On a complex Matsumoto space

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Abstract. In this paper we studied complex Matsumoto space. The expressions for fundamental metric tensor, angular metric tensor, Chern-Finsler connection coefficients and the formula of holomorphic curvature are obtained.
Keywords: complex Finsler space, complex Matsumoto space, Chern-Finsler connection coefficients, holomorphic curvature.

1. Introduction

The Finsler geometry has its origins in the famous dissertation of Germann mathematician Finsler [1] in 1918. A Finsler manifold is a manifold M where each tangent space is equipped with a Minkowski norm. J. H. Taylor and J. L. Synge introduced a special parallelism and the concept of connection in the theory of Finsler space was introduced by L. Berwald. Later on, E. Cartan, H. Rund, M. Matsumoto, D. Bao, Z. Shen etc, made effective contributions in this field. Finsler geometry has many applications in theories of physics, biology and mechanics. Especially, quantum physics has stimulated the study of complex structures.

After, that as compared to the real case in complex Finsler geometry are not known so many classes of complex Finsler metrics. Besides the significant Kobayashi and Caratheodary metrics [1], which quickened the study of such Finsler geometry and we know rather trivial classes of complex Finsler metrics, one is to Hermitian metrics on the base manifold [5] and second is to the locally Minkowski complex metrics. Therefore, any new class of complex Finsler spaces

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with some presence in both theory and applications is welcomed (see more details in ([2, 6, 9, 11, 13, 16]).

The aim of the present paper is to study the complex Matsumoto metric. In the third section to determine the fundamental metric tensor and angular metric tensor of the complex Matsumoto space. In the last section is to characterize the Chern-Finsler connection coefficients, Cartan tensor and the formula for holomorphic curvature of complex Matsumoto metric.

2. Priliminaries

Let M be a complex manifold of dimension n and $(z^k)_{k=1,2,3,..,}$ complex coordinates in a local chart. Its complexified tangent bundle $T_{\mathbb{C}}M$ splits in to holomorphic tangent bundle $T'M$ and antiholomorphic tangent bundle $T''M$, i.e $T_{\mathbb{C}}M = T'M \oplus T''M$. The holomorphic tangent bundle $T'M$ is itself a complex manifold with local coordinates (z^k, η^k) in a chart, which changes by the following rules.

$$z'^k = z'^k(z), \quad \eta^k = \frac{\partial z'^k}{\partial z^j}\eta^j \tag{1}$$

Further, $T_{\mathbb{C}}(T'M)$ decomposes into holomorphic and antiholomorphic tangent bundles $T'(T'M)$ and $T''(T'M)$ respectively.

A natural local frame $\{\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \eta^k}\}$ for $T'(T'M)$ change according to the rules from Jacobi matrix of (1). Since the changing rule of $\frac{\partial}{\partial z^k}$ contains the second order partial derivatives, the concept of complex non-linear connection (c.n.c.) was introduced.

Let $V(T'M) = ker\pi_* \subset T'(T'M)$ be the vertical bundle, spanned locally by $\{\frac{\partial}{\partial \eta^k}\}$. The complex non-linear connection (c.n.c.), determines a supplementary complex sub-bundle to $V(T'M)$ in $T'(T'M)$, i.e. $T'(T'M) = H(T'M) \oplus V(T'M)$. It determines an adapted frame $\frac{\delta}{\delta z^k}=\frac{\partial}{\partial z^k}$-$N_k^j \frac{\partial}{\partial \eta^j}$, where $N_k^j(z,\eta)$ are the coefficients of the complex nonlinear connections:[1, 11]

A complex Finsler metric F on complex manifold M is a continuous function $F : T'M \to \mathbb{R}$ satisfying following conditions [11, 14]

(i) $L := F^2$ is smooth on $\tilde{T'M} := T'M\backslash\{0\}$;

(ii) $F(z,\eta) \geq 0$, the equality holds if and only if $\eta = 0$;

(iii) $F(z,\ \lambda\eta)$=$|\lambda| F(z,\eta)$; for $\lambda \in \mathbb{C}$;

(iv) the Hermitian matrix $(g_{i\bar{j}}(z,\eta))$, with $g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$, called the fundamental metric tensor, is positive definite on $T'M \setminus \{0\}$.

Let us write $L = F^2$. Then, the pair (M, F) is called a complex Finsler space. The strongly pseudoconvexity of the Finsler metric F on complex indicatrix, $I_{F,z} = \{\eta \in T'_zM \mid F(z,\eta) < 1\}$ is implied by assumption (iv). A Hermitian connection of $(1,0)$ type named as the Chern-Finsler Connection [1] has a special meaning in a complex Finsler space. Notationally, it is $D\Gamma N = (L^i_{jk}, 0, C^i_{jk}, 0)$, where

$$\overset{CF}{N^i_j} = g^{\bar{m}i}\frac{\partial g_{l\bar{m}}}{\partial z^j}\eta^l, \quad L^i_{jk} = g^{\bar{m}i}\frac{\delta g_{j\bar{m}}}{\delta z^k} = \frac{\partial N^i_k}{\partial \eta^j}, \quad C^i_{jk} = g^{\bar{m}i}\frac{\partial g_{j\bar{m}}}{\partial \eta^k}, \tag{2}$$

where $(g^{\bar{m}i})$ is the inverse of the metric tensor $(g_{\bar{m}i})$. The horizontal lift of the Liouville complex field (or the vertical radial vector field) $\eta^k\dot{\partial}_k$ is $\chi = \eta^k\dot{\partial}_k$, where $\dot{\partial}_k = \frac{\partial}{\partial \eta^k}$ and $\delta_k = \frac{\delta}{\delta z^k}$. The holomorphic curvature [11] of the complex Finsler space (M, F) in the direction η is

$$K_F(z,\eta) = \frac{2}{L^2(z,\eta)}G(R(\chi,\tilde{\chi})\chi,\tilde{\chi}), \tag{3}$$

where G is the $N-$lift of the complex Finsler metric tensor $g_{i\bar{j}}$ defined by $G = g_{i\bar{j}}dz^i\otimes d\bar{z}^j + g_{i\bar{j}}\partial\eta^i\otimes\partial\bar{\eta}^j$ and R is the curvature of Chern Finsler connection. Locally, it has the following expression [7],

$$K_F(z,\eta) = \frac{2}{L^2}R_{\bar{j}k}\bar{\eta}^j\eta^k \tag{4}$$

where

$$R_{\bar{j}k} = -g_{l\bar{j}}\delta_{\bar{h}}(\overset{CF}{N^l_k})\bar{\eta}^h. \tag{5}$$

3. Notion of complex Matsumoto space

In this section, we find the fundamental metric tensor $g_{i\bar{j}}$ and its inverse and determinant of the complex Matsumoto metric.

Let M be an n-dimensional complex manifold and $(z,\eta) \in T'M, \eta = \eta^k\frac{\partial}{\partial z^k}$. Let a purely Hermitian positive metric a, and a differential $(1,0)$ from b, be defined on M as $a = a_{i\bar{j}}(z)dz^i\otimes d\bar{z}^j$ and $b = b_i(z)dz^i$. Aldea and munteanu [5] defined the complex Finsler metric on $T'M$ by

$$F(z,\eta) = F(\alpha(z,\eta),\, |\beta(z,\eta)|), \tag{6}$$

where

(a) $\alpha(z,\eta) = \sqrt{a_{i\bar{j}}(z)\eta^i\bar{\eta}^j}$;

(b) $|\beta(z,\eta)| = \sqrt{\beta(z,\eta)\;\overline{\beta(z,\eta)}}$ *with* $\beta(z,\eta) = b_i(z)\eta^i$.

We introduce a metric function F on the complex manifold M by

$$F = \frac{\alpha^2}{\alpha - |\beta|} \quad (|\beta| \neq 0). \tag{7}$$

We call the metric F defined by (7) as a complex Matsumoto metric and the manifold together with this complex Matsumoto metric as a complex Matsumoto space. The above complex Matsumoto metric is positive and smooth on $T'M\backslash\{0\}$. This metric is purely Hermitian if and only if β vanishes identically. The function $L = F^2$ depends on z and η because of $\alpha = \alpha(z,\eta)$ and $|\beta| = |\beta(z,\eta)|$. Also, α and β are homogeneous with respect to η, i.e. $\alpha(z,\ \lambda\eta)=|\lambda|\alpha(z,\eta)$ and $\beta(z,\ \lambda\eta) = \lambda\beta(z,\eta)$ for $\forall\lambda \in \mathbb{C}$. Therefore, $L(z,\lambda\eta) = \lambda\bar{\lambda}L(z,\eta)$ for any $\lambda \in \mathbb{C}$. From the homogeneity property, we have

$$\frac{\partial\alpha}{\partial\eta^i}\eta^i = \frac{1}{2}\alpha, \quad \frac{\partial|\beta|}{\partial\eta^i}\eta^i = \frac{1}{2}|\beta|. \tag{8}$$

Differentiating $\alpha(z,\eta)$ and $|\beta(z,\eta)|$ partially with respect to η^i and $\overline{\eta}^j$, we obtain the following

$$\frac{\partial\alpha}{\partial\eta^i} = \frac{l_i}{2\alpha}, \quad \frac{\partial|\beta|}{\partial\eta^i} = \frac{\bar{\beta}b_i}{2|\beta|}, \quad \frac{\partial\alpha}{\partial\bar{\eta}^j} = \frac{l_{\bar{j}}}{2\alpha}, \quad \frac{\partial|\beta|}{\partial\bar{\eta}^j} = \frac{\beta b_{\bar{j}}}{2|\beta|}, \tag{9}$$

$$\frac{\partial^2\alpha}{\partial\eta^i\partial\bar{\eta}^j} = \frac{a_{i\bar{j}}}{2\alpha} - \frac{l_i l_{\bar{j}}}{4\alpha^3}, \quad \frac{\partial^2|\beta|}{\partial\eta^i\partial\bar{\eta}^j} = \frac{b_i b_{\bar{j}}}{4|\beta|}, \tag{10}$$

where

$$l_i = a_{i\bar{j}}\overline{\eta}^j, \quad l_{\bar{j}} = a_{k\bar{j}}\eta^k,$$

now

$$\eta_i = \frac{\partial L}{\partial\eta^i} = L_\alpha\frac{\partial\alpha}{\partial\eta^i} + L_{|\beta|}\frac{\partial|\beta|}{\partial\eta^i}.$$

Using (9) and (10) we get

$$\eta_i = \frac{(\alpha - 2|\beta|)}{(\alpha - |\beta|)^2}Fl_i + \frac{1}{(\alpha - |\beta|)}\frac{F^2\bar{\beta}b_i}{|\beta|}, \tag{11}$$

$$\bar{\eta}_{\bar{j}} = \frac{(\alpha - |\beta|)}{(\alpha - |\beta|)^2}Fl_{\bar{j}} + \frac{1}{(\alpha - |\beta|)}\frac{F^2\beta b_{\bar{j}}}{|\beta|}. \tag{12}$$

The fundamental metric tensor $g_{i\bar{j}}$ of the complex Matsumoto space (M, F) is given by

$$\begin{aligned} g_{i\bar{j}} &= \frac{\partial^2 L}{\partial\eta^i\partial\bar{\eta}^j} \\ &= L_{\alpha\alpha}\frac{\partial\alpha}{\partial\eta^i}\frac{\partial\alpha}{\partial\bar{\eta}^j} + L_{\alpha|\beta|}\left(\frac{\partial\alpha}{\partial\eta^i}\frac{\partial|\beta|}{\partial\bar{\eta}^j} + \frac{\partial|\beta|}{\partial\eta^i}\frac{\partial\alpha}{\partial\bar{\eta}^j}\right) + L_{|\beta||\beta|}\left(\frac{\partial|\beta|}{\partial\eta^i}\frac{\partial|\beta|}{\partial\bar{\eta}^j}\right) \\ &+ L_\alpha\frac{\partial^2\alpha}{\partial\eta^i\partial\bar{\eta}^j} + L_{|\beta|}\frac{\partial^2|\beta|}{\partial\eta^i\partial\bar{\eta}^j}, \end{aligned}$$

where

$$L_\alpha = \frac{\partial L}{\partial\alpha}, \quad L_{|\beta|} = \frac{\partial L}{\partial|\beta|}, \quad L_{\alpha\alpha} = \frac{\partial^2 L}{\partial\alpha^2}, \quad L_{|\beta||\beta|} = \frac{\partial^2 L}{\partial|\beta|^2}, \quad L_{\alpha|\beta|} = \frac{\partial^2 L}{\partial\alpha\partial|\beta|}.$$

Using (9), (10) and (11), (12) we have

$$\begin{aligned} g_{i\bar{j}} &= \frac{F(\alpha-2|\beta|)}{(\alpha-|\beta|)^2}\, a_{i\bar{j}} + \frac{-F|\beta|}{2\alpha^2}\left(\frac{\alpha-4|\beta|}{(\alpha-|\beta|)^3}\right) l_i l_{\bar{j}} + \frac{F^2}{2|\beta|}\left(\frac{\alpha+2|\beta|}{(\alpha-|\beta|)^2}\right) b_i b_{\bar{j}} \\ &+ \frac{F(\alpha-4|\beta|)}{2|\beta|(\alpha-|\beta|)^3}\left(l_i \beta b_{\bar{j}} + \bar{\beta} b_i l_{\bar{j}}\right), \end{aligned}$$

which may be written as

$$g_{i\bar{j}} = p_0 a_{i\bar{j}} + p_{-2} l_i l_{\bar{j}} + q_0' b_i b_{\bar{j}} + q_{-2}\left(l_i \beta b_{\bar{j}} + \bar{\beta} b_i l_{\bar{j}}\right), \tag{13}$$

where

$$\begin{aligned} p_0 &= \frac{F(\alpha-2|\beta|)}{(\alpha-|\beta|)^2}, \quad p_{-2} = \frac{-F|\beta|}{2\alpha^2}\left(\frac{\alpha-4|\beta|}{(\alpha-|\beta|)^3}\right), \\ q_{-2} &= \frac{F(\alpha-4|\beta|)}{2|\beta|(\alpha-|\beta|)^3}, \quad q_0' = \frac{F^2}{2|\beta|}\left(\frac{\alpha+2|\beta|}{(\alpha-|\beta|)^2}\right). \end{aligned}$$

A simple calculation shows that

$$q_{-2}\left(l_i \beta b_{\bar{j}} + \bar{\beta} b_i l_{\bar{j}}\right) = \frac{q_{-2}}{q_0 p_0}\eta_i \eta_{\bar{j}} - \frac{q_0 q_{-2}}{p_0}|\beta|^2 b_i b_{\bar{j}} - \frac{p_0 q_{-2}}{q_0} l_i l_{\bar{j}}, \tag{14}$$

substituting (14) in (13) we obtain a new expression for $g_{i\bar{j}}$ as

$$g_{i\bar{j}} = p_0 a_{i\bar{j}} + p_{-2}' l_i l_{\bar{j}} + q_0'' b_i b_{\bar{j}} + q_{-2}'' \eta_i \eta_{\bar{j}}, \tag{15}$$

where

$$\begin{aligned} p_{-2}' &= p_{-2} - \frac{p_0 q_{-2}}{q_0} = \frac{-F}{2\left(\alpha-|\beta|\right)^3}\left(\frac{|\beta|}{\alpha^2}(\alpha-4|\beta|) + \alpha^2 - 4|\beta|^2\right), \\ q_0'' &= q_0' - \frac{q_0 q_{-2}}{p_0}|\beta|^2 = \frac{F}{2(\alpha-|\beta|)}\left(\frac{\alpha-4|\beta|}{|\beta|(\alpha-|\beta|)^2} - \frac{F^2(\alpha+2|\beta|)}{(\alpha-2|\beta|)}\right), \\ q_{-2}'' &= \frac{q_{-2}}{q_0 p_0} = \frac{(\alpha-|\beta|)}{2F(\alpha-2|\beta|)}(\alpha+2|\beta|). \end{aligned}$$

Theorem 3.1. *Let* $F = \frac{\alpha^2}{\alpha-|\beta|}$, $(|\beta| \neq 0)$ *be the complex Matsumoto metric, then the fundamental metric tensor* $g_{i\bar{j}}$ *is given by equation (15).*

Proof. Here the expression (15) is the fundamental metric tensor of the complex Matsumoto space. Now our next aim is to find the formulas for the inverse as well as for the determinant of the fundamental metric tensor $g_{i\bar{j}}$. For this purpose we use the following proposition given below [5]. □

Proposition 3.1. *Suppose:*
- $(Q_{i\bar{j}})$ *is a non-singular* $n \times n$ *complex matrix with inverse* $(Q^{\bar{j}i})$.
- C_i *and* $C_{\bar{i}} = \bar{C}_i, i = 1, 2, 3, \ldots, n$ *are complex numbers.*
- $C^i := Q^{\bar{j}i} C_{\bar{j}}$ *and its conjugates;* $C^2 := C^i C_i = \bar{C}^i C_{\bar{i}}$; $H_{i\bar{j}} := Q_{i\bar{j}} \pm C_i C_{\bar{j}}$.

Then:

1. $det(H_{i\bar{j}}) = (1 \pm C^2) det(Q_{i\bar{j}})$,
2. *whenever* $(1 \pm C^2) \neq 0$, *the matrix* $(H_{i\bar{j}})$ *is invertible and in this case its inverse is* $H^{\bar{j}i} = Q^{\bar{j}i} \mp \frac{1}{1 \mp C^2} C^i \bar{C}^j$.

From (15) we may re written as,

$$g_{i\bar{j}} = p_0 \left(a_{i\bar{j}} + \frac{p'_{-2}}{p_0} l_i l_{\bar{j}} + \frac{q''_0}{p_0} b_i b_{\bar{j}} + \frac{q''_{-2}}{p_0} \eta_i \eta_{\bar{j}} \right).$$

Assuming $Q_{i\bar{j}} = a_{i\bar{j}}$ and $C_i = \sqrt{\frac{p'_{-2}}{p_0}} l_i$ and applying Proposition (3.1), we find

$$Q^{\bar{j}i} = a^{\bar{j}i} \text{ and } C^i = Q^{\bar{j}i} C_{\bar{j}}, \quad C^2 = \alpha^2 \frac{p'_{-2}}{p_0},$$

where $(a^{\bar{j}i})$ is the Hermitian inverse of $(a_{i\bar{j}})$ since $1 \pm C^2 \neq 0$, and $1 - C^2 = \frac{p_0 - \alpha^2 p'_{-2}}{p_0}$.

The matrix $H_{i\bar{j}} = a_{i\bar{j}} + \frac{p'_{-2}}{p_0} l_i l_{\bar{j}}$ is invertible with the inverse as,

$$H^{\bar{j}i} = a^{\bar{j}i} + R \eta^i \bar{\eta}^j \text{ and } det(H_{i\bar{j}}) = \left(1 + \frac{\alpha^2 p'_2}{p_0}\right) det(a_{i\bar{j}}) = \frac{p'_{-2}}{R p_0} det(a_{i\bar{j}})$$

where $R = \frac{p'_2}{p_0 - \alpha^2 p'_2}$.

Taking $Q_{i\bar{j}} = a_{i\bar{j}} + \frac{p'_{-2}}{p_0} l_i l_{\bar{j}}$ and $C_i = \sqrt{\frac{q''_0}{p_0}} b_i$ and again applying Proposition (3.1), we obtain $Q^{\bar{j}i} = a^{\bar{j}i} + R \eta^i \bar{\eta}^j$ and $C^i = \sqrt{\frac{q''_0}{p_0}} (b_i + R \bar{\beta} \eta^i)$, since $1 \pm C^2 \neq 0$, the inverse of $H_{i\bar{j}} = a_{i\bar{j}} + \frac{p'_{-2}}{p_0} l_i l_{\bar{j}} + \frac{q''_0}{p_0} b_i b_{\bar{j}}$ exists and is given by

$$H^{\bar{j}i} = a^{\bar{j}i} + R \eta^i \bar{\eta}^j + \frac{q''_0}{p_0} (b^i + R \bar{\beta} \eta^i)(b^{\bar{j}} + R \beta \bar{\eta}^j),$$

and also

$$det(a_{i\bar{j}} + \frac{p'_{-2}}{p_0} l_i l_{\bar{j}} + \frac{q''_0}{p_0} b_i b_{\bar{j}}) = \frac{\gamma p'_{-2}}{R p_0} det(a_{i\bar{j}}),$$

where $\gamma = 1 + \frac{q_0^2}{p_0} (\|b\|^2 + R |\beta|^2)$.

We set $Q_{i\bar{j}} = a_{i\bar{j}} + \frac{p'_{-2}}{p_0} l_i l_{\bar{j}} + \frac{q''_0}{p_0} b_i b_{\bar{j}}$ and $C_i = \sqrt{\frac{q''_{-2}}{p_0}} \eta_i$. Then, we have

$$Q^{\bar{j}i} = a^{\bar{j}i} + R \eta^i \bar{\eta}^j + \frac{q''_0}{\gamma p_0} (b^i + R \bar{\beta} \eta^i)(b^{\bar{j}} + R \beta \bar{\eta}^j),$$

$$C^2 = \frac{q''_{-2}}{p_0} \left[a^{\bar{j}i} \eta_i \eta_{\bar{j}} + R F^4 + \frac{q''_0}{\gamma p_0} \eta_i \eta_{\bar{j}} (b^i + R \bar{\beta} \eta^i)(b^{\bar{j}} + R \beta \bar{\eta}^j) \right].$$

Since $1 \pm C^2 \neq 0,\ H_{i\bar{j}} = a_{i\bar{j}} + \frac{p'_{-2}}{p_0} l_i l_{\bar{j}} + \frac{q''_0}{p_0} b_i b_{\bar{j}} + \frac{q''_{-2}}{p_0} \eta_i \eta_{\bar{j}},$ is invertible with the inverse.

$$(16) \quad H^{\bar{j}i} = \frac{a^{\bar{j}i} + R\eta^i \bar{\eta}^j + \frac{q''_0}{\gamma p_0}(b^i + R\bar{\beta}\eta^i)(b^{\bar{j}} + R\beta\bar{\eta}^j)}{1 + \frac{q''_{-2}}{p_0}\eta_m \bar{\eta}_n \left[a^{\bar{n}m} + R\eta^m \bar{\eta}^n + \frac{q''_0}{\gamma p_0}(b^m + R\bar{\beta}\eta^m)(b^{\bar{n}} + R\beta\bar{\eta}^n)\right]}.$$

In view of (11), we have

$$\begin{aligned} a^{\bar{n}m}\eta_i\bar{\eta}_j &= p_0^2\alpha^2 + 2q_0p_0|\beta|^2 + q_0^2\|b\|^2|\beta|^2, \\ b^m\eta_m &= (p_0\bar{\beta} + q_0\|b\|^2\bar{\beta}), \\ b^{\bar{n}}\bar{\eta}_n &= (p_0\beta + q_0\|b\|^2\beta). \end{aligned}$$

Therefore,

$$(17) \quad \begin{aligned} (b^m\eta_m + R\bar{\beta}F^2)(b^{\bar{n}}\bar{\eta}_n + R\beta F^2) &= p_0^2|\beta|^2 + 2p_0q_0|\beta|^2\|b\|^4 \\ &\quad + q_0^2|\beta|^2\|b\|^4 + 2p_0RF^2|\beta|^2 \\ &\quad + 2q_0RF^2|\beta|^2\|b\|^2 + R^2F^4|\beta|^2, \end{aligned}$$

substitute (17), in (16), we get

$$(18) \qquad H^{\bar{j}i} = \frac{1}{M}\left[a^{\bar{j}i} + R\eta^i\bar{\eta}^j + \frac{q''_0}{\gamma p_0}(b^i + R\bar{\beta}\eta^i)(b^{\bar{j}} + R\beta\bar{\eta}^j)\right],$$

where

$$(19) \qquad \begin{aligned} M = 1 + \frac{q''_{-2}}{p_0}\Bigg\{ & p_0^2\alpha^2 + 2q_0p_0|\beta|^2 + q_0^2\|b\|^2|\beta|^2 \\ & + RF^4 + \frac{q''_0}{\gamma p_0}\Bigg(p_0^2|\beta|^2 + 2p_0q_0|\beta|^2\|b\|^4 \\ & + q_0^2|\beta|^2\|b\|^4 + 2p_0RF^2|\beta|^2 + 2q_0RF^2|\beta|^2\|b\|^2 + R^2F^4|\beta|^2\Bigg)\Bigg\}, \end{aligned}$$

and

$$det(H_{i\bar{j}}) = (1 + c^2)det(Q_{i\bar{j}}) = M det(a_{i\bar{j}} + \frac{p'_{-2}}{p_0} l_i l_{\bar{j}} + \frac{q''_0}{p_0} b_i b_{\bar{j}}).$$

On simplifying, above we have

$$(20) \qquad det(H_{i\bar{j}}) = M\gamma\left(1 + \frac{\alpha^2 p'_{-2}}{p_0}\right)det(a_{i\bar{j}}) = \frac{M\gamma}{R}\frac{p'_{-2}}{p_0}det(a_{i\bar{j}}).$$

Since $g_{i\bar{j}} = p_0 H_{i\bar{j}}$, the inverse of the fundamental metric tensor is given by

$$(21) \qquad g^{\bar{j}i} = \frac{1}{p_0}H^{\bar{j}i},$$

where $H^{\bar{j}i}$ is given by (18). Also, the determinant of the fundamental metric tensor is given by

$$det(g_{i\bar{j}}) = p_0^n det(H_{i\bar{j}}) = p_0^n \frac{M\gamma}{R}\frac{p'_{-2}}{p_0} det(a_{i\bar{j}}). \tag{22}$$

The angular metric tensors of the complex Matsumoto space (M, F) is given by

$$\begin{aligned} K_{i\bar{j}} = \frac{\partial^2 F}{\partial\eta^i\partial\bar{\eta}^j} &= F_{\alpha\alpha}\frac{\partial\alpha}{\partial\eta^i}\frac{\partial\alpha}{\partial\bar{\eta}^j} + F_{\alpha|\beta|}\left(\frac{\partial\alpha}{\partial\eta^i}\frac{\partial|\beta|}{\partial\bar{\eta}^j} + \frac{\partial|\beta|}{\partial\eta^i}\frac{\partial\alpha}{\partial\bar{\eta}^j}\right) \\ &+ F_{|\beta||\beta|}\left(\frac{\partial|\beta|}{\partial\eta^i}\frac{\partial|\beta|}{\partial\bar{\eta}^j}\right) + F_\alpha\frac{\partial^2\alpha}{\partial\eta^i\partial\bar{\eta}^j} + F_{|\beta|}\frac{\partial^2|\beta|}{\partial\eta^i\partial\bar{\eta}^j}, \end{aligned} \tag{23}$$

where

$$F_\alpha = \frac{\partial F}{\partial\alpha},\quad F_{|\beta|} = \frac{\partial F}{\partial|\beta|},\quad F_{\alpha\alpha} = \frac{\partial^2 F}{\partial\alpha^2},\quad F_{|\beta||\beta|} = \frac{\partial^2 F}{\partial|\beta^2|}\ and\ F_{\alpha|\beta|} = \frac{\partial^2 F}{\partial\alpha\partial|\beta|}.$$

On differentiating (6) with respect to α and β, we obtain

$$\begin{aligned} F_\alpha &= \frac{\alpha^2 - 2\alpha|\beta|}{(\alpha-|\beta|)^2},\quad F_{|\beta|} = \frac{\alpha^2}{(\alpha-|\beta|)^2}, \\ F_{\alpha|\beta|} &= \frac{-2\alpha|\beta|}{(\alpha-|\beta|)^3},\quad F_{\alpha\alpha} = \frac{2|\beta|^2}{(\alpha-|\beta|)^3}, \\ F_{|\beta||\beta|} &= \frac{2\alpha^2}{(\alpha-|\beta|)^3}. \end{aligned} \tag{24}$$

On substituting (24) and (9) in (23), we obtain

$$K_{i\bar{j}} = \xi_0 a_{i\bar{j}} + \xi_{-2} l_i l_{\bar{j}} + \chi_{-2}(\beta l_i b_{\bar{j}} + \bar{\beta} b_i l_{\bar{j}}) + \chi'_0 b_i b_{\bar{j}}, \tag{25}$$

where,

$$\xi_0 = \frac{\alpha - 2|\beta|}{2\,(\alpha-|\beta|)^2},\ \xi_{-2} = \frac{-(\alpha - 3|\beta|)}{4\alpha\,(\alpha-|\beta|)^3},\ \chi'_0 = \frac{F}{4|\beta|}\frac{(\alpha+3|\beta|)}{(\alpha-|\beta|)^2},\ \chi_{-2} = \frac{-1}{2(\alpha-|\beta|)^3}.$$

Again differentiatting (7) partially with respect to η^i and $\bar{\eta}^j$, respectively we have,

$$\begin{aligned} \frac{\partial F}{\partial\eta^i} &= \left(\frac{\alpha^2 - 2\alpha|\beta|}{(\alpha-|\beta|)^2}\right)\frac{l_i}{2\alpha} + \left(\frac{\alpha^2}{(\alpha-|\beta|)^2}\right)\frac{\bar{\beta}b_i}{2|\beta|}, \\ \frac{\partial F}{\partial\bar{\eta}^j} &= \left(\frac{\alpha^2 - 2\alpha|\beta|}{(\alpha-|\beta|)^2}\right)\frac{l_{\bar{j}}}{2\alpha} + \left(\frac{\alpha^2}{(\alpha-|\beta|)^2}\right)\frac{\beta b_{\bar{j}}}{2|\beta|}. \end{aligned} \tag{26}$$

From (24) and (26), we obtain

$$\begin{aligned} &\frac{\frac{\partial F}{\partial\eta^i}\frac{\partial F}{\partial\bar{\eta}^j}}{\frac{F_\alpha}{2\alpha}\frac{F_{|\beta|}}{2|\beta|}} - \frac{F_{|\beta|}/2|\beta|}{F_\alpha/2\alpha}|\beta|^2 b_i b_{\bar{j}} - \frac{F_\alpha/2\alpha}{F_{|\beta|}/2|\beta|} l_i l_{\bar{j}} = (l_i\beta b_{\bar{j}} + l_{\bar{j}}\bar{\beta} b_i), \\ &\left\{\frac{4|\beta|(\alpha-|\beta|)^4\frac{\partial F}{\partial\eta^i}\frac{\partial F}{\partial\bar{\eta}^j}}{\alpha^3(\alpha-2|\beta|)} - \frac{|\beta|(\alpha-2|\beta|)}{\alpha^2} l_i l_{\bar{j}} - \frac{\alpha^2|\beta|}{(\alpha-2|\beta|)} b_i b_{\bar{j}}\right\} \\ &\qquad = (l_i\beta b_{\bar{j}} + l_{\bar{j}}\bar{\beta} b_i). \end{aligned} \tag{27}$$

Substituting the value of $(l_i\beta b_{\bar{j}} + l_{\bar{j}}\bar{\beta} b_i)$ from (27) in (25), then we have

$$(28)\ K_{i\bar{j}} = \frac{\alpha - 2|\beta|}{2\left(\alpha - |\beta|\right)^2} a_{i\bar{j}} + \frac{-(\alpha - 3|\beta|)}{4\alpha\left(\alpha - |\beta|\right)^3} l_i l_{\bar{j}} + \frac{F}{4|\beta|}\frac{(\alpha + 3|\beta|)}{\left(\alpha - |\beta|\right)^2} b_i b_{\bar{j}} + \frac{-1}{2\left(\alpha - |\beta|\right)^3} \left\{ \frac{4|\beta|(\alpha - |\beta|)^4 \frac{\partial F}{\partial \eta^i}\frac{\partial F}{\partial \bar{\eta}^j}}{\alpha^3(\alpha - 2|\beta|)} - \frac{|\beta|(\alpha - 2|\beta|)}{\alpha^2} l_i l_{\bar{j}} - \frac{\alpha^2|\beta|}{(\alpha - 2|\beta|)} b_i b_{\bar{j}} \right\}.$$

Since $\eta_i = \frac{\partial L}{\partial \eta^i} = 2F\frac{\partial F}{\partial \eta^i}$ and $\bar{\eta}_j = \frac{\partial L}{\partial \bar{\eta}^j} = 2F\frac{\partial F}{\partial \bar{\eta}^j}$, we have $\eta_i\bar{\eta}_j = 4L\frac{\partial F}{\partial \eta^i}\frac{\partial F}{\partial \bar{\eta}^j}$.

Substituting these values in (28) we get

$$(29) \qquad K_{i\bar{j}} = \xi_0 a_{i\bar{j}} + \xi'_{-2} l_i l_{\bar{j}} + \frac{\chi''_{-2}}{2L}\eta_i\eta_{\bar{j}} + \chi''_0 b_i b_{\bar{j}},$$

where

$$\xi'_{-2} = \frac{1}{2(\alpha - |\beta|)^3}\left(\frac{3|\beta| - \alpha}{2\alpha} + \frac{|\beta|(\alpha - 2|\beta|)}{F(\alpha - |\beta|)}\right), \quad \chi''_{-2} = \frac{-2|\beta|(\alpha - |\beta|)}{F(\alpha - 2|\beta|)},$$

$$\chi''_0 = \frac{F}{2|\beta|(\alpha - |\beta|)^2}\left(\frac{\alpha + 3|\beta|}{2} + \frac{1}{\alpha - 2|\beta|}\right),$$

or, in the equivalent form:

$$(30) \qquad K_{i\bar{j}} = \xi_0\left(a_{i\bar{j}} + \frac{\xi'_{-2}}{\xi_0} l_i l_{\bar{j}} + \frac{\chi''_0}{\xi_0} b_i b_{\bar{j}} + \frac{\chi''_{-2}}{2L\xi_0}\eta_i\eta_{\bar{j}}\right).$$

Notice that (30) we obtain following lemma;

Lemma 3.1. *Let (M, F) be a complex Matsumoto space then the angular metric tensor is given by* (30)*.*

Remark 3.1. Apply the same procedure of Proposition (3.1) then we obtain the inverse and determinant value of angular metric tensor $K_{i\bar{j}}$ as in Proposition (3.2).

Proposition 3.2. *Let $F = \frac{\alpha^2}{\alpha - |\beta|}$ be a complex Matsumoto metric with $|\beta| \neq 0$. Then, they have the following:*

(i) The inverse tensor $K^{\bar{j}i}$ of the angular tensor fields $K_{i\bar{j}}$ is.

$$(31) \quad K^{\bar{j}i} = \frac{1}{\xi_0 M_1}\left\{a^{\bar{j}i} + R_1\eta^i\bar{\eta}^j + \frac{\chi''_0}{\gamma_1\xi_0}(b^i + R_1\bar{\beta}\eta^i)(b^{\bar{j}} + R_1\beta\bar{\eta}^j)\right\}$$

where $R_1 = \frac{\xi'_{-2}}{\xi_0+\alpha^2\xi'_{-2}}$, $\gamma_1 = 1 + \frac{\chi''_0}{\xi_0}(\|b\|^2 + R_1|\beta|^2)$, *and*

$$M_1 = 1 + \frac{\chi''_{-2}}{2L\xi_0}\Big\{\xi_0^2\alpha^2 + 2\chi_0\xi_0|\beta|^2 + \chi_0^2\|b\|^2|\beta|^2 + R_1F^4 + \frac{\chi''_0}{\gamma_1\xi_0}\Big[\xi_0^2|\beta|^2 + 2\xi_0\chi_0|\beta|^2\|b\|^4 + \chi_0^2|\beta|^2\|b\|^4 + 2\xi_0R_1F^2|\beta|^2 + 2\chi_0R_1F^2|\beta|^2\|b\|^2 + R_1^2F^4|\beta|^2\Big]\Big\}. \tag{32}$$

(*ii*)

$$det(K_{i\bar{j}}) = (\xi_0)^n det(H_{i\bar{j}}) = (\xi_0)^n \frac{M_1\xi'_{-2}}{\gamma_1\xi_0} det(a_{i\bar{j}}). \tag{33}$$

4. Holomorphic curvature of complex Matsumoto metric

In this section, we study the Chern-Finsler connection, complex Cartan tensor and holomorphic curvature of complex Matsumoto metric.

Now, the Chern–Finsler connection coefficients (c.n.c.) and the horizontal and vertical coefficients are computed. By definition,

$$\overset{CF}{N^i_j} = g^{\bar{m}i}\frac{\partial g_{l\bar{m}}}{\partial z^j}\eta^l = g^{\bar{m}i}\frac{\partial \bar{\eta}_m}{\partial z^j}. \tag{34}$$

From (9) and (10), we compute the following

$$\begin{aligned} \bar{\eta}_m &= (p_0 l_{\bar{m}} + q_0 b_{\bar{m}}\beta), \\ \bar{\eta}_m &= \frac{(\alpha - 2|\beta|)}{(\alpha-|\beta|)^2}Fl_{\bar{m}} + \frac{1}{(\alpha-|\beta|)}\frac{F^2\beta b_{\bar{m}}}{|\beta|}. \end{aligned} \tag{35}$$

Differentiating (35) with respect to z^j we have

$$\begin{aligned} \frac{\partial\bar{\eta}_m}{\partial z^j} = \frac{(\alpha-2|\beta|)}{(\alpha-|\beta|)^2}\Big\{ & F\frac{\partial a_{i\bar{m}}}{\partial z^j}\eta^i + \frac{1}{(\alpha-|\beta|)^2}\Big[\frac{\alpha}{2}\frac{\partial a_{i\bar{s}}}{\partial z^j}\eta^i\bar{\eta}^s \\ & - \frac{\alpha^2}{2|\beta|}(\beta\frac{\partial b_{\bar{s}}}{\partial z^j}\bar{\eta}^s + \bar{\beta}\frac{\partial b_s}{\partial z^j}\eta^s) - (\alpha-|\beta|)\frac{\partial a_{i\bar{s}}}{\partial z^j}\eta^i\bar{\eta}^s\Big]l_{\bar{m}} \\ & + \frac{F}{(\alpha-|\beta|)}\Big[\frac{1}{|\beta|}(\beta\frac{\partial b_{\bar{m}}}{\partial z^j}\bar{\eta}^m + \bar{\beta}\frac{\partial b_m}{\partial z^j}\eta^m) - \frac{1}{\alpha}\frac{\partial a_{i\bar{s}}}{\partial z^j}\eta^i\bar{\eta}^s\Big]l_{\bar{m}}\Big\} \\ & + \frac{F}{(\alpha-|\beta|)^2}\Big[\frac{\partial a_{i\bar{s}}}{\partial z^j}\eta^i\bar{\eta}^s - \frac{1}{|\beta|}(\beta\frac{\partial b_{\bar{m}}}{\partial z^j}\bar{\eta}^m + \bar{\beta}\frac{\partial b_m}{\partial z^j}\eta^m)\Big]l_{\bar{m}} \end{aligned} \tag{36}$$

$$+\frac{F^2}{|\beta|(\alpha-|\beta|)}\left[\beta\frac{\partial b_{\bar m}}{\partial z^j}+\frac{b_{\bar m}}{2\beta}\frac{\partial b_i}{\partial z^j}\eta^i-\frac{\beta b_{\bar m}}{2|\beta|^2}(\beta\frac{\partial b_{\bar m}}{\partial z^j}\bar\eta^m+\bar\beta\frac{\partial b_m}{\partial z^j}\eta^m)\right]$$
$$+\frac{\beta b_{\bar m}}{|\beta|}\Bigg\{\frac{2F}{(\alpha-|\beta|)^3}\left[\frac{\alpha}{2}\frac{\partial a_{i\bar s}}{\partial z^j}\eta^i\bar\eta^s-\frac{\alpha^2}{2|\beta|}(\beta\frac{\partial b_{\bar s}}{\partial z^j}\bar\eta^s+\bar\beta\frac{\partial b_s}{\partial z^j}\eta^s)-(\alpha-|\beta|)\frac{\partial a_{i\bar s}}{\partial z^j}\eta^i\bar\eta^s\right]$$
$$+F^2\left[\frac{\partial a_{i\bar s}}{\partial z^j}\eta^i\bar\eta^s-\frac{1}{2|\beta|}(\beta\frac{\partial b_{\bar m}}{\partial z^j}\bar\eta^m+\bar\beta\frac{\partial b_m}{\partial z^j}\eta^m)\right]\Bigg\}.$$

Using (21) and (36) in (34), we have

$$\begin{aligned}\overset{CF}{N^i_j} &= \overset{a}{N^i_j}+\frac{1}{p_0}a^{\bar m i}\left(\frac{\partial\bar\eta_m}{\partial z^j}-p_0\frac{\partial a_{l\bar m}}{\partial z^j}\eta^l\right)\\ &-\frac{1}{p_0}\Bigg\{R\eta^i\bar\eta^m+\frac{q_0''}{\gamma p_0}(b^i+R\bar\beta\eta^i)(b^{\bar m}+R\beta\bar\eta^m)+\frac{q''_{-2}}{p_0M}\\ &\left[a^{\bar m i}+R\eta^i\bar\eta^m+\frac{q_0''}{\gamma p_0}(b^i+R\bar\beta\eta^i)(b^{\bar m}+R\beta\bar\eta^m)\right]^2\Bigg\}\frac{\partial\bar\eta_n}{\partial z^j},\end{aligned}\tag{37}$$

where $\frac{\partial\bar\eta_n}{\partial z^j}$ is same as in (36) and $\overset{a}{N^i_j}=a^{\bar m i}\frac{\partial a_{l\bar m}}{\partial z^j}\eta^l$.

Next, we have to find the expression for the vertical and horizontal coefficients of Chern–Finsler connection. Consider the following complex Cartan tensor [2]

$$C_{j\bar h k}=\frac{\partial g_{j\bar h}}{\partial\eta^k}=\frac{\partial g_{j\bar h}}{\partial\alpha}\frac{\partial\alpha}{\partial\eta^k}+\frac{\partial g_{j\bar h}}{\partial|\beta|}\frac{\partial|\beta|}{\partial\eta^k}.\tag{38}$$

On calculating values of $\frac{\partial g_{j\bar h}}{\partial\alpha}$ and $\frac{\partial g_{j\bar h}}{\partial|\beta|}$ and substituing in (38), we get

$$\begin{aligned}C_{j\bar h k} &= \left\{\left(\frac{\alpha^2-\alpha^3+4\alpha|\beta|^2}{(\alpha-|\beta|)^4}\right)\frac{l_k}{2\alpha}+\left(\frac{3\alpha^3-2\alpha^2-6\alpha^2|\beta|}{(\alpha-|\beta|)^4}\right)\frac{\beta b_{\bar k}}{2|\beta|}\right\}a_{j\bar h}\\ &+\left\{\left(\frac{4|\beta|^2-\alpha|\beta|}{2(\alpha-|\beta|)^5}\right)\frac{l_k}{2\alpha}+\left(\frac{12|\beta|^2-\alpha^2+4\alpha|\beta|}{2(\alpha-|\beta|)^6}\right)\frac{\beta b_{\bar k}}{2|\beta|}\right\}l_jl_{\bar h}\\ &+\left\{\left(\frac{\alpha^4-4\alpha^3|\beta|}{|\beta|(\alpha-|\beta|)^4}-\frac{12\alpha^3}{(\alpha-|\beta|)^5}\right)\frac{l_k}{2\alpha}+\left(\frac{(3\alpha+9|\beta|)}{2(\alpha-|\beta|)^3}\right.\right.\\ &\left.\left.+\frac{(\alpha+2|\beta|)}{(\alpha-|\beta|)^2}\right)\frac{\beta b_{\bar k}}{2|\beta|}\right\}b_jb_{\bar h}+\left\{\left(\frac{-\alpha^3+5\alpha^2|\beta|+8\alpha|\beta|^2}{2|\beta|(\alpha-|\beta|)^5}\right)\frac{l_k}{2\alpha}\right.\\ &\left.+\left(\frac{5F^2}{2\alpha(\alpha-|\beta|)^3}-\frac{F^2}{2|\beta|^2(\alpha-|\beta|)^3}-\frac{8F}{(\alpha-|\beta|)^4}\right)\frac{\beta b_{\bar k}}{2|\beta|}\right\}\\ &(l_j\beta b_{\bar h}+\bar\beta b_jl_{\bar h}).\end{aligned}\tag{39}$$

Also, the vertical coefficients of Chern–Finsler connections are defined as

$$C^i_{jk} = g^{\bar{m}i}\frac{\partial g_{k\bar{m}}}{\partial \eta^j} = g^{\bar{m}i}\frac{\partial g_{j\bar{m}}}{\partial \eta^k} = g^{\bar{m}i}C_{j\bar{m}k}. \tag{40}$$

Using (22) and (39) in (40), we have

$$\begin{aligned} C^i_{jk} &= \frac{1}{M}\Bigg[a^{\bar{m}i} + \left(1 + \frac{q_0'' R|\beta|^2}{p_0\gamma}\right)R\eta^i\bar{\eta}^m + \frac{q_0''}{p_0\gamma}b^i b^{\bar{m}} \\ &\quad + \frac{q_0''R}{p_0\gamma}(\beta b^i\bar{\eta}^m + \bar{\beta}\eta^i b^{\bar{m}})\Bigg] \\ &\quad \Bigg[\Bigg\{\left(\frac{\alpha^2 - \alpha^3 + 4\alpha|\beta|^2}{(\alpha-|\beta|)^4}\right)\frac{l_k}{2\alpha} + \left(\frac{3\alpha^3 - 2\alpha^2 - 6\alpha^2|\beta|}{(\alpha-|\beta|)^4}\right)\frac{\beta b_{\bar{k}}}{2|\beta|}\Bigg\}a_{j\bar{h}} \\ &\quad + \Bigg\{\left(\frac{4|\beta|^2 - \alpha|\beta|}{2(\alpha-|\beta|)^5}\right)\frac{l_k}{2\alpha} + \left(\frac{12|\beta|^2 - \alpha^2 + 4\alpha|\beta|}{2(\alpha-|\beta|)^6}\right)\frac{\beta b_{\bar{k}}}{2|\beta|}\Bigg\}l_j l_{\bar{h}} \\ &\quad + \Bigg\{\left(\frac{\alpha^4 - 4\alpha^3|\beta|}{|\beta|(\alpha-|\beta|)^4} - \frac{12\alpha^3}{(\alpha-|\beta|)^5}\right)\frac{l_k}{2\alpha} + \Bigg(\frac{(3\alpha + 9|\beta|)}{2(\alpha-|\beta|)^3} \\ &\quad + \frac{(\alpha+2|\beta|)}{(\alpha-|\beta|)^2}\Bigg)\frac{\beta b_{\bar{k}}}{2|\beta|}\Bigg\}b_j b_{\bar{h}} + \Bigg\{\left(\frac{-\alpha^3 + 5\alpha^2|\beta| + 8\alpha|\beta|^2}{2|\beta|(\alpha-|\beta|)^5}\right)\frac{l_k}{2\alpha} \\ &\quad + \left(\frac{5F^2}{2\alpha(\alpha-|\beta|)^3} - \frac{F^2}{2|\beta|^2(\alpha-|\beta|)^3} - \frac{8F}{(\alpha-|\beta|)^4}\right)\frac{\beta b_{\bar{k}}}{2|\beta|}\Bigg\} \\ &\quad (l_j\beta b_{\bar{h}} + \bar{\beta}b_j l_{\bar{h}})\Bigg]. \end{aligned} \tag{41}$$

Also,

$$C_k = C_{k\bar{h}j}g^{\bar{h}j}. \tag{42}$$

Plugging (22) and (40) in (42) gives us

$$\begin{aligned} C_k &= \Bigg[\Bigg\{\left(\frac{\alpha^2 - \alpha^3 + 4\alpha|\beta|^2}{(\alpha-|\beta|)^4}\right)\frac{l_k}{2\alpha} + \left(\frac{3\alpha^3 - 2\alpha^2 - 6\alpha^2|\beta|}{(\alpha-|\beta|)^4}\right)\frac{\beta b_{\bar{k}}}{2|\beta|}\Bigg\}a_{j\bar{h}} \\ &\quad + \Bigg\{\left(\frac{4|\beta|^2 - \alpha|\beta|}{2(\alpha-|\beta|)^5}\right)\frac{l_k}{2\alpha} + \left(\frac{12|\beta|^2 - \alpha^2 + 4\alpha|\beta|}{2(\alpha-|\beta|)^6}\right)\frac{\beta b_{\bar{k}}}{2|\beta|}\Bigg\}l_j l_{\bar{h}} \\ &\quad + \Bigg\{\left(\frac{\alpha^4 - 4\alpha^3|\beta|}{|\beta|(\alpha-|\beta|)^4} - \frac{12\alpha^3}{(\alpha-|\beta|)^5}\right)\frac{l_k}{2\alpha} + \Bigg(\frac{(3\alpha + 9|\beta|)}{2(\alpha-|\beta|)^3} \\ &\quad + \frac{(\alpha+2|\beta|)}{(\alpha-|\beta|)^2}\Bigg)\frac{\beta b_{\bar{k}}}{2|\beta|}\Bigg\}b_j b_{\bar{h}} + \Bigg\{\left(\frac{-\alpha^3 + 5\alpha^2|\beta| + 8\alpha|\beta|^2}{2|\beta|(\alpha-|\beta|)^5}\right)\frac{l_k}{2\alpha} \end{aligned} \tag{43}$$

$$+\left(\frac{5F^2}{2\alpha(\alpha-|\beta|)^3}-\frac{F^2}{2|\beta|^2(\alpha-|\beta|)^3}-\frac{8F}{(\alpha-|\beta|)^4}\right)\frac{\beta b_{\bar{k}}}{2|\beta|}\Bigg\}$$
$$(l_j\beta b_{\bar{h}}+\bar{\beta}b_j l_{\bar{h}})\Bigg]\Bigg[a^{\bar{h}i}+\left(1+\frac{q_0''R|\beta|^2}{p_0\gamma}\right)R\eta^j\bar{\eta}^h+\frac{q_0''}{p_0\gamma}b^j b^{\bar{h}}$$
$$+\frac{q_0''R}{p_0\gamma}(\beta b^j\bar{\eta}^h+\bar{\beta}\eta^j b^{\bar{h}})\Bigg].$$

Now, we find the Holomorphic curvature, first we compute Ricci curvature $R_{\bar{j}k}$ in (4).

Substituting the values of $g_{i\bar{j}}$ and $\overset{CF}{N}{}^i_k$ in (5) then, we have

$$\begin{aligned}
R_{\bar{j}k} \;=\; & -\Bigg[\frac{F(\alpha-2|\beta|)}{(\alpha-|\beta|)^2}\,a_{l\bar{j}}+\frac{-F|\beta|}{2\alpha^2}\left(\frac{\alpha-4|\beta|}{(\alpha-|\beta|)^3}\right)l_l l_{\bar{j}}\\
& +\frac{1}{2|\beta|}\left(\frac{3F^2}{(\alpha-|\beta|)^2}+\frac{2F^2}{(\alpha-|\beta|)}\right)b_l b_{\bar{j}}+\frac{F(\alpha-4|\beta|)}{2|\beta|(\alpha-|\beta|)^3}\\
& \left(l_l\beta b_{\bar{j}}+\bar{\beta}b_l l_{\bar{j}}\right)\Bigg]\;\delta_{\bar{h}}\Bigg\{\overset{a}{N}{}^l_k+\frac{1}{p_0}a^{\bar{m}l}\left(\frac{\partial\bar{\eta}_m}{\partial z^k}-p_0\frac{\partial a_{q\bar{m}}}{\partial z^j}\eta^q\right)\\
& -\frac{1}{p_0}\Bigg[R\eta^l\bar{\eta}^m+\frac{q_0''}{\gamma p_0}(b^l+R\bar{\beta}\eta^l)(b^{\bar{m}}+R\beta\bar{\eta}^m)+\frac{q_{-2}''}{p_0 M}\\
& \left(a^{\bar{m}l}+R\eta^l\bar{\eta}^m+\frac{q_0''}{\gamma p_0}(b^l+R\bar{\beta}\eta^l)(b^{\bar{m}}+R\beta\bar{\eta}^m)\right)^2\Bigg]\frac{\partial\bar{\eta}_m}{\partial z^k}\Bigg\}\bar{\eta}^h.
\end{aligned}\tag{44}$$

Theorem 4.1. *Let (M, F) be a complex Matsumoto space with the metric (7). Then, Ricci curvature is given by equation* (44) *and holomorphic curvature is given by* (4).

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