A note on extended Hurwitz-Lerch Zeta function

M. Ghayasuddin
Department of Mathematics
Faculty of Science
Integral University
Lucknow-226026
India
ghayas.maths@gmail.com

N. U. Khan
Department of Applied Mathematics
Faculty of Engineering and Technology
Aligarh Muslim University-202002
India
nukhanmath@gmail.com

Waseem A. Khan*
Department of Mathematics and Natural Sciences
Prince Mohammad Bin Fahd University
P.O. Box: 1664, Al Khobar 31952
Saudi Arabia
wkhan1@pmu.edu.sa

Moin Ahmad
Department of Mathematics
Shree Krishna Educational Institution
Ahmad Nagar-261125
Sitapur
moinah1986@gmail.com

Abstract. In the present research note, we introduce another extension of Hurwitz-Lerch Zeta function (HLZF) by using the generalized extended Beta function defined by Parmar [7]. We investigate its integral representations, Mellin transform, generating relations and differential formula. In view of diverse applications of the Hurwitz-Lerch Zeta functions, the results presented here are potentially useful in some other related research areas.

Keywords: Generalized Hurwitz-Lerch Zeta function, extended beta function, extended hypergeometric function, Mellin transform.

*. Corresponding author

1. Introduction

The well known Hurwitz-Lerch Zeta function (HLZF) is defined by (see, [2], [10], [11]):

$$\Phi(z,s,a)=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s}, \tag{1.1}$$

$(a\in\mathbb{C}\setminus\mathbb{Z}_0; s\in\mathbb{C},\text{ when } \mid z\mid<1; \Re(s)>1,\text{when } \mid z\mid=1)$.

Due to diverse applications of Hurwitz-Lerch Zeta function (HLZF), several extensions of $\Phi(z,s,a)$ have been introduced and studied (see, for example [1], [3], [4], [5], [8], [10], [11], [12] etc).

Very recently, Parmar et al. [9] defined the following extended Hurwitz-Lerch Zeta function (HLZF):

$$\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma)}(z,s,a;p)=\sum_{n=0}^{\infty}\frac{(\lambda)_n B_p^{(\rho,\sigma)}(\mu+n,\gamma-\mu)}{n!B(\mu,\gamma-\mu)}\frac{z^n}{(n+a)^s}, \tag{1.2}$$

$(p\geq 0,\Re(\rho)>0,\Re(\sigma)>0;\lambda,\mu\in\mathbb{C};\ \gamma,a\in\mathbb{C}\setminus\mathbb{Z}_0; s\in\mathbb{C},\text{ when } \mid z\mid<1;\Re(s+\gamma-\lambda-\mu)>1,\text{ when } \mid z\mid=1)$, where $B_p^{(\rho,\sigma)}(x,y)$ is the extended Beta function (see [6]):

$$B_p^{(\rho,\sigma)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1(\rho;\sigma;\frac{-p}{t(1-t)})dt. \tag{1.3}$$

They also presented the integral representation of (1.2)

$$\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma)}(z,s,a;p)=\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}e^{-at}F_p^{(\rho,\sigma)}(\lambda,\mu;\gamma;ze^{-t})dt. \tag{1.4}$$

For $\rho=\sigma$, (1.2) reduces to the Hurwitz-Lerch Zeta function (HLZF) defined by Parmar and Raina [8], which, further for $p=0$, gives the known extension of (1.1) (see [3]).

In a sequel of above-mentioned works, we introduce a further extension of $\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma)}(z,s,a;p)$ by using the generalized Beta function defined by Parmar [7].

2. Extended Hurwitz-Lerch Zeta function (EHLZF)

Here we define the following extension of (1.2):

$$\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a;p)=\sum_{n=0}^{\infty}\frac{(\lambda)_n B_p^{(\rho,\sigma;m)}(\mu+n,\gamma-\mu)}{n!B(\mu,\gamma-\mu)}\frac{z^n}{(n+a)^s}, \tag{2.1}$$

$(p\geq 0,\Re(\rho)>0,\Re(\sigma)>0,\Re(m)>0;\lambda,\mu\in\mathbb{C};\ \gamma,a\in\mathbb{C}\setminus\mathbb{Z}_0; s\in\mathbb{C},\text{ when } \mid z\mid<1;\Re(s+\gamma-\lambda-\mu)>1,\text{ when } \mid z\mid=1)$, where $B_p^{(\rho,\sigma;m)}(a,b)$ is the extended Beta function (see, Parmar [7])

$$B_p^{(\rho,\sigma;m)}(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}{}_1F_1(\rho;\sigma;\frac{-p}{x^m(1-x)^m})\ dx. \tag{2.2}$$

He [7] also gave the following extension of Gauass hypergeometric function:

$$F_p^{(\rho,\sigma;m)}(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n B_p^{(\rho,\sigma;m)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}. \tag{2.3}$$

If we set $m=1$ in (2.1), we get the known extended Hurwitz-Lerch Zeta function given by (1.2).

Remark 2.1. The extended Hurwitz-Lerch Zeta function (EHLZF)

$$\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a;p)$$

has the following limiting case:

$$\begin{aligned}\Phi_{\mu;\gamma}^{*(\rho,\sigma;m)}(z,s,a;p)&=\lim_{|\lambda|\longrightarrow\infty}\left\{\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}\left(\frac{z}{\lambda},s,a;p\right)\right\}\\ &=\sum_{n=0}^{\infty}\frac{B_p^{(\rho,\sigma;m)}(\mu+n,\gamma-\mu)}{n!B(\mu,\gamma-\mu)}\frac{z^n}{(n+a)^s}.\end{aligned} \tag{2.4}$$

$(p\geq 0,\Re(\rho)>0,\Re(\sigma)>0,\Re(m)>0;\mu\in\mathbb{C};\ \gamma,a\in\mathbb{C}\setminus\mathbb{Z}_0;s\in\mathbb{C},$ when $|z|<1;\Re(s+\gamma-\mu)>1,$ when $|z|=1).$

3. Integral representations of $\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a;p)$

In this section we deal with some integral representations of (2.1):

Theorem 3.1. *The following integral representation of EHLZF* $\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a;p)$ *holds true:*

$$\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a;p)=\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}e^{-at}F_p^{(\rho,\sigma;m)}(\lambda,\mu;\gamma;ze^{-t})dt, \tag{3.1}$$

$(\Re(p)\geq 0,\Re(\rho)>0,\Re(\sigma)>0,\Re(m)>0;\ p=0,\Re(a)>0;\Re(s)>0,$ when $|z|\leq 1(z\neq 1);\Re(s)>1,$ when $z=1).$

Proof. We have

$$\frac{1}{(n+a)^s}=\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}e^{-(n+a)t}dt.$$

By using the above result in (2.1) and after changing the order of summation and integration, which is guaranteed under the conditions, we get

$$\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a;p)=\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}e^{-at}\left(\sum_{n=0}^{\infty}\frac{(\lambda)_n B_p^{(\rho,\sigma;m)}(\mu+\gamma,\gamma-\mu)}{B(\mu,\gamma-\mu)}\frac{(ze^{-t})^n}{n!}\right)dt,$$

which upon using (2.3), yields the required result.

Theorem 3.2. *The following integral representation of EHLZF* $\Phi^{(\rho,\sigma;m)}_{\lambda,\mu;\gamma}(z,s,a;p)$ *holds true:*

$$\Phi^{(\rho,\sigma;m)}_{\lambda,\mu;\gamma}(z,s,a;p)=\frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-t}\Phi^{*(\rho,\sigma;m)}_{\mu,\gamma}(zt,s,a;p)dt. \tag{3.2}$$

$(\Re(p)\geq 0,\Re(\rho)>0,\Re(\sigma)>0,\Re(m)>0;p=0,\Re(\lambda)>0;\ \Re(a)>0;\Re(s)>0,$ when $|\ z\ |\leq 1(z\neq 1);\Re(s)>1$, when $z=1)$.

Proof. We have

$$(\lambda)_n=\frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda+n-1}e^{-t}dt.$$

By using the above result in (2.1) and after changing the order of summation and integration, which is guaranteed under the conditions, we get

$$\Phi^{(\rho,\sigma;m)}_{\lambda,\mu;\gamma}(z,s,a;p)=\frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-t}\left(\sum_{n=0}^{\infty}\frac{B_p^{(\rho,\sigma;m)}(\mu+n,\gamma-\mu)}{B(\mu,\gamma-\mu)}\frac{(zt)^n}{n!(n+a)^s}\right)dt,$$

which upon using (2.4), we arrive at our required result.

4. Mellin transform

The Mellin transform of the function $f(u)$ is given by

$$M\{f(u);s\}=\phi(s)=\int_0^\infty u^{s-1}f(u)du. \tag{4.1}$$

Theorem 4. *For the new extended Hurwitz-Lerch Zeta function EHLZF* $\Phi^{(\rho,\sigma;m)}_{\lambda,\mu;\gamma}(z,s,a;p)$*, we have the following Mellin transform representation:*

$$\begin{aligned} &M\left\{\Phi^{(\rho,\sigma;m)}_{\lambda,\mu;\gamma}(z,s,a;p)\right\}\\ &=\frac{\Gamma^{(\rho,\sigma)}(s)\Gamma(m\alpha+\mu)}{B(\mu,\gamma-\mu)\Gamma(2m\alpha+\gamma)}\Phi_{\lambda,m\alpha+\mu;2m\alpha+\gamma}(z,s,a). \end{aligned} \tag{4.2}$$

$(\Re(s)>0,\Re(m\alpha+\mu)>0,\Re(2m\alpha+\gamma)>0,0<\Re(\mu)<\Re(\gamma))$, where $\Gamma^{(\rho,\sigma)}(s)$ and $\Phi_{\lambda,\mu;\gamma}(z,s,a)$ are the extended Gamma function and Hurwitz-Lerch Zeta function, respectively ([7] and [3, p.313]).

Proof. Using the definition (4.1) on the L.H.S of (4.2) and then expanding $\Phi^{(\rho,\sigma;m)}_{\lambda,\mu;\gamma}(z,s,a;p)$ with the help of (2.1), we get

$$\begin{aligned} &M\left\{\Phi^{(\rho,\sigma;m)}_{\lambda,\mu;\gamma}(z,s,a;p)\right\}\\ &=\int_0^\infty p^{\alpha-1}\left(\sum_{n=0}^{\infty}\frac{(\lambda)_nB_p^{(\rho,\sigma;m)}(\mu+n,\gamma-\mu)}{n!B(\mu,\gamma-\mu)}\frac{z^n}{(n+a)^s}\right)dp. \end{aligned}$$

Now, changing the order of integration and summation, we get

$$\begin{aligned}
&M\left\{\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a;p)\right\}\\
&=\sum_{n=0}^{\infty}\frac{(\lambda)_n z^n}{n!(n+a)^s B(\mu,\gamma-\mu)}\int_0^{\infty}p^{\alpha-1}B_p^{(\rho,\sigma;m)}(\mu+n,\gamma-\mu)dp\\
&=\sum_{n=0}^{\infty}\frac{(\lambda)_n z^n}{n!(n+a)^s}\frac{B(m\alpha+\mu+n,\gamma-\mu+m\alpha)}{B(\mu,\gamma-\mu)}\Gamma^{(\rho,\sigma)}(s),
\end{aligned}$$

where $\Gamma^{(\rho,\sigma)}(s)$ is the generalized Gamma function given in Parmar [7].

Now, expanding $B(m\alpha+\mu+n,\gamma-\mu+m\alpha)$ in terms of Gamma function and then by using the result $\Gamma(\lambda+n)=\Gamma(\lambda)(\lambda)_n$, we get

$$\begin{aligned}
&M\left\{\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a;p)\right\}\\
&=\frac{\Gamma^{(\rho,\sigma)}(s)\Gamma(m\alpha+\mu)}{B(\mu,\gamma-\mu)\Gamma(2m\alpha+\gamma)}\sum_{n=0}^{\infty}\frac{(\lambda)_n(m\alpha+\mu)_n}{n!(2m\alpha+\gamma)_n}\frac{z^n}{(n+a)^s}.
\end{aligned}$$

Finally, using the definition of Hurwitz-Lerch Zeta function (HLZF) given in [3, p.313], we arrive at our required result.

5. Generating relations

Theorem 5.1. *For $p\geq 0$, $\lambda\in\mathbb{C}$ and $|\, t\,|<1$, the following generating function of $\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a;p)$ holds true:*

$$\sum_{n=0}^{\infty}(\lambda)_n\Phi_{\lambda+n,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a;p)\frac{t^n}{n!}=(1-t)^{-\lambda}\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}\left(\frac{z}{1-t},s,a;p\right). \tag{5.1}$$

Proof. Let us denote the left hand side of (5.1) by L_1. By using the series expression given in (2.1) into L_1, we find that

$$L_1=\sum_{n=0}^{\infty}(\lambda)_n\left\{\sum_{k=0}^{\infty}\frac{(\lambda+n)_k B_p^{(\rho,\sigma;m)}(\mu+k,\gamma-\mu)}{B(\mu,\gamma-\mu)}\frac{z^k}{k!(k+a)^s}\right\}\frac{t^n}{n!}, \tag{5.2}$$

which by changing the order of summation, gives

$$L_1=\sum_{k=0}^{\infty}\frac{(\lambda)_k B_p^{(\rho,\sigma;m)}(\mu+k,\gamma-\mu)}{B(\mu,\gamma-\mu)}\left\{\sum_{n=0}^{\infty}(\lambda+k)_n\frac{t^n}{n!}\right\}\frac{z^k}{k!(k+a)^s}. \tag{5.3}$$

Now, applying the following binomial expansion:

$$(1-\lambda)^{-(\lambda+k)}=\sum_{n=0}^{\infty}(\lambda+k)_n\frac{t^n}{n!},(|\, t\,|<1),$$

for evaluating the inner sum in (5.3) and then by using (2.1), we get our desired result.

Theorem 5.2. *For $p \geq 0$, $\lambda \in \mathbb{C}$ and $\mid t \mid < \mid a \mid; s \neq 1$, the following generating function of $\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a;p)$ holds true:*

$$\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a-t;p) = \sum_{n=0}^{\infty} \frac{(s)_n}{n!} \Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}(z,s+n,a;p)\, t^n. \tag{5.4}$$

Proof. Let us denote the left hand side of (5.4) by L_2. Then by using (2.1), we get

$$\begin{aligned}
L_2 &= \sum_{l=0}^{\infty} \frac{(\lambda)_l B_p^{(\rho,\sigma;m)}(\mu+l,\gamma-\mu)}{B(\mu,\gamma-\mu)} \frac{z^l}{l!(l+a-t)^s} \\
&= \sum_{l=0}^{\infty} \frac{(\lambda)_l B_p^{(\rho,\sigma;m)}(\mu+l,\gamma-\mu)}{B(\mu,\gamma-\mu)} \frac{z^l}{l!(l+a)^s}(1-\frac{t}{l+a})^{-s} \\
&= \sum_{l=0}^{\infty} \frac{(\lambda)_l B_p^{(\rho,\sigma;m)}(\mu+l,\gamma-\mu)}{B(\mu,\gamma-\mu)} \frac{z^l}{l!(l+a)^s} \left\{ \sum_{n=0}^{\infty} \frac{(s)_n}{n!} \left(\frac{t}{l+a} \right)^n \right\} \\
&= \sum_{n=0}^{\infty} \frac{(s)_n}{n!} \left(\sum_{l=0}^{\infty} \frac{(\lambda)_l B_p^{(\rho,\sigma;m)}(\mu+l,\gamma-\mu)}{B(\mu,\gamma-\mu)} \frac{z^l}{l!(l+a)^{s+n}} \right) t^n.
\end{aligned}$$

Finally, by making use of (2.1), we get the desired assertion (5.4).

Remark 5.1. For $m = 1$, the generating function (5.1) and (5.4) asserted by Theorem 5.1 and Theorem 5.2, respectively, were derived earlier by Parmar et al. [9].

6. Derivation of $\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a;p)$

In this section we provide a differential formula of our new extended Hurwitz-Lerch Zeta function (EHLZF) $\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a;p)$.

Theorem 6. *The following differential formula holds true:*

$$\frac{d^r}{dz^r}\left[\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a;p)\right] = \frac{(\mu)_r(\lambda)_r}{(\gamma)_r} \Phi_{\lambda+r,\mu+r;\gamma+r}^{(\rho,\sigma;m)}(z,s,a+r;p), \tag{6.1}$$

where $r \in \mathbb{N} = \{1, 2, 3, \cdots\}$.

Proof. Taking the derivative of $\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a;p)$ with respect to z, we get

$$\begin{aligned}
\frac{d}{dz}\left[\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a;p)\right] &= \frac{d}{dz}\left[\sum_{n=0}^{\infty} \frac{(\lambda)_n B_p^{(\rho,\sigma;m)}(\mu+n,\gamma-\mu)}{n! B(\mu,\gamma-\mu)} \frac{z^n}{(n+a)^s}\right] \\
= \sum_{n=1}^{\infty} \frac{(\lambda)_n B_p^{(\rho,\sigma;m)}(\mu+n,\gamma-\mu)}{(n-1)! B(\mu,\gamma-\mu)} \frac{z^{n-1}}{(n+a)^s}.
\end{aligned}$$

Replacing n by $n+1$, we get

$$\begin{aligned}
&\frac{d}{dz}\left[\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a;p)\right]\\
&=\frac{\mu\lambda}{\gamma}\sum_{n=0}^{\infty}\frac{(\lambda+1)_n B_p^{(\rho,\sigma;m)}(\mu+n+1,\gamma-\mu)}{n!B(\mu+1,\gamma-\mu)}\frac{z^n}{((n+1+a)^s}\\
&=\frac{\mu\lambda}{\gamma}\Phi_{\lambda+1,\mu+1;\gamma+1}^{(\rho,\sigma;m)}(z,s,a+1;p).
\end{aligned}$$

Recursive of this procedure yields us the desired result (6.1).

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Accepted: February 3, 2020