n_0}$. That is x_{3n_0} is a common fixed point of S, T and G.

Case 2: Assume $M_p^*(x_{3n}, x_{3n+1}, x_{3n+2}) > 0$ for all $n \in \mathbb{N}$. Now, we want to prove

$$F(M_p^*(x_{n-1}, x_n, x_{n+1})) \leq \psi(F(M_p^*(x_{n-2}, x_{n-1}, x_n))). \tag{3.3}$$

Setting $x = x_{3n}$, $y = x_{3n+1}$ and $z = x_{3n+2}$ in (3.2), we have

$$\varphi(x_{3n}, x_{3n+1}, x_{3n+2}) = \max \left\{ \begin{array}{c} M_p^*(x_{3n}, x_{3n+1}, x_{3n+2}), \\ M_p^*(x_{3n}, x_{3n}, x_{3n+1}), \\ M_p^*(x_{3n}, x_{3n}, x_{3n+2}), \\ M_p^*(x_{3n+2}, x_{3n+2}, x_{3n+3}) \end{array} \right\}.$$

Since M_p^* is of the first type, we get

$$\varphi(x_{3n}, x_{3n+1}, x_{3n+2}) \leq \max \left\{ M_p^*(x_{3n}, x_{3n+1}, x_{3n+2}), M_p^*(x_{3n+1}, x_{3n+2}, x_{3n+3}) \right\}.$$

If $M_p^*(x_{3n+1}, x_{3n+2}, x_{3n+3})$ is maximum in the R.H.S. of the above inequuality, we have from (3.1) that

$$\begin{aligned}
F(M_p^*(x_{3n+1}, x_{3n+2}, x_{3n+3})) &\leq F(M_p^*(Sx_{3n}, Tx_{3n+1}, Gx_{3n+2})) \\
&< \frac{1}{R}\psi(RF(\varphi(x_{3n}, x_{3n+1}, x_{3n+2}))) \\
&\leq \frac{1}{R}\psi(RF(\max \left\{ M_p^*(x_{3n}, x_{3n+1}, x_{3n+2}), \right. \\
&\quad \left. M_p^*(x_{3n+1}, x_{3n+2}, x_{3n+3}) \right\})) \\
&= \frac{1}{R}\psi(RF(M_p^*(x_{3n+1}, x_{3n+2}, x_{3n+3}))) \\
&< F(M_p^*(x_{3n+1}, x_{3n+2}, x_{3n+3})),
\end{aligned}$$

which is a contradiction. Thus,

$$F(M_p^*(x_{3n+1}, x_{3n+2}, x_{3n+3})) \leq \psi(F(M_p^*(x_{3n}, x_{3n+1}, x_{3n+2}))).$$

Similarly, we have

$$F(M_p^*(x_{3n+2}, x_{3n+3}, x_{3n+4})) \leq \psi(F(M_p^*(x_{3n+1}, x_{3n+2}, x_{3n+3}))),$$

and

$$F(M_p^*(x_{3n}, x_{3n+1}, x_{3n+2})) \leq \psi(F(M_p^*(x_{3n-1}, x_{3n}, x_{3n+1}))).$$

Therefore, for every $n \in \mathbb{N}$, we have

$$F(M_p^*(x_n, x_{n+1}, x_{n+2})) \leq \psi(F(M_p^*(x_{n-1}, x_n, x_{n+1}))).$$

Now, we have $F(M_p^*(x_n, x_{n+1}, x_{n+2})) \leq \psi(F(M_p^*(x_{n-1}, x_n, x_{n+1}))) \leq \cdots \leq \psi^n(F(M_p^*(x_0, x_1, x_2)))$.

Hence

$$\lim_{n\to\infty} F(M_p^*(x_n, x_{n+1}, x_{n+2})) = 0,$$

so that

$$\lim_{n\to\infty} M_p^*(x_n, x_{n+1}, x_{n+2}) = 0. \tag{3.4}$$

Since M_p^* is first type and F is non-decreasing, we have

$$F(M_p^*(x_n, x_n, x_{n+1})) \leq F(M_p^*(x_n, x_{n+1}, x_{n+2})) \leq \psi^n(F(M_p^*(x_0, x_1, x_2))).$$

Since $F(x, y) \leq F(x) + F(y)$ and $M_{p^s}^*(x_n, x_n, x_{n+1}) \leq 2RM_p^*(x_n, x_n, x_{n+1})$, we have

$$F(M_{p^s}^*(x_n, x_n, x_{n+1})) \leq 2RF(M_p^*(x_n, x_n, x_{n+1})) \leq 2R\psi^n(F(M_p^*(x_0, x_1, x_2))).$$

Now, from

$$\begin{aligned}M_{p^s}^*(x_{n+k}, x_n, x_n) &\leq RM_{p^s}^*(x_{n+k}, x_{n+k-1}, x_{n+k-1}) \\ &\quad + RM_{p^s}^*(x_{n+k-1}, x_{n+k-2}, x_{n+k-2}) + \cdots + M_{p^s}^*(x_{n+1}, x_n, x_n),\end{aligned}$$

we have

$$\begin{aligned}F(M_{p^s}^*(x_{n+k}, x_n, x_n)) &\leq F(RM_{p^s}^*(x_{n+k}, x_{n+k-1}, x_{n+k-1})) \\ &\quad + \cdots + F(M_{p^s}^*(x_{n+1}, x_n, x_n)) \\ &\leq 2R^2\psi^{n+k-1}(M_p^*(x_0, x_1, x_2)) \\ &+ \cdots + 2R^2\psi^n(M_p^*(x_0, x_1, x_2)) \\ &\leq 2R^2\sum_{i=n}^{\infty}\psi^i(M_p^*(x_0, x_1, x_2)).\end{aligned}$$

Since $\sum_{n=0}^{\infty}\psi^n(t)$ is convergent for each $t>0$ it follows that $\{x_n\}$ is a Cauchy sequence in the M^*-metric space $(X, M^*_{p^s})$. Since (X, M^*_p) is complete, then from Lemma 2.7 follows that the sequence $\{x_n\}$ converges to some x in the M^*-metric space $(X, M^*_{p^s})$. Hence $\lim_{n\to\infty} M^*_{p^s}(x_n, x, x) = 0$. Again, from Lemma 2.7, we have

$$M^*_p(x,x,x) = \lim_{n\to\infty} M^*_p(x,x,x_n) = \lim_{n,m\to\infty} M^*_p(x_n,x_m,x_m). \tag{3.5}$$

Since $\{x_n\}$ is a Cauchy sequence in the M^*-metric space $(X, M^*_{p^s})$ and

$$M^*_{p^s}(x_n,x_m,x_m) = 2RM^*_p(x_n,x_m,x_m) - M^*_p(x_n,x_n,x_n) - M^*_p(x_m,x_m,x_m),$$

we have

$$\lim_{n,m\to\infty} M^*_{p^s}(x_n,x_m,x_m) = 0$$

and by (3.4), we have

$$\lim_{n\to\infty} M^*_p(x_n,x_n,x_n) = 0,$$

thus by definition $M^*_{p^s}$, we have

$$\lim_{n,m\to\infty} M^*_p(x_n,x_m,x_m) = 0.$$

Therefore, by (3.5), we have

$$\begin{aligned} M^*_p(x,x,x) &= \lim_{n\to\infty} M^*_p(x_n,x,x) \\ &= \lim_{n,m\to\infty} M^*_p(x_n,x_m,x_m) = 0. \end{aligned}$$

Now, by the inequality (3.1) for $x=x$, $y=x_{3n+1}$ and $z=x_{3n+2}$, then we have

$$F(M^*_p(Sx,x_{3n+2},x_{3n+3})) \le \frac{1}{R}\psi(RF(\varphi(x,x_{3n+1},x_{3n+2}))),$$

and by letting $n\to\infty$ and using Lemma 2.5, we obtain

$$F(M^*_p(Sx,x,x)) \le \frac{1}{R}\psi(RF(M^*_p(Sx,x,x))) < F(M^*_p(Sx,x,x)),$$

which is a contradiction. Hence, $M^*_p(Sx,x,x)=0$. Thus $Sx=x$. Similarly, by using the inequality (3.1) for $y=x$, $x=x_{3n}$ and $z=x_{3n+2}$, then we have

$$F(M^*_p(x_{3n},Tx,x_{3n+3})) \le \frac{1}{R}\psi(RF(\varphi(x_{3n},x,x_{3n+2}))),$$

and letting $n\to\infty$ and using Lemma 2.5, we obtain

$$F(M^*_p(x,Tx,x)) \le \frac{1}{R}\psi(RF(M^*_p(x,Tx,x)) < F(M^*_p(x,Tx,x)),$$

which is a contradiction. Hence, $Tx=x$. Similarly, by using the inequality (3.1) for $z=x$, $x=x_{3n}$ and $y=x_{3n+1}$, we can show that $Gx=x$. □

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