On closedness of rectangular bands and left[right] normal bands

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Abstract. In this paper, first we have shown that the variety of rectangular bands is closed in the variety of all left[right] semiregular bands. Further, we have shown that the variety of left[right] normal bands are closed in some containing varieties of semigroups defined by the identities $axy = a^n yax[axy = xyay^n]$ and $axy = aya^n x[axy = xy^n ay]$, where ($\mathbf{n} \in \mathbb{N}$).

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1. Introduction

Let U be a subsemigroup of a semigroup S. Following, Isbell [9], we say that U dominates an element d of S if for every semigroup P and for all homomorphisms $\alpha, \delta : S \longrightarrow P$ and $u\alpha{=}u\delta$ for every u in U implies $d\alpha{=}d\delta$. The set of all elements of S dominated by U is called the *dominion* of U in S and we denote it by $Dom(U, S)$. It can be easily verified that $Dom(U, S)$ is a subsemigroup of S containing U. A subsemigroup U of semigroup S is called closed if $Dom(U, S) = U$. A semigroup is called absolutely closed if it is closed in every containing semigroup. Let $\mathcal{D}$ be a class of semigroups. A semigroup U is said to be $\mathcal{D}$-closed if $Dom(U, S) = U$, for all $S \in \mathcal{D}$ such that $U \subseteq S$. Let $\mathcal{A}$ and $\mathcal{D}$ be

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classes of semigroups such that $\mathcal{A}$ is a subclass of $\mathcal{D}$. We say that $\mathcal{A}$ is $\mathcal{D}$-closed if every member of $\mathcal{A}$ is $\mathcal{D}$-closed. A class $\mathcal{D}$ of semigroups is said to be closed if $Dom(U,S) = U$, for all $U, S \in \mathcal{D}$ with U as a subsemigroup of S. Let $\mathcal{B}$ and $\mathcal{C}$ be two categories of semigroups with $\mathcal{B}$ as a subcategory of $\mathcal{C}$. It can be easily verified that a semigroup U is $\mathcal{B}$-closed if it is $\mathcal{C}$-closed.

A (semigroup)amalgam $\mathcal{A} = [S_i\colon\ i \in I;\, U;\, \phi_i\colon\ i \in I]$ consists of a semigroup U (called the core of the amalgam), a family S_i: $i \in I$ of semigroups disjoint from each other and from U, and a family $\phi_i\colon\ U \to S_i (i \in I)$ of monomorphisms. We shall simplify the notation to $\mathcal{U} = [S_i;\, U;\, \phi_i]$ or to $\mathcal{U} = [S_i;\, U\ ]$ when the context allows. We shall say that the amalgam $\mathcal{A}$ is embedded in a semigroup T if there exist a monomorphism $\lambda\colon\ U \to T$ and, for each $i \in I$, a monomorphism $\lambda_i\colon\ S_i \to T$ such that:

(a) $\phi_i\lambda_i{=}\lambda$ for each $i \in I$;

(b) $S_i\lambda_i \cap S_j\lambda_j = U\lambda$, for all $i, j \in I$ such that $i\ \neq j$.

A semigroup amalgam $\mathcal{U} = [\ S,\, S';\, U;\, i,\, \alpha \mid U\ ]$ consisting of a semigroup S, a subsemigroup U of S, an isomorphic copy S' of S, where α: $S{\to}\ S'$ is an isomorphism and i is the inclusion mapping of U into S, is called a special semigroup amalgam. A class $\mathcal{C}$ of semigroups is said to have the special amalgamation property if every special semigroup amalgam in $\mathcal{C}$ is embeddable in $\mathcal{C}$.

Theorem 1.1 ([8], Theorem VII.2.3)**.** *et U be a subsemigroup of a semigroup S, S' be a semigroup disjoint from S and let α: $S{\to}\ S'$ be an isomorphism. Let $P\ = S{*_U}S'$, be the free product of the amalgam*

$$\mathcal{U} = [S, S'; U; i, \alpha \mid U],$$

where i is the inclusion mapping of U into S, and let μ, μ' be the natural monomorphisms from S, S' respectively into P . Then

$$(S\mu\ \cap\ S'\mu')\mu^{-1} = Dom(U, S).$$

From the above result, it follows that a special semigroup amalgam $[S,\, S';\, U;$ $i,\, \alpha \mid U]$ is embeddable in a semigroup if and only if $Dom(U,S) = U$. Therefore, the above amalgam with core U is embeddable in a semigroup if and only if U is closed in S.

The following theorem provided by Isbell [9], known as Isbell's zigzag theorem, is a most useful characterization of semigroup dominions and is of basic importance to our investigations.

Theorem 1.2 ([9], Theorem 2.3)**.** *Let U be a subsemigroup of a semigroup S and let $d \in S$. Then $d \in Dom(U,S)$ if and only if $d \in U$ or there exists a series of factorizations of d as follows:*

(1) $\quad d = a_0t_1 = y_1a_1t_1 = y_1a_2t_2 = y_2a_3t_2 = \cdots = y_ma_{2m-1}t_m = y_ma_{2m},$

where $m \geq 1$, $a_i \in U$ $(i = 0, 1, \ldots, 2m)$, $y_i, t_i \in S$ $(i = 1, 2, \ldots, m)$, *and*

$$\begin{aligned} a_0 &= y_1 a_1, & a_{2m-1} t_m &= a_{2m}, & \\ a_{2i-1} t_i &= a_{2i} t_{i+1}, & y_i a_{2i} &= y_{i+1} a_{2i+1} & (1 \leq i \leq m-1). \end{aligned}$$

Such a series of factorizations is called a zigzag in S *over* U *with value* d, *length* m *and spine* $a_0, a_1, \ldots, a_{2m}$.

The following result is from Khan [10] and is also necessary for our investigations.

Theorem 1.3 ([10], Result 3)**.** *Let* U *and* S *be semigroups with* U *as a subsemigroup of* S. *Take any* $d \in S \backslash U$ *such that* $d \in Dom(U, S)$. *Let* (1) *be a zigzag of minimal length* m *over* U *with value* d. *Then,* $t_j, y_j \in S \backslash U$, *for all* $j = 1, 2, \ldots, m$.

Definition 1.1. *A semigroup* S *is said to be a band if* S *satisfies the identity* $a^2 = a$, *for all* $a \in S$.

Definition 1.2. *A band* S *is said to be a rectangular band if* S *satisfies the identity* $a = axa$, *for all* $a, x \in S$.

Definition 1.3. *A band* S *is said to be a left[right] normal band if* S *satisfies the identity* $axy = ayx[axy = xay]$, *for all* $a, x, y \in S$.

Definition 1.4. *A band* S *is said to be a left[right] semiregular band if* S *satisfies the identity* $axy = axyayxy[axy = axayaxy]$, *for all* $a, x, y \in S$.

The reader is referred to Petrich [11] for a complete description of all varieties of bands. The semigroup theoretic notations and conventions of Clifford and Preston [6] and Howie [8] will be used throughout without explicit mention.

2. Closedness of rectangular bands

In general, varieties of bands containing the varieties of rectangular bands are not absolutely closed as Higgins [7, Chapter 4] gave an example to show that the variety of all rectangular bands is not absolutely closed. Therefore, for the varieties of semigroups, it is worthwhile to find largest subvarieties of the variety of all semigroups in which rectangular bands is closed. In this direction, we show that the variety of rectangular bands is closed in the variety of all left[right] semiregular bands.

Proposition 2.1. *Let* U *be any rectangular band and* S *be any semiregular band containing* U. *Assume that* $d \in Dom(U, S) \backslash U$. *If* (1) *is a zigzag in* S *over* U *with value* d *of minimal length* m, *then:*

(*a*) $y_i a_{2i-1} t_i = y_i a_{2i-1} a_{2i} y_{i+1} a_{2i+1} t_{i+1}$;

(*b*) $(y_i a_{2i-1} a_{2i})(y_{i+1} a_{2i+1} a_{2i+2}) = y_i a_{2i-1} a_{2i} a_{2i+2}$,

for all $i = 1, 2, ..., m-1$.

Proof.

$$\begin{aligned}
(a)\quad y_i a_{2i-1} t_i &= y_i a_{2i-1} a_{2i-1} t_i \text{ (since } U \text{ is a band)}\\
&= y_i a_{2i-1} a_{2i} t_{i+1} \text{ (by zigzag equations)}\\
&= (y_i a_{2i-1} a_{2i}) t_{i+1}\\
&= (y_i a_{2i-1} a_{2i} y_i a_{2i} a_{2i-1} a_{2i}) t_{i+1} \text{ (since } S \text{ is a left semi-regular band)}\\
&= y_i a_{2i-1} a_{2i} y_i (a_{2i} a_{2i-1} a_{2i}) t_{i+1}\\
&= y_i a_{2i-1} a_{2i} y_i (a_{2i}) t_{i+1} \text{ (since } U \text{ is a rectangular band)}\\
&= y_i a_{2i-1} a_{2i} y_{i+1} a_{2i+1} t_{i+1} \text{ (by zigzag equations)},
\end{aligned}$$

as required.

$$\begin{aligned}
(b)\quad (y_i a_{2i-1} a_{2i})(y_{i+1} a_{2i+1} a_{2i+2}) &= y_i a_{2i-1} a_{2i} (y_i a_{2i}) a_{2i+2} \text{ (by zigzag equations)}\\
&= y_i a_{2i-1} a_{2i} y_i (a_{2i}) a_{2i+2}\\
&= y_i a_{2i-1} a_{2i} y_i (a_{2i} a_{2i-1} a_{2i}) a_{2i+2}\\
&\quad \text{(since } U \text{ is a rectangular band)}\\
&= (y_i a_{2i-1} a_{2i} y_i a_{2i} a_{2i-1} a_{2i}) a_{2i+2}\\
&= (y_i a_{2i-1} a_{2i}) a_{2i+2}\\
&\quad \text{(since } S \text{ is a left semi-regular band)},
\end{aligned}$$

as required. □

Theorem 2.1. *Rectangular bands are closed in left semiregular bands.*

Proof. Let U be a rectangular band and S be any left semiregular band containing U as a subband. We have to show that $Dom(U, S) = U$. Take any $d \in Dom(U, S) \backslash U$. Then d has zigzag of type (1) in S over U with value d of minimal length m. Now

$$\begin{aligned}
d &= a_0 t_1 \text{ (by zigzag equations)}\\
&= y_1 a_1 t_1 \text{ (by zigzag equations)}\\
&= y_1 a_1 a_2 y_2 a_3 t_2 \text{ (by Proposition 2.1 (a))}\\
&= y_1 a_1 a_2 (y_2 a_3 t_2)\\
&= y_1 a_1 a_2 (y_2 a_3 a_4 y_3 a_5 t_3) \text{ (by Proposition 2.1 (a))}\\
&\vdots\\
&= y_1 a_1 a_2 y_2 a_3 a_4 \cdots y_{m-1} a_{2m-3} a_{2m-2} y_m (a_{2m-1}) t_m
\end{aligned}$$

$= y_1a_1a_2y_2a_3a_4 \cdots y_{m-1}a_{2m-3}a_{2m-2}y_m(a_{2m-1}a_{2m-1})t_m$ (since U is a band)
$= y_1a_1a_2y_2a_3a_4 \cdots y_{m-1}a_{2m-3}a_{2m-2}y_ma_{2m-1}a_{2m}$ (by zigzag equations)
$= y_1a_1a_2y_2a_3a_4 \cdots y_{m-2}a_{2m-5}a_{2m-4}((y_{m-1}a_{2m-3}a_{2m-2})(y_ma_{2m-1}a_{2m}))$
$= y_1a_1a_2y_2a_3a_4 \cdots y_{m-2}a_{2m-5}a_{2m-4}(y_{m-1}a_{2m-3}a_{2m-2}a_{2m})$
(by Proposition 2.1 (b))
$= y_1a_1a_2y_2a_3a_4 \cdots y_{m-3}a_{2m-7}a_{2m-6}((y_{m-2}a_{2m-5}a_{2m-4})(y_{m-1}a_{2m-3}a_{2m-2}))a_{2m}$
$= y_1a_1a_2y_2a_3a_4 \cdots y_{m-3}a_{2m-7}a_{2m-6}(y_{m-2}a_{2m-5}a_{2m-4}a_{2m-2})a_{2m}$
(by Proposition 2.1 (b))
$\vdots$
$= y_1a_1a_2a_4 \cdots a_{2m-4}a_{2m-2}a_{2m}$
$= a_0a_2a_4 \cdots a_{2m-4}a_{2m-2}a_{2m}$ (by zigzag equations)
$\in U$

$\Rightarrow Dom(U, S) = U$.

Hence, rectangular bands are closed in left semiregular bands. Thus, the proof of the theorem is complete. □

Dually, we can prove the following Theorem

Theorem 2.2. *Rectangular bands are closed in right semiregular bands.*

Corollary 2.1. *The variety of all rectangular bands is closed in the variety of all left[right] semiregular bands.*

Corollary 2.2. *The variety of all rectangular bands is closed in the following varieties of bands:*

(*i*) *The variety of all regular bands.*

(*ii*) *The variety of all left[right] seminormal bands.*

(*iii*) *The variety of all left[right] quasinormal bands.*

(*iv*) *The variety of all normal bands.*

3. Closedness of left[right] normal bands

In general, varieties of bands containing the variety of normal bands are not absolutely closed as Higgins [7, Chapter 4] had shown that variety of right [left] normal bands is not absolutely closed. Therefore, for the varieties of semigroups, it is worthwhile to find largest subvarieties of the variety of all semigroups in which the variety of right [left] normal bands is closed. As a first step in this direction, one attempts to find those varieties of semigroups that are closed in itself. Encouraged by the fact that Scheiblich [12] had shown that the variety of all normal bands was closed, Alam and Khan in [3, 4, 5] had shown that the variety of left [right] regular bands, left [right] quasi-normal bands and left [right] semi-normal bands were closed. In [2], Ahanger and Shah had proved a stronger fact that the variety of left [right] regular bands was closed in the variety of all bands and, recently, Abbas and Ashraf [1] had shown that a variety of left [right] normal bands was closed in some containing homotypical varieties (varieties admitting an identity containing same variables on both sides) of semigroups.

To this end, we first note that Petrich [11, Theorem II.5.1] has classified an identity on bands in atmost three variables. Therefore, on the class of bands, varieties of semigroups defined by the identities $axy = a^n yax$ $[axy = xyay^n]$ and $axy = aya^n x$ $[axy = xy^n ay]$ are equivalent to left[right] normal bands. In this section, we have shown that varieties of semigroups defined by the identities $axy = a^n yax[axy = xyay^n]$ and $axy = aya^n x[axy = xy^n ay]$, where ($\mathbf{n} \in \mathbb{N}$), are closed in itself and, as an application and consequence of these results, we conclude that the varieties of semigroups defined by the identities $axy = a^n yax[axy = xyay^n]$ and $axy = aya^n x[axy = xy^n ay]$ have special amalgamation property and left[right] normal bands are closed in $axy = a^n yax[axy = xyay^n]$ and $axy = aya^n x[axy = xy^n ay]$ and, thus a modest, but an important step towards the solution of the above problem. However the problem of finding out largest varieties of semigroups in which the varieties of semigroups defined by the identities $axy = a^n yax[axy = xyay^n]$ and $axy = aya^n x[axy = xy^n ay]$ are closed still remains open.

Lemma 3.1. *Let U be a subsemigroup of semigroup S such that S satisfies an identity $axy = a^n yax$ $[axy = aya^n x]$ and let $d \in Dom(U,S) \setminus U$ has a zigzag of type (1) in S over U with value d of shortest possible length m. Then*

$$d = y_k a_{2k-1} t_k \Big(\prod_{i=1}^{k} a^n_{2k-(2i-1)}\Big),$$

for each $k = 1, 2, \ldots, m$.

Proof. Let $\mathcal{V}_1 = [axy = a^n yax]$ and $\mathcal{V}_2 = [axy = aya^n x]$ be the varieties of semigroups.

First, we show that in both cases whether $S \in \mathcal{V}_1$ or $S \in \mathcal{V}_2$, S satisfies $xyz = xyzy^n$.

Case (i). When $S \in \mathcal{V}_1$, then for any $x, y, z \in S$, we have

$$\begin{aligned} xyz &= (x^n(zx)y) \text{ (as } S \in \mathcal{V}_1 \text{)} \\ &= (x^n)^n(yx^n z)x \text{ (as } S \in \mathcal{V}_1 \text{)} \\ &= ((x^n)^n(y^n zy)x^n x) \text{ (as } S \in \mathcal{V}_1 \text{)} \\ &= ((x^n x)(y^n z)y) \text{ (as } S \in \mathcal{V}_1 \text{)} \\ &= ((x^n x)^n y(x^n x)y^n)z \text{ (as } S \in \mathcal{V}_1 \text{)} \end{aligned}$$

$$
\begin{aligned}
&= x^n x y^n y z \text{ (as } S \in \mathcal{V}_1 \text{)} \\
&= (x^{n-1}(xx)y^n)yz \text{ (for } n = 1, \text{ we treat } x^{n-1}x \text{ as } x) \\
&= ((x^{n-1})^n y^n x^{n-1} x)xyz \text{ (as } S \in \mathcal{V}_1 \text{)} \\
&= x^{n-1} x y^n x y z \text{ (as } S \in \mathcal{V}_1 \text{)} \\
&= (x^n y^n x(yz)) \\
&= xyzy^n \text{ (as } S \in \mathcal{V}_1 \text{).} \qquad (2)
\end{aligned}
$$

Case (ii): When $S \in \mathcal{V}_2$, then for any $x, y, z \in S$, we have

$$
\begin{aligned}
xyz &= (x(zx^n)y) \text{ (as } S \in \mathcal{V}_2 \text{)} \\
&= x(y(x^n z)x^n) \text{ (as } S \in \mathcal{V}_2 \text{)} \\
&= (xyx^n y^n)x^n z \text{ (as } S \in \mathcal{V}_2 \text{)} \\
&= xy^n y x^n z \text{ (as } S \in \mathcal{V}_2 \text{)} \\
&= (x(yy^n)(x^n z)) \\
&= (x(x^n z)x^n y)y^n \text{ (as } S \in \mathcal{V}_2 \text{)} \\
&= (xyx^n)zy^n \text{ (as } S \in \mathcal{V}_2 \text{)} \\
&= x(x^n(x^n y)z)y^n \text{ (as } S \in \mathcal{V}_2 \text{)} \\
&= x(x^n z(x^n)^n x^n)yy^n \text{ (as } S \in \mathcal{V}_2 \text{)} \\
&= (xx^n x^n z)yy^n \text{ (as } S \in \mathcal{V}_2 \text{)} \\
&= (xzx^n y)y^n \text{ (as } S \in \mathcal{V}_2 \text{)} \\
&= xyzy^n \text{ (as } S \in \mathcal{V}_2 \text{).}
\end{aligned}
$$

Thus, the claim is proved.

Now, we shall prove the lemma by using induction on k. Let U be a subsemigroup of semigroup S such that S belongs to either $\mathcal{V}_1$ or $\mathcal{V}_2$ and let $d \in Dom(U, S)\backslash U$ has a zigzag of type (1) in S over U with value d of shortest possible length m.

Now, for $k = 1$, we have

$$
\begin{aligned}
d &= y_1 a_1 t_1 \text{ (by zigzag equations)} \\
&= y_1 a_1 t_1 a_1^n \text{ (by equation (2)).}
\end{aligned}
$$

Thus, the result holds for $k = 1$. Assume inductively that the result holds for $k = j < m$. Then, we shall show that it also holds for $k = j + 1$. Now,

$$
\begin{aligned}
d &= y_j a_{2j-1} t_j (\prod_{i=1}^{j} a^n_{2j-(2i-1)}) \text{ (by inductive hypothesis)} \\
&= y_{j+1} a_{2j+1} t_{j+1} (\prod_{i=1}^{j} a^n_{2j-(2i-1)}) \text{ (by zigzag equations)}
\end{aligned}
$$

$$= y_{j+1}a_{2j+1}t_{j+1}a^n_{2j+1}(\prod_{i=1}^{j} a^n_{2j-(2i-1)}) \text{ (by equation (2))}$$
$$= y_{j+1}a_{2j+1}t_{j+1}(\prod_{i=1}^{j+1} a^n_{2(j+1)-(2i-1)}),$$

as required and, by induction, the lemma is established. □

Theorem 3.1. *The variety $\mathcal{V} = [axy = a^n yax]$ of semigroups, i.e. the class of all semigroups satisfying the identity $axy = a^n yax$, is closed.*

Proof. Take any U, $S \in \mathcal{V}$ with U as a subsemigroup of S such that $d \in Dom(U, S)\backslash U$. Let d has zigzag of type (1) in S over U of shortest possible length m. Now,

$$d = y_m a_{2m-1} t_m(\prod_{i=1}^{m} a^n_{2m-(2i-1)}) \text{ (by Lemma 3.1)}$$
$$= (y_m(a_{2m-1}t_m)a_{2m-1})a^{n-1}_{2m-1}(\prod_{i=2}^{m} a^n_{2m-(2i-1)}) \text{ (for } n = 1,$$
$$\text{we treat } a_{2m-1}a^{n-1}_{2m-1} \text{ as } a_{2m-1})$$
$$= (y^n_m a_{2m-1}y_m a_{2m-1})t_m a^{n-1}_{2m-1}(\prod_{i=2}^{m} a^n_{2m-(2i-1)}) \text{ (as } S \in \mathcal{V})$$
$$= (y_m a_{2m-1})(a_{2m-1}t_m)a^{n-1}_{2m-1}(\prod_{i=2}^{m} a^n_{2m-(2i-1)}) \text{ (as } S \in \mathcal{V})$$
$$= y_{m-1}a_{2m-2}a_{2m}a^{n-1}_{2m-1}(\prod_{i=2}^{m} a^n_{2m-(2i-1)}) \text{ (by zigzag equations)}$$
$$= (y_{m-1}(a_{2m-2}a_{2m})a^{n-1}_{2m-1})a_{2m-3}a^{n-1}_{2m-3}(\prod_{i=3}^{m} a^n_{2m-(2i-1)})$$
$$= y^n_{m-1}(a^{n-1}_{2m-1}(y_{m-1}a_{2m-2}a_{2m})a_{2m-3})a^{n-1}_{2m-3}(\prod_{i=3}^{m} a^n_{2m-(2i-1)}) \text{ (as } S \in \mathcal{V})$$
$$= y^n_{m-1}((a^{n-1}_{2m-1})^n a_{2m-3}a^{n-1}_{2m-1}y_{m-1})a_{2m-2}a_{2m}a^{n-1}_{2m-3}(\prod_{i=3}^{m} a^n_{2m-(2i-1)}) \text{ (as } S \in \mathcal{V})$$
$$= (y^n_{m-1}a^{n-1}_{2m-1}y_{m-1}(a_{2m-3}a_{2m-2}a_{2m}))a^{n-1}_{2m-3}(\prod_{i=3}^{m} a^n_{2m-(2i-1)}) \text{ (as } S \in \mathcal{V})$$
$$= (y_{m-1}a_{2m-3})a_{2m-2}a_{2m}a^{n-1}_{2m-1}a^{n-1}_{2m-3}(\prod_{i=3}^{m} a^n_{2m-(2i-1)}) \text{ (as } S \in \mathcal{V})$$

$$= y_{m-2}a_{2m-4}a_{2m-2}a_{2m}a_{2m-1}^{n-1}a_{2m-3}^{n-1}(\prod_{i=3}^{m} a_{2m-(2i-1)}^{n}) \text{ (by zigzag equations)}$$

$\vdots$

$$= y_1a_2a_4\cdots a_{2m-2}a_{2m}a_{2m-1}^{n-1}a_{2m-3}^{n-1}\cdots a_3^{n-1}a_1^{n}$$
$$= (y_1(a_2a_4\cdots a_{2m-2}a_{2m})(a_{2m-1}^{n-1}a_{2m-3}^{n-1}\cdots a_3^{n-1}))a_1a_1^{n-1}$$
$$= y_1^n((a_{2m-1}^{n-1}a_{2m-3}^{n-1}\cdots a_3^{n-1})(y_1a_2a_4\cdots a_{2m-2}a_{2m})a_1)a_1^{n-1} \text{ (as } S\in\mathcal{V}\text{)}$$
$$= y_1^n((a_{2m-1}^{n-1}a_{2m-3}^{n-1}\cdots a_3^{n-1})^n a_1(a_{2m-1}^{n-1}a_{2m-3}^{n-1}\cdots a_3^{n-1})y_1)a_2a_4\cdots a_{2m-2}a_{2m}a_1^{n-1} \text{ (as } S\in\mathcal{V}\text{)}$$
$$= (y_1^n(a_{2m-1}^{n-1}a_{2m-3}^{n-1}\cdots a_3^{n-1})y_1(a_1a_2a_4\cdots a_{2m-2}a_{2m}))a_1^{n-1} \text{ (as } S\in\mathcal{V})$$
$$= (y_1a_1)a_2a_4\cdots a_{2m-2}a_{2m}a_{2m-1}^{n-1}a_{2m-3}^{n-1}\cdots a_3^{n-1}a_1^{n-1} \text{ (as } S\in\mathcal{V})$$
$$= a_0a_2a_4\cdots a_{2m-2}a_{2m}a_{2m-1}^{n-1}a_{2m-3}^{n-1}\cdots a_3^{n-1}a_1^{n-1} \text{ (by zigzag equations)}$$
$$\in U$$

$\Rightarrow Dom(U,S) = U$.

Thus, the proof of the theorem is complete. □

The following corollary is an immediate consequence of Theorem 3.1:

Corollary 3.1. *The variety of all left normal bands is closed in the variety* $\mathcal{V} = [axy = a^nyax]$ *of semigroups.*

Dually, we may prove the following results.

Theorem 3.2. *The variety* $\mathcal{V} = [axy = xyay^n]$ *of semigroups, i.e. the class of all semigroups satisfying the identity* $axy = xyay^n$*, is closed.*

Corollary 3.2. *The variety of all right normal bands is closed in the variety* $\mathcal{V} = [axy = xyay^n]$ *of semigroups.*

Theorem 3.3. *The variety* $\mathcal{V} = [axy = aya^nx]$ *of semigroups, i.e. the class of all semigroups satisfying the identity* $axy = aya^nx$*, is closed.*

Proof. Take any U, $S \in \mathcal{V}$ with U as a subsemigroup of S such that $d \in Dom(U,S)\backslash U$. Let d has zigzag of type (1) in S over U of shortest possible length m. Now,

$$d = y_ma_{2m-1}t_m(\prod_{i=1}^{m} a_{2m-(2i-1)}^{n}) \text{ (by Lemma 3.1)}$$
$$= (y_m(a_{2m-1}t_m)a_{2m-1})a_{2m-1}^{n-1}(\prod_{i=2}^{m} a_{2m-(2i-1)}^{n}) \text{ (for } n\text{=1, we treat } a_{2m-1}a_{2m-1}^{n-1} \text{ as } a_{2m-1})$$

$$= (y_m a_{2m-1} y_m^n a_{2m-1}) t_m a_{2m-1}^{n-1} (\prod_{i=2}^{m} a_{2m-(2i-1)}^n) \text{ (as } S \in \mathcal{V} \text{)}$$

$$= (y_m a_{2m-1})(a_{2m-1} t_m) a_{2m-1}^{n-1} (\prod_{i=2}^{m} a_{2m-(2i-1)}^n) \text{ (as } S \in \mathcal{V} \text{)}$$

$$= y_{m-1} a_{2m-2} a_{2m} a_{2m-1}^{n-1} (\prod_{i=2}^{m} a_{2m-(2i-1)}^n) \text{ (by zigzag equations)}$$

$$= (y_{m-1}(a_{2m-2} a_{2m}) a_{2m-1}^{n-1}) a_{2m-3} a_{2m-3}^{n-1} (\prod_{i=3}^{m} a_{2m-(2i-1)}^n)$$

$$= y_{m-1}(a_{2m-1}^{n-1}(y_{m-1}^n a_{2m-2} a_{2m}) a_{2m-3}) a_{2m-3}^{n-1} (\prod_{i=3}^{m} a_{2m-(2i-1)}^n) \text{ (as } S \in \mathcal{V})$$

$$= y_{m-1}(a_{2m-1}^{n-1} a_{2m-3} (a_{2m-1}^{n-1})^n y_{m-1}^n) a_{2m-2} a_{2m} a_{2m-3}^{n-1} (\prod_{i=3}^{m} a_{2m-(2i-1)}^n) \text{ (as } S \in \mathcal{V})$$

$$= (y_{m-1} a_{2m-1}^{n-1} y_{m-1}^n (a_{2m-3} a_{2m-2} a_{2m})) a_{2m-3}^{n-1} (\prod_{i=3}^{m} a_{2m-(2i-1)}^n) \text{ (as } S \in \mathcal{V})$$

$$= (y_{m-1} a_{2m-3}) a_{2m-2} a_{2m} a_{2m-1}^{n-1} a_{2m-3}^{n-1} (\prod_{i=3}^{m} a_{2m-(2i-1)}^n) \text{ (as } S \in \mathcal{V})$$

$$= y_{m-2} a_{2m-4} a_{2m-2} a_{2m} a_{2m-1}^{n-1} a_{2m-3}^{n-1} (\prod_{i=3}^{m} a_{2m-(2i-1)}^n) \text{ (by zigzag equations)}$$

$$\vdots$$

$$\begin{aligned}
&= y_1 a_2 a_4 \cdots a_{2m-2} a_{2m} a_{2m-1}^{n-1} a_{2m-3}^{n-1} \cdots a_3^{n-1} a_1^n \\
&= (y_1 (a_2 a_4 \cdots a_{2m-2} a_{2m})(a_{2m-1}^{n-1} a_{2m-3}^{n-1} \cdots a_3^{n-1})) a_1 a_1^{n-1} \\
&= (y_1 (a_{2m-1}^{n-1} a_{2m-3}^{n-1} \cdots a_3^{n-1} y_1^n a_2 a_4 \cdots a_{2m-2} a_{2m}) a_1) a_1^{n-1} \text{ (as } S \in \mathcal{V} \text{)} \\
&= (y_1 a_1 y_1^n (a_{2m-1}^{n-1} a_{2m-3}^{n-1} \cdots a_3^{n-1} y_1^n)) a_2 a_4 \cdots a_{2m-2} a_{2m} a_1^{n-1} \text{ (as } S \in \mathcal{V}) \\
&= (y_1 (a_{2m-1}^{n-1} a_{2m-3}^{n-1} \cdots a_3^{n-1}) y_1^n (a_1 a_2 a_4 \cdots a_{2m-2} a_{2m})) a_1^{n-1} \text{ (as } S \in \mathcal{V}) \\
&= (y_1 a_1) a_2 a_4 \cdots a_{2m-2} a_{2m} a_{2m-1}^{n-1} a_{2m-3}^{n-1} \cdots a_3^{n-1} a_1^{n-1} \text{ (as } S \in \mathcal{V}) \\
&= a_0 a_2 a_4 \cdots a_{2m-2} a_{2m} a_{2m-1}^{n-1} a_{2m-3}^{n-1} \cdots a_3^{n-1} a_1^{n-1} \text{ (by zigzag equations)} \\
&\in U
\end{aligned}$$

$\Rightarrow Dom(U, S) = U$.

Thus, the proof of the theorem is complete. □

The following corollary is an immediate consequence of Theorem 3.3:

Corollary 3.3. *The variety of all left normal bands is closed in the variety* $\mathcal{V} = [axy = aya^n x]$ *of semigroups.*

Dually, we may prove the following results.

Theorem 3.4. *The variety $\mathcal{V} = [axy = xy^n ay]$ of semigroups, i.e. the class of all semigroups satisfying the identity $axy = xy^n ay$, is closed.*

Corollary 3.4. *The variety of all right normal bands is closed in the variety $\mathcal{V} = [axy = xy^n ay]$ of semigroups.*

In the view of Section 2, we propose an important open problem.

Problem 1. Is the variety of normal bands closed in the variety of left[right] semiregular bands?

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