Junction interface conditions for asymptotic gradient full-observer in Hilbert space

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Abstract. The fundamentals concept of boundary asymptotic gradient observer of full order type $\partial\Omega AGFO$-observer via internal case in link with the strategic sensors in different system domains have been presented. The results so obtained for linear dynamical systems which is created by a strongly continuous semi-group (SCS-group) in Hilbert space $H^{1/2}(\partial\Omega)$ have been analyzed. Consequently, the existence of sufficient conditions for $\partial\Omega AGFO$-estimator in parabolic infinite dimensional systems have been studied and scrutinized. In addition to that, we have observed at the junction interface that the interior solution is harmonized with the exterior solution for asymptotic gradient full observation.

Keywords: $\partial\Omega AGFO$-observer, $\partial\Omega AG$-detectability, $\partial\Omega G$-strategic sensor, junction conditions.

1. Introduction

In literature, a distributed parameter systems and observability concepts on a special domain Ω have been widely developed and tackled by several authors [1-2]. The determination of Luenberger observer is to offer an asymptotic for-

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mal approximation for the current state of deliberated system [3-4]. Recently, Al-Saphory and El Jai et al. have explored a new direction of regional analysis for distributed parameter systems in finite time interval and infinite, with regional or regional boundary cases associated with strategic sensors and actuators as in [5-15]. In this paper, we familiarize and explore the notion of $\partial\Omega AGFO$-observer connected to extended internal region approach of the considered system domain [7, 12]. Therefore the usual boundary case have been developed through an extension to previous works as in [13-14]. In addition to that, boundary detectability and boundary strategic sensors have been deliberated and analyzed.

The incentive of studying this notion is there exist several problem in the real world needs to be studied as in [4, 16]. Indeed, the authors have obtained a more general mathematical model of the $BAGFO$-observer which characterized by internal gradient strategic (zone, pointwise or filament) sensors (Figure 1).

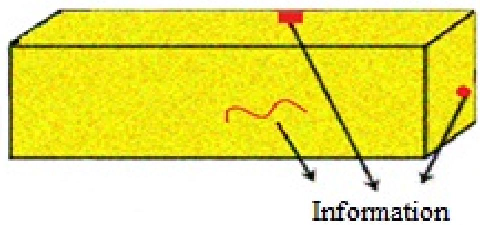

Figure 1: Mathematical modeling with positions of sensors.

The rest of the paper is prearranged as follows. Section 2 is enthusiastic to the considered system and preliminaries. In Section 3, we study $\partial\Omega G$-observability and $\partial\Omega AG$-detectability and extant some original results. In section 4 we familiarize $\partial\Omega AGFO$-observer concepts in terms of $\partial\Omega AG$-detectability and $\partial\Omega G$-strategic sensors. Also, the matching of inside to the outside solution at a junction interface has been studied in the sense of Banerjee et al. [16]. Finally, some applications for distributed diffusion systems with the devoted different domains and strategic sensors have been demonstrated.

2. Preliminaries of system inceptions

Assume Ω be an open set in $\mathbb{R}^n$, through smooth boundary $\partial\Omega$ with the following sets

$$\Pi = \Omega \times (0, \infty); \ \Xi = \partial\Omega \times (0, \infty).$$

Suppose that the following spaces specify as separable Hilbert type given by

$$\mathbb{W} = H^1(\Omega); \ \mathbb{U} = L^2(0, \infty, R^p); \ \mathbb{Y} = L^2(0, \infty, R^q).$$

So, these spaces represent respectively as state space; input space and measurement space such that p with q the numbers of controls and information [17].

Thus, the system can be written as:

$$(1)\qquad \begin{cases} \dfrac{\partial w}{\partial t}(\xi,t) = \mathcal{A}w(\xi,t) + Bu(t) & \Pi \\ w(\mu,t) = 0 & \Xi \\ w(\zeta,0) = w(\xi) & \Omega \end{cases}$$

augmented with the output function

$$(2)\qquad y(.,t) = Cw(.,t) \qquad \Pi$$

where Ω grips for the closure of Ω and $w_0(\xi)$ which is made-up to be unidentified in the state space $\mathbb{W} = H^1(\Omega)$. Therefore, $\mathcal{A}$ is a linear self-adjoint transformation of 2^{nd} differential case, with compact resolvent. Now, operators $B \in L(\mathbb{R}^p, \mathbb{W})$ and $C \in L(H^1(\Omega), \mathbb{R}^q)$ be contingent on the structures of control and information [18], which means, in various situations [12]. Consequently, we obtain $B \notin L(\mathbb{R}^p, \mathbb{W})$ and $C \notin L(H^{1/2}(\partial\Omega), \mathbb{R}^q)$. Accordingly, the system (1) has a unique solution [17-18] which is assumed as

$$(3)\qquad w(\zeta,t) = S_{\mathcal{A}}(t)w_0(\zeta) + \int_0^t S_{\mathcal{A}}(t-\tau)Bu(\tau)\,d\tau \qquad \Pi$$

. K an operator is defined by following

$$\begin{aligned} K &: \mathbb{W} \longrightarrow \mathbb{Y}, \\ w &\longrightarrow CS_{\mathcal{A}}(\cdot)w \end{aligned}$$

and,

$$y(\cdot,t) = K(t)w(\cdot,0),$$

where K is bounded linear operator [14-15].

. $K^* : \mathbb{Y} \longrightarrow \mathbb{W}$ is the adjoint operator of K, is defined by

$$K^*y^* = \int_0^t S^*_{\mathcal{A}}(\tau)C^*y^*(\cdot,\tau)\,d\tau.$$

. The operator ∇ signifies the gradient, which is assumed to have the form

$$(4)\qquad \begin{cases} \nabla : H^1(\Omega) \longrightarrow (H^1(\Omega))^n \\ w \longrightarrow \nabla_w = \left(\dfrac{\partial w}{\partial \zeta_1}, \cdots, \dfrac{\partial w}{\partial \zeta_n}\right) \end{cases}$$

∇^* is the adjoint of ∇ and specified by

$$\begin{cases} \nabla^* : (H^1(\Omega))^n \longrightarrow H^1(\Omega) \\ w \longrightarrow \nabla^*_w = v \end{cases}$$

Such that v is characterized a solution of the Dirichlet problem

$$\begin{cases} \Delta_v = -\operatorname{div}(w) & \Omega \\ v = 0 & \partial\Omega \end{cases}$$

Thus, an extension of the trace operator [19] which is denoted by γ defined as

$$\gamma : (H^1(\Omega))^n \longrightarrow (H^{1/2}(\partial\Omega))^n$$

and the adjoints is correspondingly given by γ^*.

. Systems (1)-(2) are assumed to be exactly observable (or $\mathbb{E}\Omega$-observable) and weakly observable (or $\mathbb{W}\Omega$- observable) on $[0,T]$ if $\operatorname{Im} K^* = H^1(\Omega)$ and $\overline{\operatorname{Im} K^*} = H^1(\Omega)$ respectively.

. The semi-group $(S_{\mathcal{A}}(t))_{t\geq 0}$ is asymptotically stable in $H^1(\Omega)$ (or ΩA-stable), if, for +ve constants M_Ω and α_Ω, then

$$\|S_{\mathcal{A}}(\cdot)\|_{L(H^1(\Omega),\mathbb{W})} \leq M_\Omega e^{-\alpha_\Omega t},\ t \geq 0.$$

. System (1) is called ΩA-stable if the transformation $\mathcal{A}$ produces SCS-group $(S_{\mathcal{A}}(t))_{t\geq 0}$ which is ΩA-stable.

. Systems (1)-(2) are assumed to be asymptotically detectable (ΩA-detectable) if the transformation $H_\Omega : \mathbb{Y} \longrightarrow H^1(\Omega)$ such that the operator $(\mathcal{A} - H_\Omega C)$ creates a SCS-group $(S_{H_\Omega}(t))_{t\geq 0}$, which is ΩA-stable.

3. $\partial\Omega G$-observability and $\partial\Omega AG$-detectability

The observability definitions to boundary case for parabolic, hyperbolic linear, semi-linear and nonlinear have been extended [20-23] with duel concept [24]. Though, we come across to some definitions and theorems to elucidate the concept of $\partial\Omega AG$-detectability and $\partial\Omega G$-observability in the state space $H^{1/2}(\partial\Omega)$ in it's an extension from [5,14].

Definition 3.1. *System* (1) *together with information (2) is assumed to be exactly gradient observable (or $\mathbb{E}\Omega G$-observable) on* $[0,T]$ *if:*

$$\operatorname{Im}\nabla K^* = \left(H^1(\Omega)\right)^n.$$

Definition 3.2. *System* (1) *together with information (2) is assumed to be weakly gradient observable (or $\mathbb{W}\Omega G$-observable) on* $[0,T]$ *if:*

$$\overline{\operatorname{Im}\nabla K^*} = \left(H^1(\Omega)\right)^n.$$

Definition 3.3. *System* (1) *together with information (2) is assumed to be exactly boundary gradient observable ($\mathbb{E}\partial\Omega G$-observable) on* $[0,T]$, *if:*

$$\operatorname{Im}\gamma\nabla K^* = \left(H^{1/2}(\partial\Omega)\right)^n.$$

Definition 3.4. *System* (1) *together with information (2) is assumed to be weakly gradient observable* $\mathbb{W}\partial\Omega G$*-observable on [0,T], if:*

$$\overline{\mathrm{Im}\gamma\nabla K^*} = \left(H^{1/2}(\partial\Omega)\right)^n.$$

Remark 3.1. We can deduced that, the equation:

$$\overline{\mathrm{Im}\gamma\nabla K^*} = \left(H^{1/2}(\partial\Omega)\right)^n$$

is equivalent to:

$$\ker \nabla K^* \gamma^* = \{0\}.$$

From previous results, we present the characterization of exactly boundary gradient observable system in Ω ($EG\partial\Omega$-observable) in the following result.

Proposition 3.1. *System* (1) *together with information (2) is said to be* $E\partial\Omega G$*-observable on* $[0,T]$ *if and only if* $\exists\ \alpha_{E\partial\Omega G} \geq 0$*, such that:*

$$\|\gamma\nabla w\|_{L\left(H^1(\Omega),\left(H^{1/2}(\partial\Omega)\right)^n\right)} \leq \alpha_{E\partial\Omega G}\,\|Kw_0\|_Y\,,\ for\ all\ w_0 \in \mathbb{W} \tag{5}$$

Now, we give the concept of boundary gradient strategic sensor ($\partial\Omega G$-strategic sensor).

Definition 3.5. *Sensor* (D, f) *is* $\partial\Omega G$*-strategic, if the corresponding system is* $\mathbb{W}\partial\Omega G$*-observable.*

Definition 3.6. *The semi-group* $(S_{\mathcal{A}}(t))_{t\geq 0}$ *is supposed to be boundary asymptotic gradient stable on* $(H^{1/2}(\partial\Omega))^n$ *(*$\partial\Omega AG$*-stable), if for some positive constants* $M_{\partial\Omega AG}, \alpha_{\partial\Omega AG} > 0$*, then:*

$$\|\gamma\nabla S_A(t)\|_{\left(H^{1/2}(\partial\Omega)\right)^n} \leq M_{\partial\Omega AG}e^{-\alpha_{\partial\Omega AG}t},\ for\ all\ t \geq 0.$$

Remark 3.2. If the semi-group $(S_{\mathcal{A}}(t))_{t\geq 0}$ is $\partial\Omega AG$-stable, then for all $w_0 \in (H^{1/2}(\partial\Omega))^n$ the solutions associated to the autonomous system of (1) satisfies:

$$\lim_{t\to\infty} \|\gamma\nabla S_{\mathcal{A}}w(\cdot,t)\|_{\left(H^{1/2}(\partial\Omega)\right)^n} = \lim_{t\to\infty} \|\gamma\nabla S_{\mathcal{A}}w(\cdot)\|_{\left(H^{1/2}(\partial\Omega)\right)^n} = 0 \tag{6}$$

Definition 3.7. *System* (1) *is assumed to be* $\partial\Omega AG$*-stable, if the transformation* $\mathcal{A}$ *produces* SCS*-group* $(S_{\mathcal{A}}(t))_{t\geq 0}$ *which is* $\partial\Omega AG$*-stable.*

Definition 3.8. *System* (1) *together with the information (2) is assumed to be* $\partial\Omega AG$*-detectable if there is transformation such that* $(\mathcal{A} - H_{\partial\Omega AG}C)$ *, produces a* SCS*-group* $(S_{H_{\partial\Omega AG}}(t))_{t\geq 0}$ *which is* $\partial\Omega AG$*-stable.*

Though, one can assume the following results. Consequently, the notion of $\partial\Omega AG$-detectability is a weaker property than the exact $E\partial\Omega G$-observability [1,14].

4. Boundary asymptotic gradient full-order observer

A methodology that permits a construction and reconstruct asymptotically gradient in full order estimator ($\partial\Omega AGFO$-estimator) of $\hat{T}w(\xi,t)$ has been presented in this section. This technique evades the evaluation inverse problem, and related to calculate the unknown initial state [3,10], which permits to guess a current state in $\partial\Omega$ with no needs to the outcome of the initial state of the main system.

4.1 Modernization of $\partial\Omega AGFO$-estimator

Assume the following system:

$$\tag{7} \begin{cases} \dfrac{\partial w}{\partial t}(\zeta,t) = Aw(\zeta,t) + Bu(t) & \Pi \\ w(\mu,t) = 0 & \Xi \\ w(\zeta,0) = w_0(\zeta) & \Omega \\ y(\cdot,t) = Cw(\cdot,t) & \Pi \end{cases}$$

For a region $\partial\Omega$, assume that for $\hat{T} \in L((H^{1/2}(\Omega))^n, (H^{1/2}(\partial\Omega))^n)$ and $\hat{T} = \gamma T$, $\exists\ \mathcal{V}(\cdot,t)$, such that:

$$\tag{8} \mathcal{V}(\zeta,t) = \hat{T}w(\zeta,t) \qquad \Pi$$

where $\mathcal{V}(\cdot,t)$ is a state system. Therefore, if we can form a system which is an asymptotic approach for $\mathcal{V}(\cdot,t)$, then it will be give an asymptotic estimation for $\hat{T}w(\zeta,t)$ (i.e. it structure an asymptotic observer to the restriction of $Tw(\zeta,t)$ on $\partial\Omega$).

Equations (2)-(8) provides:

$$\tag{9} \begin{bmatrix} y \\ \mathcal{V} \end{bmatrix} = \begin{bmatrix} C \\ \hat{T} \end{bmatrix} w$$

Suppose there exists two linear bounded operators $\mathbb{R}$ and $\mathbb{S}$, where $\mathbb{R} : \mathcal{R} \longrightarrow (H^{1/2}(\partial\Omega))^n$ and $\mathbb{S} : (H^{1/2}(\partial\Omega))^n \longrightarrow (H^{1/2}(\partial\Omega))^n$, such that $\mathbb{R}C + \mathbb{S}\hat{T} = I$, then by deriving $\mathcal{V}(\zeta,t)$, we have:

$$\begin{aligned} \tfrac{\partial \mathcal{V}}{\partial t} &= \hat{T}\tfrac{\partial w}{\partial t}(\zeta,t) = \hat{T}\mathcal{A}w(\zeta,t) + \hat{T}Bu(t) \\ &= \hat{T}\mathcal{A}\mathbb{S}\mathcal{V}(\zeta,t) + \hat{T}\mathcal{A}\mathbb{S}\mathbb{R}y(\zeta,t) + \hat{T}Bu(t) \qquad \Pi \end{aligned}$$

Consider ($\partial\Omega AGFO$-estimator for x) as:

$$\tag{10} \begin{cases} \dfrac{\partial \mathcal{V}}{\partial t}(\zeta,t) = F_{\partial\Omega AG}\mathcal{V}(\zeta,t) + G_{\partial\Omega AG}u(t) + H_{\partial\Omega AG}y(\cdot,t) & \Pi \\ \mathcal{V}(\zeta,0) = \mathcal{V}_0(\zeta) & \Omega \\ \mathcal{V}(\mu,t) = 0 & \Xi \end{cases}$$

with $F_{\partial\Omega\Omega AG}$ generates SCS-group $(S_{F_{\partial\Omega}}(t))_{t\geq 0}$, that is $\partial\Omega AG$-stable on $\mathbb{W} = H^{1/2}(\partial\Omega)$, that means $\exists\ M_{F_{\partial\Omega}},\ \alpha_{F_{\partial\Omega}} > 0$, such that:

$$(11) \qquad \|\chi_{\partial\Omega} S_{F_{\partial\Omega}}(.)\|_{L\left(\left(H^{1/2}(\partial\Omega)\right)^n,\left(H^{1/2}(\partial\Omega)\right)^n\right)} \leq M_{F_{\partial\Omega}} e^{-\alpha_{F_{\partial\Omega}} t},\ \ t \geq 0 \qquad \Pi$$

and $G_{\partial\Omega} \in L(\mathbb{R}^p, (H^{1/2}(\partial\Omega))^n)$ and $H_{\partial\Omega} \in L(\mathbb{R}^p, (H^{1/2}(\partial\Omega))^n)$. The solution of (10) is given by:

$$(12) \quad \mathcal{V}(.,t) = S_{F_{\partial\Omega}}(t)\mathcal{V}_0(\cdot) + \int_0^t S_{F_{\partial\Omega}}(t-\tau)\left[G_{\partial\Omega}u(\tau) + H_{\partial\Omega}y(\cdot,\tau)\right] d\tau \qquad \Pi$$

Now, in the case when $\hat{T} = I$ and $\mathbb{W} = \mathbb{V}$ in equation (8), the operator equation [4]:

$$\hat{T}\mathcal{A} - F_{\partial\Omega AG}\hat{T} = H_{\partial\Omega}\mathcal{A}GC$$

of the $\partial\Omega EFO$-observer becomes to:

$$F_{\partial\Omega AG}\mathcal{A}G = \mathcal{A} - H_{\partial\Omega AG}C,$$

where $\mathcal{A}$ and C are identified. Hence, the operator $H_{\partial\Omega AG}$ has to be known such that the operator $F_{\partial\Omega AG}$ is $\partial\Omega AG$-stable.

Also, for the equation (7), the dynamical system can be deliberated as:

$$(13) \qquad \begin{cases} \dfrac{\partial\mathcal{V}}{\partial t}(\zeta,t) = \mathcal{A}\mathcal{V}(\zeta,t) + Bu(t) + H_{\partial\Omega AG}\left(y(.,t) - C\mathcal{V}(\zeta,t)\right) & \Pi \\ \mathcal{V}(\mu,t) = 0 & \Xi \\ \mathcal{V}(\zeta,0) = 0 & \Omega \end{cases}$$

which is named $\partial\Omega EFO$-observer.

4.2 Junction interface conditions

We examine the three regions E, E_1 and E_2 as in (Figure 2) of junction conditions [16] used to generalize an approach may be called asymptotic observer to build the gradient of current state on the $\partial\Omega$. Thus, the boundary observer on

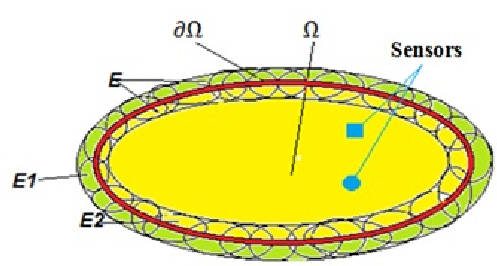

Figure 2: Ω, $\partial\Omega$ and regions junction conditions.

$\partial\Omega$ might be gotten as an observer of internal regional type in E_2. If we have the following mapping $\mathfrak{R}$ holds an extension of continuous linear operator [19], $\mathfrak{R} : (H^{1/2}(\partial\Omega))^n \longrightarrow (H^1(\Omega))^n$, such that:

$$\gamma\nabla\mathfrak{R}h(\mu,t) = h(\mu,t), \quad \textit{for all } h \in \left(H^{1/2}(\partial\Omega)\right)^n \qquad \Xi \tag{14}$$

Let $\forall\, w_0 \in \partial\Omega$ there exists $r > 0$ is an random and appropriately small real with the following sets:

$$E = \bigcup_{w_0\in\partial\Omega} B(w_0,r) = \{w \in \Omega \textit{ or } w \in E_1 : \|w - w_0\| < r,\ w_0 \in \partial\Omega\},$$

where:

$$E_1 = \bigcup_{w_0\in\partial\Omega} B(w_0,r) = \{w \in E \textit{ or } w \notin \Omega : \|w - w_0\| < r,\ w_0 \in \partial\Omega\}$$

and

$$E_2 = \bigcup_{w_0\in\partial\Omega} B(w_0,r) = \{w \in E \textit{ or } w \notin E_1 : \|w - w_0\| < r,\ w_0 \in \partial\Omega\} \subset \Omega$$

and then we have:

$$E = E_1 \cup E_2,\ \partial\Omega = E_1 \cap \bar{E}_2 \textit{ and } E_2 = E \cap \Omega,$$

where $B(w_0,r)$ represents a ball of radius r centered in $w_0(\mu,t)$ and $\partial\Omega$ is boundary of the domain Ω.

For the a region E_2 of the domain Ω and let χ_{E_2} be a function assumed as:

$$\chi_{E_2} : \left(H^1(\Omega)\right)^n \to \left(H^1(E_2)\right)^n,$$
$$w : \chi_{E_2} w = w|_{E_2},$$

where $w|_{E_2}$ is the restriction of w to E_2 with adjoint operator $\chi^*_{E_2}$ (for more details see references [7-9]).

Definition 4.1. *System* (7) *is E_2AG-stable, then the autonomous system solution linked to* (7)*, asymptotically converges to 0 when t approaches to ∞.*

Definition 4.2. *System* (7) *is called E_2AG-detectable, if there exists an operator $H_{E_2AG} : \mathcal{O} \longrightarrow (H^1(E_2))^n$, such that the operator $A - H_{E_2AG}C$ produces a SCS-group $(S_{E_2AG}(t))_{t>0}$, which is E_2AG-stable.*

So, the process of junction conditions from interior to exterior of E_2AG-detectability might be assumed as follows [23-26]:

Proposition 4.1. *If the system* (7) *is $\bar{E}_2AG$-detectable, then it is $\partial\Omega AG$-detactable.*

Proof. Suppose that $w(\zeta,t) \in H^{1/2}(\partial\Omega)$ and $\bar{w}(\zeta,t)$ be an extension to $H^{1/2}(\bar{E}_2)$ with $\partial\Omega \subset \bar{E}_2$.

Trace theorem [19] with equation (14) tells, there exist $\mathfrak{R}\bar{w}(\zeta,t) \in (H^1(\Omega))^n$ with a bounded support such that:

$$\gamma\left(\mathfrak{R}\mathfrak{R}_{E_2}\bar{w}(\zeta,t)\right) = w(\zeta,t) \qquad \Pi \tag{15}$$

where $\mathfrak{R}_{E_2} : (H^1(E_2))^n \longrightarrow (H^{1/2}(\partial\Omega))^n$. Since the system (7) is $\bar{E}_2$-detectable, then it is E_2-detectable [18, 25]. Accordingly, there exists an operator $\chi_{E_2}\nabla K^* : \mathcal{O} \longrightarrow (H^1(E_2))^n$ specified by:

$$H_{\partial\Omega AG}w(\cdot,t) = \gamma\nabla K^* y(\zeta,t) \qquad \Pi \tag{16}$$

such that the operator $A - H_{\partial\Omega AG}C$ produces a SCS-group $(S_{\partial\Omega}(t))_{t>0}$ which is $\partial\Omega AG$-stable. For every $\in \mathcal{O}$, then we get:

$$\chi_{E_2}\nabla K^* y(\xi,t) = \chi_{E_2}\mathfrak{R}\mathfrak{R}_{E_2}\bar{w}(\xi,t)$$

and hence:

$$H_{\partial\Omega AG} = \left(\gamma\chi_{E_2}^*\nabla K^* y\right) : \mathbb{Y} \longrightarrow \left(H^{1/2}(\partial\Omega)\right)^n$$

such that $A - H_{\partial\Omega AG}C$produces a semi-group $(S_{\partial\Omega}(t))_{t>0}$, which is $\partial\Omega AG$-stable. To conclude, the system (7) is $\partial\Omega AG$-detectable. □

Proposition 4.2. *If the dynamical system (13) is $\bar{E}_2AGFO$-observer for the systems* (7) *then, its $\partial\Omega AGFO$-observer.*

Proof. In view of assumptions as in Proposition 4.3 with equations (15) and (16) and since the dynamical systems (13) $\bar{E}_2AGFO$-observer, so we can assume that:

I- The systems (13) is E_2AGFO-observer [25-26], thus there exists a dynamical system with $w(\xi,t) \in \mathbb{W}$, such that:

$$\chi_{E_2}\hat{T}w(\zeta,t) = \chi_{E_2}\mathfrak{R}\mathfrak{R}_{E_2}\bar{w}(\zeta,t).$$

Then, we have:

$$\left(\gamma\chi_{E_2}^*\chi_{E_2}\mathfrak{R}\hat{T}w\right)(\zeta,t) = w(\zeta,t) \qquad \Pi \tag{17}$$

II- The equations (2) and (16) allow:

$$\begin{bmatrix} y \\ \mathcal{V} \end{bmatrix}(\zeta,t) = \begin{bmatrix} C \\ \left(\gamma\chi_{E_2}^*\chi_{E_2}\mathfrak{R}\hat{T}\right) \end{bmatrix} w(\zeta,t) \qquad \Pi$$

and there exists two linear bounded operator $\bar{R}$ and C satisfy the relation:

$$\bar{R}C\left(\gamma\chi_{E_2}^*\chi_{E_2}\mathfrak{R}\hat{T}\right) + \gamma\chi_{E_2}^*\chi_{E_2}\mathfrak{R}\hat{T} = I_{\partial\Omega AG}.$$

III- There exist an operator $F_{\bar{E}_2}$ is $\bar{E}_2AGFO$-observer, such that $\partial\Omega AG$-stable (see [28]). To end with the dynamical system (13) is $\partial\Omega AGFO$-observer for the system (7).

□

4.3 Sensors and $\partial\Omega AG$ -detectability

The boundary asymptotic gradient detectability concept with the spatial structure of sensors can be linked [6]. Now, for that determination assume that J has unstable modes to have a clear picture of this concept with respect to sensor structures [5].

Proposition 4.3. *Assume that there is q zone sensors $(D_i, f_i)_{1\leq i\leq q}$ and $\rho(\mathcal{A})$ is the spectrum of A holds for finite J eigenvalues of $Re\lambda_J \geq 0$. Then,* (7) *is $\partial\Omega AG$-detectable if and only if:*

I- $q \geq m$,

II- $rank\, G_i = m_i, i = 1, 2, \cdots, J$, *where* $supm_i = m < \infty$ *and* $j = 1, 2, \cdots, m_i$,

$$G = (G)_{ij} = \begin{cases} < \varphi_{nj}, f_i(.) >_{L^2(D_i)}, & \textit{zone case} \\ \varphi_{nj}(b_i), & \textit{poitwise case} \\ < \Phi_{nj}, f_i(.) >_{L^2(\Gamma_i)}, & \textit{boundary zone case} \end{cases}$$

Proof. The proof can be established as in [25] with case of state gradient $w(\xi, t)$ belong to sub region Γ, such that $\Gamma = \partial\Omega$. □

Remark 4.1. If the system (7) is $\partial\Omega AG$-detectable, then it is possible to construct an $\partial\Omega AGFO$-observer for the original system [5, 25].

Proposition 4.4. *If the systems* (7) *is $\partial\Omega AG$-detectable, then the dynamical system* (13) *is $\partial\Omega AGFO$-observer of the systems* (7) *that means:*

$$\lim_{t\to\infty} [w(\zeta, t) - \mathcal{V}(\zeta, t)] = 0 \qquad \Pi \tag{18}$$

Proof. Assume $\varphi(\zeta, t) = w(\zeta, t) - \mathcal{V}(\zeta, t)$, where $\mathcal{V}(\zeta, t)$ is the solution of (13). Differentiate equation (18) and use of equations (7) and (13), we attain:

$$\begin{aligned} \tfrac{\partial\varphi}{\partial t}(\zeta, t) &= \tfrac{\partial w}{\partial t}(\zeta, t) - \tfrac{\partial V}{\partial t}(\zeta, t) \\ &= (A - H_{\partial\Omega AG} C)\, \varphi(\zeta, t) \qquad \Pi. \end{aligned}$$

The system (7) is $\partial\Omega AG$-detectable. Hence, there exists an operator $H_{\partial\Omega AG} \in L(\mathcal{O}, (H^{1/2}(\partial\Omega))^n)$, such that $A - H_{\partial\Omega AG}C$ produces a SCS-group $(S_{\partial\Omega AG}(t))_{t\geq 0}$m which is $\partial\Omega AG$-stable on $(H^{1/2}(\partial\Omega))^n$, and there exists $M_{\partial\Omega AG}$, $\omega_{\partial\Omega AG} > 0$, such that:

$$\|\varphi\|_{(H^{1/2}(\partial\Omega))^n} \leq \|\gamma\nabla S_{\partial\Omega AG}(t)\|_{(H^{1/2}(\partial\Omega))^n} \|\varphi_0\| \leq M_{\partial\Omega AG} e^{-\omega_{\partial\Omega AG} t} \|\varphi_0\|$$

with

$$\varphi_0(\zeta) = w_0(\zeta) - \mathcal{V}_0(\zeta)$$

and hereafter, we got the following:

$$\lim_{t\to\infty} [w(\zeta, t) - \mathcal{V}(\zeta, t)] = 0 \qquad \Pi.$$

□

5. Applications to $\partial\Omega AGFO$-Observer

The distributed diffusion systems defined in the domain Ω have been considered as an application to $\partial\Omega AGFO$-observer [12, 27]. Several applications in real life problems associated with different types of sensor have been prolonged. For two-dimensional system, the domain:

$$\Omega =]0, a_1[\times]0, a_2[$$

with the boundary is given by the following form:

$$\partial\Omega = [0, a_1] \times \{a_2\} \cup [0, a_1] \times \{0\} \cup \{0\} \times [0, a_2] \cup a_1 \times [0, a_2]$$

is a region of $\bar{\Omega}$.

The eigenfunctions of (16) are defined by:

$$\varphi_{nm}(\zeta_1, \zeta_2) = \left(\frac{4}{a_1 a_2}\right)^{1/2} \cos n\pi \left(\frac{\zeta_1}{a_1}\right) \cos n\pi \left(\frac{\zeta_2}{a_2}\right) \tag{19}$$

associated with eigenvalues:

$$\lambda_{nm} = -\left(\frac{n^2}{a_1^2} + \frac{m^2}{a_2^2}\right)\pi^2, \;\; n, m \geq 1 \tag{20}$$

If we assume that $\frac{a_1^2}{a_2^2} \notin Q$ [28-30], then the multiplicity of the eigenvalues λ_{nm} is $r_{nm} = 1$ for every $n, m = 1, 2, \cdots, J$, then one sensor (D, f) may be sufficient for $\partial\Omega AGFO$-observer [26-30].

5.1 Rectangular domain

A sufficient conditions which is characterized some cases of the $\partial\Omega AGFO$-observer in the rectangular domain of system (21) with various sensor locations cases have been provided in this section.

5.1.1 Internal zone sensors case

Assume the following two dimensional system that is defined by parabolic equation:

$$\begin{cases} \dfrac{\partial w}{\partial t}(\zeta_1, \zeta_2, t) = \dfrac{\partial^2 w}{\partial w \partial \zeta_1^2}(\zeta_1, \zeta_2, t) + \dfrac{\partial^2 w}{\partial \zeta_1^2}(\zeta_1, \zeta_2, t) & \Pi \\ w(\mu_1, \mu_2, t) = 0 & \Xi \\ w(\zeta_1, \zeta_2, 0) = w_0(\zeta_1, \zeta_2) & \Omega \end{cases} \tag{21}$$

together with the information is represented via internal pointwise or zone sensors

$$y(\cdot, t) = \int_D w(\zeta_1, \zeta_2, t) f(\zeta_1, \zeta_2)\, d\zeta_1 d\zeta_2 \qquad \Pi \tag{22}$$

where the zone sensor is situated interior to the domain Ω (Figure 3), with support of:

$$D = [\zeta_{1_0} - l_1, \zeta_{1_0} + l_1] \times [\zeta_{2_0} - l_2, \zeta_{2_0} + l_2] \subset \Omega \text{ and } L^2(D)$$

In this case the system (21) together with the information (22) have an associ-

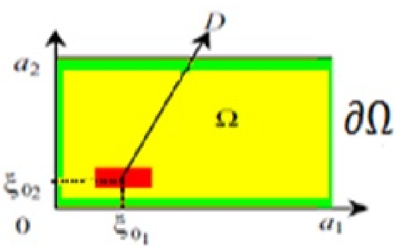

Figure 3: Ω, $\partial\Omega$ with sensor position D of internal zone type.

ated dynamical system, that is specified by the following formula:

$$(23) \qquad \begin{cases} \dfrac{\partial \mathcal{V}}{\partial t}(\zeta_1, \zeta_2, t) = \dfrac{\partial^2 \mathcal{V}}{\partial w \partial \zeta_1^2}(\zeta_1, \zeta_2, t) + \dfrac{\partial^2 \mathcal{V}}{\partial \zeta_1^2}(\zeta_1, \zeta_2, t) & \\ \qquad\qquad -H_{\partial\Omega GA}(C\mathcal{V}(\zeta_1, \zeta_2, t) - y(t)) & \Pi \\ \mathcal{V}(\mu_1, \mu_2, t) = 0 & \Xi \\ \mathcal{V}(\zeta_1, \zeta_2, 0) = z_0(\zeta_1, \zeta_2) & \Omega \end{cases}$$

Hence, the following important result is obtained.

Proposition 5.1. *Assume that f_1 and f_2 are symmetric about $\zeta = \zeta_{01}$ and $\zeta = \zeta_{02}$ respectively, then the process* (23) *is $\partial\Omega AGFO$-observer for systems* (21)-(22) *if $n\zeta_{01}/a_1$ and $m\zeta_{02}/a_2 \notin N$, for every $n, m = 1, 2, \cdots, J$.*

5.1.2 Pointwise sensors case

Assume the system (21) together with information (24) which is measured by internal pointwise sensors. Then, the output function can be formulated as:

$$(24) \qquad y(t) = \int_\Omega w(\zeta_1, \zeta_2, t)\delta(\zeta_1 - b_1, \zeta_2 - b_2)\, d\zeta_1 d\zeta_2 \qquad \Pi$$

So, the following result is prophesied.

Proposition 5.2. *Let $b = (b_1, b_2)$ is the sensor positioned in Ω, then the dynamic system* (23) *is $\partial\Omega AGFO$-observer for the system* (21)-(24), *if $(nb_1)/(a_1)$ and $(mb_2)/(a_2 \notin N)$, for every $n, m = 1, 2, \cdots, J$.*

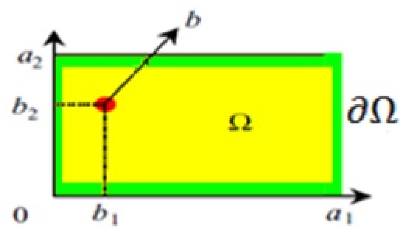

Figure 4: Ω, $\partial\Omega$ with sensor position b of pointwise zone type.

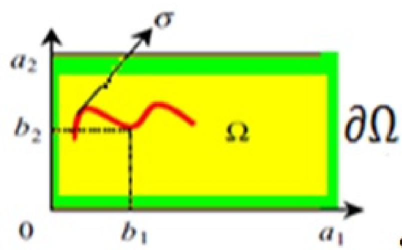

Figure 5: Ω, $\partial\Omega$ with sensor position σ of filament zone type.

5.1.3 Filament pointwise sensors case

Assume that the filament sensor positioned in Ω, where $\sigma = Im(\gamma) \subset \Omega$ is symmetric with respect to the line $b = (b_1, b_2)$ (Figure 5). More precisely, the sensor is line of pointwise positioned in Ω, then the output function still given by equation (21).

Proposition 5.3. *Let the sensor is located in* $b = (b_1, b_2)$*, then the process* (23) *is* $\partial\Omega AGFO$*-observer to* (21)-(24)*, if* $(nb_1)/(a_1)$ *and* $(mb_2)/(a_2 \notin N)$*, for every* $n, m = 1, 2, \cdots, J$.

5.2 Circular domain

Remark 5.1. The results in 5.1 can be extended to the case of circular domain with the internal zone and pointwise sensor as in [27-28].

6. Conclusion

The crossing problem from interior to exterior of asymptotic gradient full order observer have been explored and achieved in rigorous results. Thus, the characterizations of this approach are presented in connection with corresponding notions as stability, detectability, strategic sensor and considered domain. Then, the boundary asymptotic gradient reconstruction state via full-order observer in parabolic distributed parameter systems is examined and proved. Many interesting results concerning the choice of sensors structure are given and illustrated in specific situations to diffusion systems. Moreover, many problem still opened for instance, hyperbolic distributed parameter systems and it's development of the sense of these results as in [22] with another operators (see [31-32]).

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