On bi-univalent functions involving Srivastava-Attiya operator

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Abstract. Two subclasses $L\sum_{\Sigma}^{b,\delta}(\mu,\alpha)$ and $L\sum_{\Sigma}^{b,\delta}(\mu,\beta)$ of the class $\sum$ of Bi-univalent functions have been introduced by making use of the Srivastava-Attiya operator. The estimates of the coefficients $|a_2|$ and $|a_3|$ of functions have been found for these subclasses. The results obtained are quit interesting and new.

Keywords: univalent function, bi-univalent function, coefficients bounds, and Srivastava-Attiya operator.

1. Introduction

Let Σ denotes the class of functions f of the form [1]

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which satisfy the following two continuous:

i. Holomorphic in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$,

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ii. Normalized by $f(0) = f'(0) - 1 = 0$.

In addition, with $z \in U$, the general form of Hurwitz-Lerch Zeta function $\varphi(\delta, b, z)$ which are used with the convolution of holomorphic function can be defined by:

$$\varphi(\delta, b, z) = \sum_{K=0}^{\infty} \frac{z^k}{(k+b)^\delta} = b^{-\delta} + \frac{z}{(1+b)^\delta} + \sum_{k=2}^{\infty} \frac{z^k}{(k+b)^\delta}, \tag{2}$$

such that b is a complex number with $b \neq 0, -1, -2, \ldots, \mu \in \mathbb{C}$, and $\mathrm{Re}(\delta) > 1$.

Also, Srivastava and Attiya [2] defined the linear operators $Q_{\delta,b} : \Sigma \longrightarrow \Sigma$ by means of:

$$Q_{\delta,b} f(z) = G_{\delta,b} * f(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+b}{k+b} \right)^\delta a_k z^k, \tag{3}$$

where $G_{\delta,b} \in \Sigma$ to be:

$$G_{\delta,b} = (1+b)^\delta \left[\varphi(\delta, b, z) - b^{-\delta} \right] = z + \sum_{k=2}^{\infty} \left(\frac{1+b}{k+b} \right)^\delta z^k. \tag{4}$$

Remark 1.1. $Q_{0,b}$ and $Q_{-\delta,b}$ denotes the identity and inverse operator of $Q_{\delta,b}$ respectively.

Koebe one-quarter theorem includes the image of U under every univalent functions $f \in A$ with an open disk centered at origin and radius $\frac{1}{4}$. Therefore, the inverse of every univalent function $f \in A$ can be written as $f^{-1} : f(U) \longrightarrow U$ satisfying:

$$f^{-1}(f(z)) = z, \; z \in U,$$
$$f\left(f^{-1}(\omega)\right) = \omega, \; |\omega| < r_0(f), r_0(f) \geq \frac{1}{4}.$$

Furthermore, we notice that the inverse function has the series expansion which can be written in the form:

$$f^{-1}(\omega) = \omega - a_2\omega^2 + \left(2a_2^2 - a_3\right)\omega^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)\omega^4 + \ldots \tag{5}$$

In addition, a function $f \in A$ is bi-univalent if both f and the inverse $g = f^{-1}$ are univalent in U.

Recently, several authors are concentrating on these functions, which are defined to the class Σ composed with various other features of the bi-univalent function class Σ, considering the most two important subclasses of univalent functions $S^*(\beta)$ and $C(\beta)$ of order β (see [2-7]). Consequently via definition, the classes $S^*(\beta)$ and $C(\beta)$ can be written as:

$$S^*(\beta) = \left\{ f \in S : Re\left(\frac{zf'(z)}{f(z)} \right) > \beta, \, z \in U \text{ and } 0 \leq \beta < 1 \right\}$$

and

$$C(\beta) = \left\{ f \in S : Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta,\ z \in U \text{ and } 0 \leq \beta < 1 \right\}.$$

For $0 \leq \beta < 1$, if both f and its inverse f^{-1} are starlike and convex function of order β, then a function $f \in \sum d$ is in the class $S^*_\Sigma(\beta)$, or $C_\Sigma(\beta)$. These classes are introduced and investigated by Brannan and Taha [3]. Moreover, the coefficients $|a_2|$ and $|a_3|$ for functions in the classes $S^*_\Sigma(\beta)$ and $C_\Sigma(\beta)$ have been found.

The main objective of this study is to present new two subclasses of the class Σ related to the Srivastava-Attiya operator [2,7,10] and accordingly to find the estimation of the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses [11-13].

In order to prove our main results, we have to remembrance the following lemma.

Lemma 1.1. *If a function $h(z) \in P$ ([6])*

$$h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots,\ z \in U. \tag{6}$$

Then

$$|c_k| \leq 2,\ k \in N. \tag{7}$$

P is the family of all functions p, that an analytic in U for which $h(0) = 1$ and $Re(h(z)) > 0$.

2. Coefficient estimates of $L\sum_\Sigma^{b,\delta}(\mu, \alpha)$

Definition 2.1. *Let $f(z)$ related by (1). Then, it be in the class $L\sum_\Sigma^{b,\delta}(\mu, \alpha)$, if the following are fulfilled [4]:*

$$f \in \sum,\ \arg\left|\left\{(1-\mu)\frac{Q_{\delta,b}f(z)}{z} + \mu\left(Q_{\delta,b}f(z)\right)'\right\}\right| < \frac{\alpha\pi}{2} \tag{8}$$

with $0 < \alpha \leq 1, \mu \geq 1, z \in U$, and

$$arg\left|\left\{(1-\mu)\frac{Q_{\delta,b}g(\omega)}{\omega} + \mu\left(Q_{\delta,b}g(\omega)\right)'\right\}\right| < \frac{\alpha\pi}{2}, \tag{9}$$

where $0 < \alpha \leq 1$, $\mu \geq 1$, $z \in U$, and the function g is extended by $g = f^{-1}$ and given by:

$$f^{-1}(\omega) = \omega - a_2\omega^2 + \left(2a_2^2 - a_3\right)\omega^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)\omega^4 + \ldots \tag{10}$$

For the functions in the class $L\sum_\Sigma^{b,\delta}(\mu, \alpha)$, we find the coefficients $|a_2|$ and $|a_3|$.

Theorem 2.1. *Let $f(z)$ which is given by (1) supposed to be in the class $L\sum_{\Sigma}^{b,\delta}(\mu,\alpha)$, $0<\alpha\leq 1$ and $\mu\geq 1$, $\delta\geq 1$. Then:*

$$|a_2| \leq \frac{2a}{\sqrt{\alpha\left(\left(\frac{1+b}{3+b}\right)^{\delta}(2+4\mu)\right)+(1-\alpha)\left(\frac{1+b}{2+b}\right)^{2\delta}(1+\mu)^2}} \tag{11}$$

and

$$|a_3| \leq \frac{2\alpha}{\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)}. \tag{12}$$

Proof. The inequalities (11) and (12) are equivalent to:

$$(1-\mu)\frac{Q_{\delta,b}f(z)}{z}+\mu\left(Q_{\delta,b}f(z)\right)'=(p(z))^{\alpha} \tag{13}$$

and

$$(1-\mu)\frac{Q_{\delta,b}f(\omega)}{\omega}+\mu\left(Q_{\delta,b}f(\omega)\right)'=(q(\omega))^{\alpha}, \tag{14}$$

where $p(z)$ and $q(w)$ satisfies the inequalities:

$$\mathrm{Re}(p(z))>0,\ z\in U \text{ and } \mathrm{Re}(q(w))>0,\ w\in U.$$

Moreover, the functions $p(z)$ and $q(w)$ can be written as:

$$p(z)=1+p_1z+p_2z^2+\dots \tag{15}$$

and

$$q(\omega)=1+q_1\omega+q_2\omega^2+\dots \tag{16}$$

As well, $g(w)$ is given as in (3).

Now, by equating the coefficients in equations (11) and (12), we get:

$$\left(\frac{1+b}{2+b}\right)^{\delta}(1+\mu)a_2=p_1\alpha, \tag{17}$$

$$\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)a_3=p_2\alpha+\frac{\alpha(\alpha-1)}{2}p_1^2, \tag{18}$$

$$\left(\frac{1+b}{2+b}\right)^{\delta}(1+\mu)a_2=-q_1\alpha, \tag{19}$$

$$\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)\left(2a_2^2-a_3\right)=q_2\alpha+\frac{\alpha(\alpha-1)}{2}q_1^2. \tag{20}$$

From equations (18) and (19), we obtained:

$$p_1 = -q_1 \tag{21}$$

and

$$2\left(\frac{1+b}{3+b}\right)^{\delta}(2\mu+1)^2 a_2^2 = \alpha\left(p_1+q_1\right) + \frac{\alpha(\alpha-1)}{2}\left(q_1^2+p_1^2\right). \tag{22}$$

From (20), (21) and (22), we got:

$$a_2^2 = \frac{\alpha^2\left(q_2^2+p_2^2\right)}{\alpha\left(\left(\frac{1+b}{3+b}\right)^{\delta}(2+4\mu)\right)+(1-\alpha)\left(\frac{1+b}{2+b}\right)^{2\delta}(1+\mu)^2}. \tag{23}$$

By applying lemma (1) on the coefficients p_2 and q_2, we obtained:

$$|a_2| \le \frac{2\alpha}{\sqrt{\alpha\left(\left(\frac{1+b}{3+b}\right)^{\delta}(2+4\mu)\right)+(1-\alpha)\left(\frac{1+b}{2+b}\right)^{2\delta}(1+\mu)^2}}. \tag{24}$$

Now, to find the bound on $|a_3|$, subtract (20) from (18) to get:

$$2\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)a_3 - 2\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)a_2^2 = \alpha\left(p_2-q_2^2\right). \tag{25}$$

From (23) and with the help of $p_1^2 = q_1^2$, substitute the value of a_2^2 to get:

$$a_3 = \frac{\alpha\left[\left(4\alpha(1+2\mu)\left(\frac{1+b}{3+b}\right)^{\delta}+(1-\alpha)(1+\mu)^2\left(\frac{1+b}{3+b}\right)^{\delta}\right)p_2 - (1-\alpha)(1+\mu)^2\left(\frac{1+b}{2+b}\right)^{2\delta}q_2^2\right]}{2\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)\left[2\alpha(1+2\mu)\left(\frac{1+b}{3+b}\right)^{\delta}+(1-\alpha)(1+\mu)^2\left(\frac{1+b}{2+b}\right)^{2\delta}\right]}. \tag{26}$$

Now, considering Lemma 1.1 again and using the substations of coefficients p_1, p_2, q_1 and q_2, to get:

$$|a_3| \le \frac{2\alpha}{\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)}. \tag{27}$$

Hence, the proof of the Theorem 2.2 is completed. □

Now, assuming $\mu = 1$ and $b = 1$ in above theorem, then we have:

Corollary 2.1. *If $f(z)$ given by (1) is in $L_{\Sigma}^{1,\delta}(1,\alpha)$, $0<\alpha\leq 1$ and $\mu\geq 1$, $\delta\geq 1$, then we have:*

$$|a_2|\leq \frac{2\alpha}{\sqrt{6\alpha\left(\frac{1}{2}\right)^{\delta}+4(1-\alpha)\left(\frac{1}{2}\right)^{2\delta}}} \tag{28}$$

and

$$|a_3|\leq \frac{2\alpha}{\left(\frac{1}{2}\right)^{\delta}(1+2\mu)}. \tag{29}$$

Assuming $\alpha=1$ in theorem 2.2, we have the following corollary:

Corollary 2.2. *Let $f(z)$ which is given by (1) belonged to the class $L\sum_{\Sigma}^{b,\delta}(\mu,1)$, $0<\alpha\leq 1$ and $\mu\geq 1$, $\delta\geq 1$. Then:*

$$|a_2|\leq \frac{2}{\sqrt{\left(\left(\frac{1+b}{3+b}\right)^{\delta}(2+4\mu)\right)}} \tag{30}$$

and

$$|a_3|\leq \frac{2}{\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)}. \tag{31}$$

3. Coefficient estimates of $L\sum_{\Sigma}^{b,\delta}(\beta,\mu)$

Definition 3.1. *Let $f(z)$ related by (1). Then it be in the class $L\sum_{\Sigma}^{b,\delta}(\beta,\mu)$ if the following conditions are fulfilled:*

$$f\in\sum \text{ and } \operatorname{Re}\left\{(1-\mu)\frac{Q_{\delta,b}f(z)}{z}+\mu\left(Q_{\delta,b}f(z)\right)'\right\}>\beta, \tag{32}$$

where $0<\beta\leq 1$, $\mu\geq 1$, $z\in U$ and

$$\operatorname{Re}\left\{(1-\mu)\frac{Q_{\delta,b}g(\omega)}{\omega}+\mu\left(Q_{\delta,b}g(\omega)\right)\right\}>\beta, \tag{33}$$

such that $0<\beta\leq 1$, $\mu\geq 1$, $z\in U$. Thus, the function g is introduced that the inverse of f given as in (5).

Theorem 3.1. *If $f(z)$ which is given by (1) supposed to be in $L\sum_{\Sigma}^{b,\delta}(\beta,\mu)$, $0\leq\beta<1$ and $\mu\geq 0$, then:*

$$|a_2|\leq \sqrt{\frac{2(1-\beta)}{\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)}} \tag{34}$$

and

$$|a_3|\leq \frac{2(1-\beta)}{\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)}. \tag{35}$$

Proof. The inequalities (32) and (33) are equivalent to:

$$(1-\mu)\frac{Q_{\delta,b}f(z)}{z}+\mu\left(Q_{\delta,b}f(z)\right)'=\beta+(1-\beta)p(z) \tag{36}$$

and

$$(1-\mu)\frac{Q_{\delta,b}g(\omega)}{\omega}+\mu\left(Q_{\delta,b}g(\omega)\right)'=\beta+(1-\beta)q(z). \tag{37}$$

By equating coefficients in equations (36) and (37) produces:

$$\left(\frac{1+b}{2+b}\right)^{\delta}(1+\mu)a_2=p_1(1-\beta), \tag{38}$$

$$\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)a_3=p_2(1-\beta) \tag{39}$$

and

$$\left(\frac{1+b}{2+b}\right)^{\delta}(1+\mu)a_2=-q_1(1-\beta), \tag{40}$$

$$\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)\left(2a_2^2-a_3\right)=q_2(1-\beta). \tag{41}$$

From equations (39) and (40), we obtained:

$$p_1=-q_1 \tag{42}$$

and also from (39) and (41), we obtain:

$$2\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)a_2^2=(1-\beta)\left(p_2+q_2\right). \tag{43}$$

So, we get:

$$a_2^2=\frac{\left(p_2+q_2\right)(1-\beta)}{2\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)}. \tag{44}$$

By applying lemma 1.1 for the coefficients p_2 and q_2, we obtained:

$$\left|a_2^2\right|\leq\sqrt{\frac{2(1-\beta)}{\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)}} \tag{45}$$

which is the looked-for inequality as given in the (34).

Now, by subtracting (41) from (39), we have:

$$2\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)a_3^2=(1-\beta)\left(p_2-q_2\right)+2\left(\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)a_2^2\right. \tag{46}$$

Then, substitute the value of a_2^2 from (43), to obtain:

$$a_3^2 = \frac{(p_2 - q_2)(1-\beta)}{2\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)}. \tag{47}$$

Now, with the help of the Lemma 1.1, we got:

$$\left|a_3^2\right| \leq \frac{2(1-\beta)}{\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)}, \tag{48}$$

which is the bound on $|a_3^2|$ as stated in (35). □

Assuming $\beta = 0$ in theorem 3.2, we have the following corollary:

Corollary 3.1. *Let $f(z)$ given by (1) supposed to be in the class $L_{\Sigma}^{b,\delta}(0,\mu)$, $0 \leq \beta < 1$ and $\mu \geq 0$, $z \in U$. Then:*

$$\left|a_2^2\right| \leq \sqrt{\frac{2}{\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)}}, \tag{49}$$

$$\left|a_3^2\right| \leq \frac{2}{\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)}. \tag{50}$$

Conclusion

We have been shown the existence of novel two subclasses types paly an interested roll to the Srivastava-Attiya operator with their original results. Consequently, the obtained outcomes have demonstrated the estimation of the coefficients $|a2|$ and $|a3|$ for associated complex functions in new subclasses. Many problem still opened for example extend the obtained results to the case of differential operator in Hebert space as in [14-15] or with another operator types (see [16-17]).

References

[1] P. L. Duren, *Univalent functions, in: Grunddlehren der mathematischen Wissenschaften*, Band 259, Springer -Verlag, New York, Berlin, Hidelberg and Tokyo, 1983.

[2] H. M. Srivastava, A. A. Attiya, *An integral operator associated with the Hurwitz-Lerch zeta function and differential subordination*, Integral Transforms Spec. Funct., 18 (2007), 207–216.

[3] D.A. Brannan, T.S. Taha, *On some classes of bi-univalent functions*, Mathematical Analysis and its Applications, (1985), 53-60.

[4] T. Panigrahi, *Coefficient bounds for a of bi-univalent analytic function associated with Hohlov operator*, 16 (2013), 91-100.

[5] H. M. Srivastava, A. K. Mishra, P. Gochhayat, *Certain subclasses of analytic and bi-univalent functions*, Appl. Math. Lett., 23 (2010), 1188–1192.

[6] S. Bulut, *Coefficient estimates for a class of analytic and bi-univalent functions*, Novi Sad J. Math., 43 (2013), 59–65.

[7] D. Breaz, N. Breaz, H.M. Srivastava, *An extension of the univalent condition for a family of integral operators*, Appl. Math. Lett., 22 (2009), 41-44.

[8] A.S. Juma, F,S, Aziz, *Appling Ruschewwyh derivative on two subclasses of bi-univalent functions*, IJBAS-IJENS, 12 (2012), 68-74.

[9] X-F. Li, A. P. Wang, *Two new subclasses of bi-univalent functions*, Int. Math. Forum., 7 (2012), 1495–1504.

[10] R. J. Libera, *Some classes of regular univalent functions*, Proc. Amer. Math. Soc, 135 (1969), 429-449.

[11] S. Khan, N. Khan, S. Hussain, Q. Ahmad, M. Zaighum, *Some subclass of bi-univalent function associated to Srivastava-Attya operator*, Bulletan Mathematical Analysis and Applications, 9 (2017), 37-44.

[12] H. M. Srivastava, D. Bansal, *Coefficient estimates for a subclass of analytic and bi-univalent functions*, J. Egyptian Math. Soc., 23 (2015), 242–246.

[13] Q. H. Xu, Y. C. Gui, H. M. Srivastava, *Coefficient estimates for a certain subclass of analytic and bi-univalent functions*, Appl. Math. Lett., 25 (2012), 990-994.

[14] R. Al-Saphory, Z. Khalid, M. Jasim, *Junction interface conditions for asymptotic gradient full-observer in Hilbert space*, Italian Journal of Pure and Applied Mathematics, 49 (2023), 1-16.

[15] S. Rekkab, S. Benhadid, R. Al-Saphory, *An asymptotic analysis of the gradient remediability problem for disturbed distributed linear systems,* Baghdad Science Journal, 19 (2022), 1623–1635.

[16] M. Abdul Ameer, A. R. Juma, R. Al-Saphory, *On differential subordination of higher-order derivatives of multivalent functions*, Journal of Physics: Conference Series, 1818 (012188) (2021), 1-10.

[17] M. Abdul Ameer, A. S. Juma, R. Al-Saphory, *Harmonic meromorphic starlike functions of complex order involving Mittag-Leffler operator*, Italian Journal of Pure and Applied Mathematics, 48 (2022), 1-9.

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