Nonexistence of global solutions of some nonlinear ultra-parabolic equations on the Heisenberg group

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Abstract. This article provides sufficient conditions for non existence Global weak solutions for non-local and non-linear equivalent equations on $\mathbb{H}^{\mathbb{N}} \times (0, \infty) \times (0, \infty)$, where $\mathbb{H}^{\mathbb{N}}$ is the Heisenberg group. Our method of proof relies on a suitable choice of a test function and the weak formulation approach of the sought for solutions.
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1. Introduction

In this article we are concerned with the nonexistence of global solutions of nonlocal nonlinear ultra-parabolic two-times equation posed on the Heisenberg group.

We start with the equation:

$$\mathbf{D}^{\alpha_1}_{0|t_1}(u) + \mathbf{D}^{\alpha_2}_{0|t_2}(u) + (-\Delta_{\mathbb{H}})^{\alpha/2}(|u|^m) = |u|^p \tag{1.1}$$

posed for $\omega = (\eta, t_1, t_2) \in Q = \mathbb{H}^N \times \mathbb{R}^+ \times \mathbb{R}^+$, $N \in \mathbb{N}$ and supplemented with the initial conditions

$$u(\eta, t_1, 0) = u_1(\eta, t_1) \ , u(\eta, 0, t_2) = u_2(\eta, t_2)$$

Here, $p > 1$ are real number, $m \in \mathbb{N}$ and where for $0 < \alpha_1 < \alpha_2 < 1$ and $\mathbf{D}^{\alpha_1}_{0|t_1}$, $\mathbf{D}^{\alpha_2}_{0|t_2}$ is the fractional derivative in the sense of the so-called Caputo's. Then, we extend our results to the system of two equations

$$\begin{cases} \mathbf{D}^{\alpha_1}_{0|t_1}(u) + \mathbf{D}^{\alpha_2}_{0|t_2}(u) + (-\Delta_{\mathbb{H}})^{\alpha/2}(|u|^m) = |v|^p, \\ \mathbf{D}^{\beta_1}_{0|t_1}(v) + \mathbf{D}^{\beta_2}_{0|t_2}(v) + (-\Delta_{\mathbb{H}})^{\beta/2}(|v|^n) = |u|^q \end{cases} \tag{1.2}$$

posed for $\omega = (\eta, t_1, t_2) \in Q = \mathbb{H}^N \times \mathbb{R}^+ \times \mathbb{R}^+$, $N \in \mathbb{N}$ and supplemented with the initial conditions

$$u(\eta, t_1, 0) = u_1(\eta, t_1) \ u(\eta, 0, t_2) = u_2(\eta, t_2),$$
$$v(\eta, t_1, 0) = v_1(\eta, t_1) \ v(\eta, 0, t_2) = v_2(\eta, t_2).$$

Here p, q are positive real numbers and $0 < \alpha_1 < \alpha_2 < 1$, $0 < \beta_1 < \beta_2 < 1$,$0 < \alpha, \beta \leq 2$.

Heisenberg group. The Heisenberg group $\mathbb{H}^N$, whose points will be denoted by $\eta = (x, y, \tau)$, is the Lie group $(\mathbb{R}^{2N+1}, \circ)$ with the non-commutative group operation $\circ$ defined by

$$\eta \circ \tilde{\eta} = (x + \tilde{x},\ y + \tilde{y},\ \tau + \tilde{\tau} + 2\left(\langle x, \tilde{y}\rangle - \langle \tilde{x}, y\rangle\right)),\quad \eta^{-1} = (-x,\ -y,\ \tau),$$

where $\langle ., . \rangle$ is the usual inner product in $\mathbb{R}^{N}$[1].

The Laplacian $\Delta_{\mathbb{H}}$ over $\mathbb{H}$ is obtained from the vector fields

$$X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial \tau} \qquad \text{and} \qquad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial \tau}$$

as

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{N} \left(X_i^2 + Y_i^2\right), \tag{1.3}$$

An explicit calculation gives us the expression

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{N} \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial \tau} - 4x_i \frac{\partial^2}{\partial y_i \partial \tau} + 4\left(x_i^2 + y_i^2\right) \frac{\partial^2}{\partial \tau^2} \right). \tag{1.4}$$

The operator $\Delta_{\mathbb{H}}$ is a degenerate elliptic operator verifying the so-called Hormander condition of order 1. It is invariant by left multiplication in the group since

$$\Delta_{\mathbb{H}} \left(u \left(\eta \circ \tilde{\eta}\right)\right) = \left(\Delta_{\mathbb{H}} u\right) \left(\eta \circ \tilde{\eta}\right),\ \forall \left(\eta, \tilde{\eta}\right) \in \mathbb{H}^N \times \mathbb{H}^N.$$

The homogeneous norm of the space is

$$|\eta|_{\mathbb{H}} = \left(\tau^2 + \left(\sum_{i=1}^{N} \left(x_i^2 + y_i^2\right) \right)^2 \right)^{1/4} \tag{1.5}$$

and the natural distance is accordingly defined $d\left(\eta, \tilde{\eta}\right) = |\tilde{\eta}^{-1} \circ \eta|_{\mathbb{H}}$. It is also important to observe that, $\eta \to |\eta|_{\mathbb{H}}$ is homogeneous of degree one compared to the natural group of dilatations

$$\delta_\lambda\left(\eta\right) = \left(\lambda x, \lambda y, \lambda^2 \tau\right), \tag{1.6}$$

whose Jacobian determinant is λ^Λ, where $\Lambda = 2N + 2$, is the homogeneous dimension of $\mathbb{H}$. Note also, that the $\Delta_{\mathbb{H}}$ operator is homogeneous of degree 2 with respect to the dilatation δ_λ defined in (1.6), namely

$$\Delta_{\mathbb{H}} = \lambda^2 \delta_\lambda \left(\Delta_{\mathbb{H}}\right).$$

1. here $\langle x, y\rangle = \sum_{i=1}^{N} x_i y_i$

See [3], [4], [9], [11], [13].

Now, we call *sub-elliptic gradient*

$$\nabla_{\mathbb{H}} = (X, Y) = (X_1, \ldots, X_N, Y_1, \ldots, Y_N) . \tag{1.7}$$

A remarkable property of the Kohn Laplacian is that a fundamental solution of $-\Delta_{\mathbb{H}}$ with pole at zero is given by

$$\Gamma(\eta) = \frac{C_\Lambda}{|\eta|_{\mathbb{H}}^{\Lambda-2}}, \tag{1.8}$$

where C_Λ is a suitable positive constant.

A basic role in the functional analysis on the Heisenberg group is played by the following Sobolev-type inequality

$$\|v\|_{\Lambda^*}^2 \leq c\|\nabla_{\mathbb{H}} v\|_2^2, \forall v \in \mathrm{C}_0^\infty\left(\mathbb{H}^N\right), \tag{1.9}$$

where $\Lambda^* = \frac{2\Lambda}{\Lambda-2}$ and c is a positive constant.

This inequality ensures in particular that for every domain Ω the function

$$\|v\| \leq \|\nabla_{\mathbb{H}} v\|_2$$

is a norm on $\mathrm{C}_0^\infty(\Omega)$. We denote by $\mathrm{S}_0^1(\Omega)$ the closure of $\mathrm{C}_0^\infty(\Omega)$ with respect to this norm; $\mathrm{S}_0^1(\Omega)$ becomes a Hilbert space with the inner product

$$< u, v >_{\mathrm{S}_0^1} = \int_\Omega < \nabla_{\mathbb{H}} u, \nabla_{\mathbb{H}} v > .$$

Fractional powers of sub-elliptic Laplacians. Here, we recall a result on fractional powers of sub-Laplacian in the Heisenberg group. Let $\mathcal{N}(t, x)$ be the fundamental solution of $\Delta_{\mathbb{H}} + \frac{\partial}{\partial t}$. For all $0 < \beta < 4$, the integral

$$R_\beta(x) = \frac{1}{\Gamma\left(\frac{\beta}{2}\right)} \int_0^{+\infty} t^{\frac{\beta}{2}-1} \mathcal{N}(t, x)\, dt$$

converges absolutely for $x \neq 0$. If $\beta < 0, \beta \notin \{0, -2, -4, \ldots\}$, then

$$\tilde{R}_\beta(x) = \frac{\frac{\beta}{2}}{\Gamma\left(\frac{\beta}{2}\right)} \int_0^{+\infty} t^{\frac{\beta}{2}-1} \mathcal{N}(t, x)\, dt$$

defines a smooth function in $\mathbb{H}\backslash\{0\}$, since $t \mapsto \mathcal{N}(t, x)$, vanishes of infinite order as $t \to 0$ if $x \neq 0$. In addition, $\tilde{R}_\beta$ is positive and $\mathbb{H}$-homogeneous of degree $\beta - 4$.

1.1 Theorem

For every $v \in \mathcal{S}(\mathbb{H})^2$ $(-\Delta_{\mathbb{H}})^s v \in L^2(\mathbb{H})$ and

$$(-\Delta_{\mathbb{H}})^s v(x) = \int_{\mathbb{H}} (v(x \circ y) - v(x) - \chi(y) < \nabla_{\mathbb{H}} v(x), y >) \tilde{R}_{-2s}(y)\, dy,$$

where χ is the characteristic function of the unit ball $B_\rho(0,1)$, ($\rho(x) = R_{2-\alpha}^{\frac{-1}{2-\alpha}}(x)$, $0 < \alpha < 2$, ρ is an $\mathbb{H}$-homogeneous norm in $\mathbb{H}$ smooth outside the origin).

1.2 Note

Proof of the Theorem 1.1 (see [2]).

1.3 Note

For $\alpha = 2$ in equation (1.1) (see, ([1], [6], [11]).

2. Preliminaries

The nonlocal operator (the left-sided Riemann-Liouville) $D^\alpha_{0|t}$ is defined, for a an absolutely continuous function $g : \mathbb{R}^+ \longrightarrow \mathbb{R}$ by

$$\left(D^\alpha_{0|t}\right) g(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{g(\tau)}{(t-\tau)^\alpha} d\tau$$

and $\Gamma(\alpha) = \int_0^\infty r^{\alpha-1} e^{-r} dr$ is the Euler gamma function. And the right-sided Riemann-Liouville derivatives of order $0 < \alpha < 1$. are defined, by:

$$\left(D^\alpha_{t|T}\right) g(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T \frac{g(\tau)}{(\tau - t)^\alpha} d\tau.$$

Note that for a differentiable function g, we have the left-sided Caputo derivatives of order α are defined as:

$$\mathbf{D}^\alpha_{0|t}(g)(t) = D^\alpha_{0|t}(g - g(0))(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{g'(\tau)}{(\tau - t)^\alpha} d\tau.$$

Finally, taking into account the following integration by parts formula:

$$\int_0^T f(t)\, D^\alpha_{0|t} g(t)\, dt = \int_0^T D^\alpha_{t|T} f(t)\, g(t)\, dt.$$

Now, we define the regular function $\psi : \psi \in C_0^2(\mathbb{R}^+)^3$ by

$$\psi(\xi) = \begin{cases} 1, & \text{if } 0 \le \xi \le 1, \\ \searrow, & \text{if } 1 \le \xi \le 2, \\ 0, & \text{if } \xi \ge 2, \end{cases} \tag{2.1}$$

2. Schwartz's class
3. Space defined and continuous functions and differentiable twice and compact support on $\mathbb{R}^+$

which will be used hereafter.

3. Résults

3.1 Definition

A locally integrable function $u \in L^m_{loc}(Q_T) \cap L^p_{loc}(Q_T)$ is called a local weak solution of (1.1) in Q_T $\left(Q_T = \mathbb{H}^N \times [0,T] \times [0,T]\right)$ subject to the initial data $u_1, u_2 \in L^1_{loc}\left(\mathbb{H}^N \times [0,T]\right)$ if the equality

$$\begin{aligned} &\int_{Q_T} |u|^p \varphi d\omega + \int_{Q_T} u_2 D^{\alpha_1}_{t_1|T} \varphi d\omega + \int_{Q_T} u_1 D^{\alpha_2}_{t_2|T} \varphi d\omega \\ &= \int_{Q_T} u D^{\alpha_1}_{t_1|T} \varphi d\omega + \int_{Q_T} u D^{\alpha_2}_{t_2|T} \varphi d\omega + \int_{Q_T} |u|^m \left(-\Delta_{\mathbb{H}}\right)^{\alpha/2} \varphi d\omega \end{aligned} \tag{3.1}$$

is satisfied for any φ be a smooth test function ($\varphi \in C^\infty_0(Q_T)$) with

$$\varphi\left(.,T,.\right) = \varphi\left(.,.,T\right) = 0, \ \ \varphi \geq 0, \ \ d\omega = d\eta dt_1 dt_2$$

and the solution is called global if $T = +\infty$.

3.2 Theorem

Let $1 < m < p < p_c$, where

$$p_c = m + \frac{m\alpha - (m-1)(\frac{\alpha}{\alpha_1} + \frac{\alpha}{\alpha_2})}{2N + 2 - \alpha + (\frac{\alpha}{\alpha_1} + \frac{\alpha}{\alpha_2})}, \ \ \text{(c for critical)}$$

and

$$\int_Q u_2 D^{\alpha_1}_{t_1|T} \varphi d\omega > 0, \quad \int_Q u_1 D^{\alpha_2}_{t_2|T} \varphi d\omega > 0.$$

Then, (1.1) does not have a nontrivial global weak solution. For the proof, we need to recall the following proposition from Proposition 2.3.

3.3 Proposition

Consider a convex function $F \in C^2(\mathbb{R})$. Assume that $\varphi \in C^\infty_0(\mathbb{R}^{2N+1})$, then

$$F'(\varphi)(-\Delta_{\mathbb{H}})^{\alpha/2}\varphi \geq (-\Delta_{\mathbb{H}})^{\alpha/2} F(\varphi). \tag{3.2}$$

In particular, if $F(0) = 0$ and $\varphi \in C^\infty_0(\mathbb{R}^{2N+1})$, then

$$\int_{\mathbb{R}^{2N+1}} F'(\varphi)(-\Delta_{\mathbb{H}})^{\alpha/2}\varphi d\eta \geq 0. \tag{3.3}$$

Let us mention that hereafter we will use inequality (2.1) for $F(\varphi) = \varphi^\sigma$, $\sigma \gg 1$[4], $\varphi \geq 0$; in this case it reads

$$\sigma\varphi^{\sigma-1}(-\Delta_{\mathbb{H}})^{\alpha/2}\varphi \geq (-\Delta_{\mathbb{H}})^{\alpha/2}\varphi^\sigma. \tag{3.4}$$

We need the following Lemma taken from [32].

4. σ is much larger than 1

3.4 Lemma

Let $f \in L^1\left(\mathbb{R}^{2N+1}\right)$ and $\int_{\mathbb{R}^{2N+1}} f d\eta \geq 0$. Then there exists a test function $0 \leq \varphi \leq 1$ such that

$$\int_{\mathbb{R}^{2N+1}} f\varphi d\eta \geq 0. \tag{3.5}$$

3.5 Note

Let us set
$$\int_{Q_T} = \int_0^T \int_0^T \int_{\mathbb{R}^{2N+1}}, \quad \int_Q = \int_0^\infty \int_0^\infty \int_{\mathbb{R}^{2N+1}}.$$

Proof of Theorem 3.2. The proof is by contradiction. For that, let u be a solution and φ be a smooth nonnegative test function such that

$$\begin{aligned}
&\mathcal{A}(\varphi) = \int_Q |D^{\alpha_1}_{t_1|T}\varphi^\sigma|^{\frac{p}{p-1}} \varphi^{\frac{-\sigma}{p-1}} d\omega < \infty,\\
&\mathcal{B}(\varphi) = \int_Q |D^{\alpha_2}_{t_2|T}\varphi^\sigma|^{\frac{p}{p-1}} \varphi^{\frac{-\sigma}{p-1}} d\omega < \infty,\\
&\mathcal{K}(\varphi) = \int_Q |(-\Delta_{\mathbb{H}})^{\alpha/2}\varphi|^{\frac{p}{p-m}} \varphi^{(\sigma-\frac{p}{p-m})} d\omega < \infty.
\end{aligned} \tag{3.6}$$

Then, taking φ^σ, $\sigma \gg 1$ instead of φ in (3.1) and using inequality (3.4), we obtain

$$\begin{aligned}
&\int_Q |u|^p \varphi^\sigma d\omega + \int_Q u_2 D^{\alpha_1}_{t_1|T}\varphi^\sigma d\omega + \int_Q u_1 D^{\alpha_2}_{t_2|T}\varphi^\sigma d\omega\\
&= \int_Q u D^{\alpha_1}_{t_1|T}\varphi^\sigma d\omega + \int_Q u D^{\alpha_2}_{t_2|T}\varphi^\sigma d\omega + \int_Q |u|^m \left(-\Delta_{\mathbb{H}}\right)^{\alpha/2} \varphi^\sigma d\omega\\
&\leq \int_Q u D^{\alpha_1}_{t_1|T}\varphi^\sigma d\omega + \int_Q u D^{\alpha_2}_{t_2|T}\varphi^\sigma d\omega + \int_Q |u|^m \sigma\varphi^{\sigma-1} \left(-\Delta_{\mathbb{H}}\right)^{\alpha/2} \varphi d\omega.
\end{aligned}$$

- For $\int_Q u D^{\alpha_1}_{t_1|T}\varphi^\sigma d\omega$ by means of the ε-Young's inequality $ab \leq \varepsilon a^p + C(\varepsilon) b^{\frac{p}{p-1}}$, $\frac{1}{p} + \frac{p-1}{p} = 1, a \geq 0, b \geq 0$, we obtain

$$u D^{\alpha_1}_{t_1|T}\varphi^\sigma \leq |u||D^{\alpha_1}_{t_1|T}\varphi^\sigma| = \varphi^{\frac{\sigma}{p}}\varphi^{\frac{-\sigma}{p}}|u||D^{\alpha_1}_{t_1|T}\varphi^\sigma| = |u|\varphi^{\frac{\sigma}{p}}|D^{\alpha_1}_{t_1|T}\varphi^\sigma|\varphi^{\frac{-\sigma}{p}}$$

because $\varphi^{\frac{\sigma}{p}}\varphi^{\frac{-\sigma}{p}} = \varphi^{\frac{\sigma}{p}-\frac{\sigma}{p}} = \varphi^0 = 1$ if we pose $a = |u|\varphi^{\frac{\sigma}{p}}$ and $b = |D^{\alpha_1}_{t_1|T}\varphi^\sigma|\varphi^{\frac{-\sigma}{p}}$, then

$$u D^{\alpha_1}_{t_1|T}\varphi^\sigma \leq ab \leq \varepsilon a^p + C(\varepsilon) b^{\frac{p}{p-1}} = \varepsilon\left(|u|\varphi^{\frac{\sigma}{p}}\right)^p + C(\varepsilon)\left(|D^{\alpha_1}_{t_1|T}\varphi^\sigma|\varphi^{\frac{-\sigma}{p}}\right)^{\frac{p}{p-1}}$$

$$\Updownarrow$$

$$u D^{\alpha_1}_{t_1|T}\varphi^\sigma \leq \varepsilon|u|^p\varphi^\sigma + C(\varepsilon)|D^{\alpha_1}_{t_1|T}\varphi^\sigma|^{\frac{p}{p-1}}\varphi^{\frac{-\sigma}{p-1}}$$

$$\Downarrow$$

$$\int_Q uD^{\alpha_1}_{t_1|T}\varphi^\sigma d\omega \leq \varepsilon\int_Q |u|^p\varphi^\sigma d\omega + C(\varepsilon)\int_Q |D^{\alpha_1}_{t_1|T}\varphi^\sigma|^{\frac{p}{p-1}}\varphi^{\frac{-\sigma}{p-1}}d\omega$$

$$\Updownarrow$$

$$(I)\qquad \int_Q uD^{\alpha_1}_{t_1|T}\varphi^\sigma d\omega \leq \varepsilon\int_Q |u|^p\varphi^\sigma d\omega + C(\varepsilon)\mathcal{A}(\varphi)$$

• For $\int_Q uD^{\alpha_2}_{t_2|T}\varphi^\sigma d\omega$ we take the previous method with placing t_2 place t_1 and α_2 place α_1 to get the

$$(II)\qquad \int_Q uD^{\alpha_2}_{t_2|T}\varphi^\sigma d\omega \leq \varepsilon\int_Q |u|^p\varphi^\sigma d\omega + C(\varepsilon)\mathcal{B}(\varphi)$$

• For $\int_Q |u|^m\sigma\varphi^{\sigma-1}\left(-\Delta_{\mathbb{H}}\right)^{\alpha/2}\varphi d\omega$ by means of the ε-Young's inequality $ab \leq \varepsilon a^{\frac{p}{m}} + C(\varepsilon)b^{\frac{p}{p-m}}$, $\frac{m}{p}+\frac{p-m}{p} = 1, a \geq 0, b \geq 0$, we obtain $|u|^m\sigma\varphi^{\sigma-1}\left(-\Delta_{\mathbb{H}}\right)^{\alpha/2}\varphi = 1.|u|^m\sigma\varphi^{\sigma-1}\left(-\Delta_{\mathbb{H}}\right)^{\alpha/2}\varphi = \varphi^{\frac{m\sigma}{p}}\varphi^{\frac{-m\sigma}{p}}|u|^m\sigma\varphi^{\sigma-1}\left(-\Delta_{\mathbb{H}}\right)^{\alpha/2}\varphi$ because $\varphi^{\frac{m\sigma}{p}}\varphi^{\frac{-m\sigma}{p}} = \varphi^{\frac{m\sigma}{p}-\frac{m\sigma}{p}} = \varphi^0 = 1$, if we pose $a = |u|^m\varphi^{\frac{m\sigma}{p}}$ and $b = |\left(-\Delta_{\mathbb{H}}\right)^{\alpha/2}\varphi|\sigma\varphi^{\sigma-1-\frac{m\sigma}{p}}$, then

$$|u|^m\sigma\varphi^{\sigma-1}\left(-\Delta_{\mathbb{H}}\right)^{\alpha/2}\varphi \leq ab \leq \varepsilon a^{\frac{p}{m}} + C(\varepsilon)b^{\frac{p}{p-m}}$$

$$\varepsilon a^{\frac{p}{m}} + C(\varepsilon)b^{\frac{p}{p-m}} = \varepsilon\left(|u|^m\varphi^{\frac{m\sigma}{p}}\right)^{\frac{p}{m}} + C(\varepsilon)\left(|\left(-\Delta_{\mathbb{H}}\right)^{\alpha/2}\varphi|\sigma\varphi^{\sigma-1-\frac{m\sigma}{p}}\right)^{\frac{p}{p-m}}$$

$$\Updownarrow$$

$$|u|^m\sigma\varphi^{\sigma-1}\left(-\Delta_{\mathbb{H}}\right)^{\alpha/2}\varphi \leq \varepsilon|u|^p\varphi^\sigma + C(\varepsilon)|\left(-\Delta_{\mathbb{H}}\right)^{\alpha/2}\varphi|^{\frac{p}{p-m}}\sigma^{\frac{p}{p-m}}\varphi^{(\sigma-1-\frac{m\sigma}{p})\frac{p}{p-m}}$$

$$\Updownarrow$$

$$|u|^m\sigma\varphi^{\sigma-1}\left(-\Delta_{\mathbb{H}}\right)^{\alpha/2}\varphi \leq \varepsilon|u|^p\varphi^\sigma + C(\varepsilon)|\left(-\Delta_{\mathbb{H}}\right)^{\alpha/2}\varphi|^{\frac{p}{p-m}}\sigma^{\frac{p}{p-m}}\varphi^{(\sigma-\frac{p}{p-m})}$$

$$\Downarrow$$

$$\begin{aligned}&\int_Q |u|^m\sigma\varphi^{\sigma-1}\left(-\Delta_{\mathbb{H}}\right)^{\alpha/2}\varphi d\omega\\ &\leq \varepsilon\int_Q |u|^p\varphi^\sigma d\omega + C(\varepsilon)\sigma^{\frac{p}{p-m}}\int_Q |\left(-\Delta_{\mathbb{H}}\right)^{\alpha/2}\varphi|^{\frac{p}{p-m}}\varphi^{(\sigma-\frac{p}{p-m})}d\omega\end{aligned}$$

$$\Updownarrow$$

$$(III)\qquad \int_Q |u|^m\sigma\varphi^{\sigma-1}\left(-\Delta_{\mathbb{H}}\right)^{\alpha/2}\varphi d\omega \leq \varepsilon\int_Q |u|^p\varphi^\sigma d\omega + C(\varepsilon)\sigma^{\frac{p}{p-m}}\mathcal{K}(\varphi).$$

Now, we choose $\varepsilon = \frac{1}{6}$ and $C = \max\left\{C(\varepsilon), C(\varepsilon)\sigma^{\frac{p}{p-m}}\right\}$ and the (I), (II), (III) we obtain

$$\int_Q |u|^p \varphi^\sigma d\omega + \int_Q u_2 D^{\alpha_1}_{t_1|T} \varphi^\sigma d\omega + \int_Q u_1 D^{\alpha_2}_{t_2|T} \varphi^\sigma d\omega$$

$$\leq \frac{1}{2}\int_Q |u|^p \varphi^\sigma d\omega + C\left(\mathcal{A}(\varphi) + \mathcal{B}(\varphi) + \mathcal{K}(\varphi)\right). \tag{3.7}$$

We choose the test function $\varphi(\eta, t_1, t_2)$, in the form

$$\varphi(\eta, t_1, t_2) = \varphi_1(\eta)\,\varphi_2(t_1)\,\varphi_3(t_2)\,, \tag{3.8}$$

where $\varphi_1(\eta) = \psi(\frac{\tau^2+|x|^4+|y|^4}{R^4})$, and $\varphi_2(t_1) = \psi(\frac{t_1}{R^{\rho_1}})$, and $\varphi_3(t_2) = \psi(\frac{t_2}{R^{\rho_2}})$, and $\rho_1 = \frac{\alpha(p-1)}{\alpha_1(p-m)}$, and $\rho_2 = \frac{\alpha(p-1)}{\alpha_2(p-m)}$. Set

$$\Omega_1 = \left\{\tilde{\eta} \in \mathbb{H};\ 0 < \tilde{\tau}^2 + |\tilde{x}|^4 + |\tilde{y}|^4 \leq 2\right\},$$
$$\Omega_2 = \left\{\tilde{t_1};\ 0 \leq \tilde{t_1} \leq 2\right\},$$
$$\Omega_3 = \left\{\tilde{t_2};\ 0 \leq \tilde{t_2} \leq 2\right\},$$

we apply the change of next variables $\tilde{\tau} = R^{-2}\tau$, $\tilde{x} = R^{-1}x$, $\tilde{y} = R^{-1}y$, $\tilde{t}_1 = R^{-\rho_1}t_1$, $\tilde{t}_2 = R^{-\rho_2}t_2$, we obtain the estimates,

$$\mathcal{A}(\varphi) \leq \mathbf{A}R^{\mathbf{a}},\quad \mathcal{B}(\varphi) \leq \mathbf{B}R^{\mathbf{b}},\quad \mathcal{K}(\varphi) \leq \mathbf{K}R^{\mathbf{k}} \tag{3.9}$$

with

$$\mathbf{a} = -\frac{\alpha_1\rho_1 p}{p-1} + 2N + 2 + \rho_1 + \rho_2 = -\frac{\alpha p}{p-m} + 2N + 2 + \rho_1 + \rho_2,$$
$$\mathbf{b} = -\frac{\alpha_2\rho_2 p}{p-1} + 2N + 2 + \rho_1 + \rho_2 = -\frac{\alpha p}{p-m} + 2N + 2 + \rho_1 + \rho_2,$$
$$\mathbf{k} = -\frac{\alpha p}{p-m} + 2N + 2 + \rho_1 + \rho_2,$$

we put $\mathbf{v} = -\frac{\alpha p}{p-m} + 2N + 2 + \rho_1 + \rho_2$. Then

$$\mathcal{A}(\varphi) \leq \mathbf{A}R^{\mathbf{v}},\quad \mathcal{B}(\varphi) \leq \mathbf{B}R^{\mathbf{v}},\quad \mathcal{K}(\varphi) \leq \mathbf{K}R^{\mathbf{v}} \tag{3.10}$$

the constants $\mathbf{A}$; $\mathbf{B}$; $\mathbf{K}$ are $\mathcal{A}(\varphi)$ and $\mathcal{B}(\varphi)$ and $\mathcal{K}(\varphi)$ evaluated on $\Omega_1 \times \Omega_2 \times \Omega_3$. Now, if

$$-\frac{\alpha p}{p-m} + 2N + 2 + \rho_1 + \rho_2 < 0 \iff p < p_c$$

by letting $R \to \infty$ in (2.6), we obtain

$$\int_Q |u|^p d\omega = 0 \quad \Rightarrow \quad u \equiv 0$$

this is a contradiction. □

4. System of fractional equations

We consider

$$\begin{cases}\mathbf{D}^{\alpha_1}_{0|t_1}(u)+\mathbf{D}^{\alpha_2}_{0|t_2}(u)+(-\Delta_{\mathbb{H}})^{\alpha/2}(|u|^m)=|v|^p\\ \mathbf{D}^{\beta_1}_{0|t_1}(v)+\mathbf{D}^{\beta_2}_{0|t_2}(v)+(-\Delta_{\mathbb{H}})^{\beta/2}(|v|^n)=|u|^q\end{cases}$$

posed for $\omega=(\eta,t_1,t_2)\in Q=\mathbb{H}^N\times\mathbb{R}^+\times\mathbb{R}^+$, $N\in\mathbb{N}$ and supplemented with the initial conditions $u(\eta,t_1,0)=u_1(\eta,t_1)$, $u(\eta,0,t_2)=u_2(\eta,t_2)$, $v(\eta,t_1,0)=v_1(\eta,t_1)$, $v(\eta,0,t_2)=v_2(\eta,t_2)$, Here, p,q are positive real numbers and $0<\alpha_1<\alpha_2<1$, $0<\beta_1<\beta_2<1$,$0<\alpha,\beta\leq 2$.

Let us set

$$I_0=\int_Q u_2 D^{\alpha_1}_{t_1|T}\varphi d\omega+\int_Q u_1 D^{\alpha_2}_{t_2|T}\varphi d\omega,$$
$$J_0=\int_Q v_2 D^{\beta_1}_{t_1|T}\varphi d\omega+\int_Q v_1 D^{\beta_2}_{t_2|T}\varphi d\omega$$

where $d\omega=d\eta dt_1dt_2$.

4.1 Definition

We say that $(u,v)\in\left(L^q_{loc}(Q)\cap L^m_{loc}(Q)\right)\times\left(L^p_{loc}(Q)\cap L^n_{loc}(Q)\right)$ is a weak formulation to system (1.2) if

$$\begin{cases}\int_Q |v|^p\varphi d\omega+I_0=\int_Q uD^{\alpha_1}_{t_1|T}\varphi d\omega+\int_Q uD^{\alpha_2}_{t_2|T}\varphi d\omega+\int_Q |u|^m(-\Delta_{\mathbb{H}})^{\alpha/2}\varphi d\omega,\\ \int_Q |u|^q\varphi d\omega+J_0=\int_Q vD^{\beta_1}_{t_1|T}\varphi d\omega+\int_Q vD^{\beta_2}_{t_2|T}\varphi d\omega+\int_Q |v|^n(-\Delta_{\mathbb{H}})^{\beta/2}\varphi d\omega\end{cases}$$

for any test function φ (see the equality (3.1)). Now, set

$$\sigma_1=-\frac{1}{pq-1}\left[pq\alpha_1+p\beta_1-2(pq-1)-\left(\frac{(pq-p)\alpha_1}{\alpha}+\frac{(p-1)\beta_1}{\beta}\right)(2N+2)\right],$$
$$\sigma_2=-\frac{1}{pq-1}\left[pq\alpha_1+p\beta_2-2(pq-p)-\left(\frac{(pq-p)\alpha_1}{\alpha}+\frac{(p-1)\beta_1}{\beta}\right)(2N+2)\right],$$
$$\sigma_3=-\frac{1}{pq-n}\left[pq\alpha_1+p\beta_1-2(2pq-nq-p)\right.$$
$$\left.-\left(\frac{(pq-p)\alpha_1}{\alpha}+\frac{(pq-nq)\beta_1}{\beta}\right)(2N+2)\right],$$
$$\sigma_4=-\frac{1}{pq-1}\left[pq\alpha_2+p\beta_1-2(pq-1)-\left(\frac{(pq-p)\alpha_1}{\alpha}+\frac{(p-1)\beta_1}{\beta}\right)(2N+2)\right],$$
$$\sigma_5=-\frac{1}{pq-1}\left[pq\alpha_2+p\beta_2-2(pq-1)-\left(\frac{(p-1)\alpha_1}{\alpha}+\frac{(pq-p)\beta_1}{\beta}\right)(2N+2)\right],$$

$$\sigma_6 = -\frac{1}{pq-n}\left[pq\alpha_2 + p\beta_1 - 2\,(pq-n) - \left(\frac{(pq-p)\alpha_1}{\alpha} + \frac{(p-n)\beta_1}{\beta}\right)(2N+2)\right],$$

$$\sigma_7 = -\frac{1}{pq-m}\left[pq\alpha_1 + pm\beta_1 - 2\,(pq-m) - \left(\frac{(pq-pm)\alpha_1}{\alpha} + \frac{(mp-m)\beta_1}{\beta}\right)(2N+2)\right],$$

$$\sigma_8 = -\frac{1}{pq-m}\left[pq\alpha_1 + pm\beta_2 - 2\,(pq-m) - \left(\frac{(pq-pm)\alpha_1}{\alpha} + \frac{(mp-m)\beta_1}{\beta}\right)(2N+2)\right],$$

$$\sigma_9 = -\frac{1}{pq-nm}\left[pq\alpha_1 + pm\beta_1 - 2\,(pq-nm) - \left(\frac{(pq-pm)\alpha_1}{\alpha} + \frac{(mp-nm)\beta_1}{\beta}\right)(2N+2)\right].$$

Note. The way we calculate $\{\sigma_1, \ldots, \sigma_9\}$ is the same as the way we calculate $\{\delta_1, \ldots, \delta_9\}$.

4.2 Theorem

Let $q > 1$, $p > 1$, $q > m$, $p > n$ and suppose that

$$\int_Q u_2 D^{\alpha_1}_{t_1|T}\varphi^\mu d\omega > 0, \ \int_Q u_1 D^{\alpha_2}_{t_2|T}\varphi^\mu d\omega > 0,$$
$$\int_Q v_2 D^{\beta_1}_{t_1|T}\varphi^\mu d\omega > 0 \ \ \int_Q v_1 D^{\beta_2}_{t_2|T}\varphi^\mu d\omega > 0.$$

If $\max\{\sigma_1, \ldots, \sigma_9, \delta_1, \ldots, \delta_9\} \le 0$.

Then, the system (1.2) does not admit local nontrivial weak solution[5].

Proof. As in the proof of Theorem 1, we reason by the absurd. Suppose (u; v) is a weak non-trivial solution that exists globally in time. Next,replacing φ by φ^μ in (4.1). Since the initial conditions u_0 and v_0 are positive, the variational formulation (4.1) leads to

$$(4.1)\qquad \begin{cases} \int_Q |v|^p \varphi^\mu d\omega \le \int_Q u D^{\alpha_1}_{t_1|T}\varphi^\mu d\omega + \int_Q u D^{\alpha_2}_{t_2|T}\varphi^\mu d\omega \\ + \int_Q |u|^m \left(-\Delta_{\mathbb{H}}\right)^{\alpha/2} \varphi^\mu d\omega, \\ \int_Q |u|^q \varphi^\mu d\omega \le \int_Q v D^{\beta_1}_{t_1|T}\varphi^\mu d\omega + \int_Q v D^{\beta_2}_{t_2|T}\varphi^\mu d\omega \\ + \int_Q |v|^n \left(-\Delta_{\mathbb{H}}\right)^{\beta/2} \varphi^\mu d\omega. \end{cases}$$

5. Then solutions to system (1.2) blow-up whenever $\max\{\sigma_1, \ldots, \sigma_9, \delta_1, \ldots, \delta_9\} \le 0$

Applying Hölder's inequality, we obtain the following estimates:

• For $q > m$

$$\int_Q |u|^m \left|(-\Delta_{\mathbb{H}})^{\alpha/2} \varphi^\mu\right| d\omega \le \mu \left(\int_Q |u|^q \varphi^\mu d\omega\right)^{\frac{m}{q}}$$

$$\times \left(\int_Q \varphi^{\mu - \frac{q}{q-m}} \left|(-\Delta_{\mathbb{H}})^{\alpha/2} \varphi\right|^{\frac{q}{q-m}} d\omega\right)^{\frac{q-m}{q}}. \tag{4.2}$$

• For $q > 1$:

$$\int_Q u \left|D^{\alpha_1}_{t_1|T} \varphi^\mu\right| d\omega \le \left(\int_Q |u|^q \varphi^\mu d\omega\right)^{\frac{1}{q}} \times \left(\int_Q \left|D^{\alpha_1}_{t_1|T} \varphi^\mu\right|^{\frac{q}{q-1}} \varphi^{\frac{-\mu}{q-1}} d\omega\right)^{\frac{q-1}{q}} \tag{4.3}$$

and

$$\int_Q u \left|D^{\alpha_2}_{t_2|T} \varphi^\mu\right| d\omega \le \left(\int_Q |u|^q \varphi^\mu d\omega\right)^{\frac{1}{q}} \times \left(\int_Q \left|D^{\alpha_2}_{t_2|T} \varphi^\mu\right|^{\frac{q}{q-1}} \varphi^{\frac{-\mu}{q-1}} d\omega\right)^{\frac{q-1}{q}}. \tag{4.4}$$

Similarly, we have:

• For $p > n$:

$$\int_Q |v|^n \left|(-\Delta_{\mathbb{H}})^{\beta/2} \varphi^\mu\right| d\omega \le \mu \left(\int_Q |v|^p \varphi^\mu d\omega\right)^{\frac{n}{p}}$$

$$\times \left(\int_Q \varphi^{\mu - \frac{p}{p-n}} \left|(-\Delta_{\mathbb{H}})^{\beta/2} \varphi\right|^{\frac{p}{p-n}} d\omega\right)^{\frac{p-n}{p}} \tag{4.5}$$

• For $p > 1$:

$$\int_Q v \left|D^{\beta_1}_{t_1|T} \varphi^\mu\right| d\omega \le \left(\int_Q |v|^p \varphi^\mu d\omega\right)^{\frac{1}{p}}$$

$$\times \left(\int_Q \left|D^{\beta_1}_{t_1|T} \varphi^\mu\right|^{\frac{p}{p-1}} \varphi^{\frac{-\mu}{p-1}} d\omega\right)^{\frac{p-1}{p}} \tag{4.6}$$

and

$$\int_Q v \left|D^{\beta_2}_{t_2|T} \varphi^\mu\right| d\omega \le \left(\int_Q |v|^p \varphi^\mu d\omega\right)^{\frac{1}{p}} \times \left(\int_Q \left|D^{\beta_2}_{t_2|T} \varphi^\mu\right|^{\frac{p}{p-1}} \varphi^{\frac{-\mu}{p-1}} d\omega\right)^{\frac{p-1}{p}}. \tag{4.7}$$

If, we set

$$I_u = \int_Q |u|^q \varphi^\mu d\omega, \;\; I_v = \int_Q |v|^p \varphi^\mu d\omega,$$

$$A(q,m) = \mu \left(\int_Q \varphi^{\mu - \frac{q}{q-m}} \left|(-\Delta_{\mathbb{H}})^{\alpha/2} \varphi\right|^{\frac{q}{q-m}} d\omega\right)^{\frac{q-m}{q}},$$

$$A(p,n) = \mu \left(\int_Q \varphi^{\mu - \frac{p}{p-n}} \left|(-\Delta_{\mathbb{H}})^{\beta/2} \varphi\right|^{\frac{p}{p-n}} d\omega\right)^{\frac{p-n}{p}},$$

$$B(q) = \left(\int_Q \left| D^{\alpha_1}_{t_1|T} \varphi^\mu \right|^{\frac{q}{q-1}} \varphi^{\frac{-\mu}{q-1}} d\omega \right)^{\frac{q-1}{q}},$$

$$B(p) = \left(\int_Q \left| D^{\beta_1}_{t_1|T} \varphi^\mu \right|^{\frac{p}{p-1}} \varphi^{\frac{-\mu}{p-1}} d\omega \right)^{\frac{p-1}{p}},$$

$$C(q) = \left(\int_Q \left| D^{\alpha_2}_{t_2|T} \varphi^\mu \right|^{\frac{q}{q-1}} \varphi^{\frac{-\mu}{q-1}} d\omega \right)^{\frac{q-1}{q}},$$

$$C(p) = \left(\int_Q \left| D^{\beta_2}_{t_2|T} \varphi^\mu \right|^{\frac{p}{p-1}} \varphi^{\frac{-\mu}{p-1}} d\omega \right)^{\frac{p-1}{p}},$$

$$I_0^\mu = \int_Q u_2 D^{\alpha_1}_{t_1|T} \varphi^\mu d\omega + \int_Q u_1 D^{\alpha_2}_{t_2|T} \varphi^\mu d\omega,$$

$$J_0^\mu = \int_Q v_2 D^{\beta_1}_{t_1|T} \varphi^\mu d\omega + \int_Q v_1 D^{\beta_2}_{t_2|T} \varphi^\mu d\omega$$

then, using estimates (4.3)-(4.4)-(4.5), we can write (4.1) as

$$I_v + I_0^\mu \le I_u^{\frac{1}{q}} B(q) + I_u^{\frac{1}{q}} C(q) + I_u^{\frac{m}{q}} A(q,m),$$

$$I_u + J_0^\mu \le I_v^{\frac{1}{p}} B(p) + I_v^{\frac{1}{p}} C(p) + I_v^{\frac{n}{p}} A(p,n).$$

Since $I_0^\mu, J_0^\mu > 0$, we have

$$I_v \le I_u^{\frac{1}{q}} B(q) + I_u^{\frac{1}{q}} C(q) + I_u^{\frac{m}{q}} A(q,m), \tag{4.8}$$

$$I_u \le I_v^{\frac{1}{p}} B(p) + I_v^{\frac{1}{p}} C(p) + I_v^{\frac{n}{p}} A(p,n). \tag{4.9}$$

Now, from (4.8) and (4.9), we have

$$\begin{aligned} I_v + I_0^\mu \le & \left(I_v^{\frac{1}{pq}} B^{\frac{1}{q}}(p) + I_v^{\frac{1}{pq}} C^{\frac{1}{q}}(p) + I_v^{\frac{n}{pq}} A^{\frac{1}{q}}(p,n) \right) B(q), \\ & + \left(I_v^{\frac{1}{pq}} B^{\frac{1}{q}}(p) + I_v^{\frac{1}{pq}} C^{\frac{1}{q}}(p) + I_v^{\frac{n}{pq}} A^{\frac{1}{q}}(p,n) \right) C(q) \\ & + \left(I_v^{\frac{m}{pq}} B^{\frac{m}{q}}(p) + I_v^{\frac{m}{pq}} C^{\frac{m}{q}}(p) + I_v^{\frac{nm}{pq}} A^{\frac{m}{q}}(p,n) \right) A(q,m). \end{aligned}$$

Then, Young's inequality implies

$$\begin{aligned} I_v + I_0^\mu \le K \Bigg[& \left(B^{\frac{1}{q}}(p) B(q) \right)^{\frac{pq}{pq-1}} + \left(C^{\frac{1}{q}}(p) B(q) \right)^{\frac{pq}{pq-1}} + \left(A^{\frac{1}{q}}(p,n) B(q) \right)^{\frac{pq}{pq-n}} \\ & + \left(B^{\frac{1}{q}}(p) C(q) \right)^{\frac{pq}{pq-1}} + \left(C^{\frac{1}{q}}(p) C(q) \right)^{\frac{pq}{pq-1}} + \left(A^{\frac{1}{q}}(p,n) C(q) \right)^{\frac{pq}{pq-n}} \\ & + \left(B^{\frac{m}{q}}(p) A(q,m) \right)^{\frac{pq}{pq-m}} + \left(C^{\frac{m}{q}}(p) A(q,m) \right)^{\frac{pq}{pq-m}} + \left(A^{\frac{m}{q}}(p,n) A(q,m) \right)^{\frac{pq}{pq-nm}} \Bigg]. \end{aligned}$$

Let's take now the test function $\varphi(\eta, t_1, t_2)$ in the form

$$\varphi(\eta, t_1, t_2) = \psi\left(\frac{\tau^{2\theta_j} + |x|^{4\theta_j} + |y|^{4\theta_j}}{R^4}\right)\psi\left(\frac{t_1}{R}\right)\psi\left(\frac{t_2}{R}\right), \quad j = 1, 2,$$

and θ_j will be determined further. Then

$$\Delta_{\mathbb{H}}\varphi(\eta) = \psi\left(\frac{t_1}{R}\right)\psi\left(\frac{t_2}{R}\right)\Delta_{\mathbb{H}}\psi(\rho),$$

where

$$\rho = \frac{\tau^{2\theta_j} + |x|^{4\theta_j} + |y|^{4\theta_j}}{R^4}$$

and

$$\Delta_{\mathbb{H}}\psi(\rho) = \sum_{i=1}^{N}\left(\frac{\partial^2\psi(\rho)}{\partial x_i^2} + \frac{\partial^2\psi(\rho)}{\partial y_i^2} + 4y_i\frac{\partial^2\psi(\rho)}{\partial x_i\partial\tau} \right.$$
$$\left. -4x_i\frac{\partial^2\psi(\rho)}{\partial y_i\partial\tau} + 4\left(x_i^2 + y_i^2\right)\frac{\partial^2\psi(\rho)}{\partial\tau^2}\right)$$

we have

$$\frac{\partial^2\psi(\rho)}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\frac{\partial\rho}{\partial x_i}\frac{\partial\psi(\rho)}{\partial\rho}\right) = \frac{\partial^2\rho}{\partial x_i^2}\psi'(\rho) + \left(\frac{\partial\rho}{\partial x_i}\right)^2\psi''(\rho),$$
$$= \frac{4\theta_j}{R^4}\left(|x|^{4\theta_j-2} + (4\theta_j - 2)\, x_i^2|x|^{4\theta_j-4}\right)\psi'(\rho) + \frac{16\theta_j^2}{R^8}x_i^2|x|^{8\theta_j-4}\psi''(\rho)$$

and

$$\frac{\partial^2\psi(\rho)}{\partial y_i^2} = \frac{\partial}{\partial y_i}\left(\frac{\partial\rho}{\partial y_i}\frac{\partial\psi(\rho)}{\partial\rho}\right) = \frac{\partial^2\rho}{\partial y_i^2}\psi'(\rho) + \left(\frac{\partial\rho}{\partial y_i}\right)^2\psi''(\rho),$$
$$= \frac{4\theta_j}{R^4}\left(|y|^{4\theta_j-2} + (4\theta_j - 2)\, y_i^2|y|^{4\theta_j-4}\right)\psi'(\rho) + \frac{16\theta_j^2}{R^8}y_i^2|y|^{8\theta_j-4}\psi''(\rho)$$

and

$$4y_i\frac{\partial^2\psi(\rho)}{\partial x_i\partial\tau} = 4y_i\frac{\partial}{\partial x_i}\left(\frac{\partial\rho}{\partial\tau}\frac{\partial\psi(\rho)}{\partial\rho}\right) = 4y_i\left[\frac{\partial^2\rho}{\partial x_i\partial\tau}\psi'(\rho) + \left(\frac{\partial\rho}{\partial\tau}\right)\left(\frac{\partial\rho}{\partial x_i}\right)\psi''(\rho)\right],$$
$$= 4y_i\left[\frac{\partial}{\partial x_i}\left(\frac{2\theta_j}{R^4}\tau^{2\theta_j-1}\right)\psi'(\rho) + \left(\frac{2\theta_j}{R^4}\tau^{2\theta_j-1}\right)\left(\frac{4\theta_j}{R^4}|x|^{4\theta_j-2}x_i\right)\psi''(\rho)\right]$$
$$= \frac{8\theta_j^2}{R^8}\tau^{2\theta_j-1}|x|^{4\theta_j-2}x_iy_i\psi''(\rho)$$

and

$$-4x_i\frac{\partial^2\psi(\rho)}{\partial y_i\partial\tau} = -4x_i\frac{\partial}{\partial y_i}\left(\frac{\partial\rho}{\partial\tau}\frac{\partial\psi(\rho)}{\partial\rho}\right)$$

$$= -4x_i\left[\frac{\partial^2\rho}{\partial y_i\partial\tau}\psi'(\rho) + \left(\frac{\partial\rho}{\partial\tau}\right)\left(\frac{\partial\rho}{\partial y_i}\right)\psi''(\rho)\right]$$
$$= -4x_i\left[\frac{\partial}{\partial y_i}\left(\frac{2\theta_j}{R^4}\tau^{2\theta_j-1}\right)\psi'(\rho) + \left(\frac{2\theta_j}{R^4}\tau^{2\theta_j-1}\right)\left(\frac{4\theta_j}{R^4}|y|^{4\theta_j-2}y_i\right)\psi''(\rho)\right]$$
$$= -\frac{8\theta_j^2}{R^8}\tau^{2\theta_j-1}|y|^{4\theta_j-2}x_i y_i\psi''(\rho)$$

and

$$4\left(x_i^2+y_i^2\right)\frac{\partial^2\psi(\rho)}{\partial\tau^2} = 4\left(x_i^2+y_i^2\right)\frac{\partial}{\partial\tau}\left(\frac{\partial\rho}{\partial\tau}\frac{\partial\psi(\rho)}{\partial\rho}\right)$$
$$= 4\left(x_i^2+y_i^2\right)\left[\left(\frac{\partial^2\rho}{\partial\tau^2}\right)\psi'(\rho) + \left(\frac{\partial\rho}{\partial\tau}\right)^2\psi''(\rho)\right]$$
$$= 4\left(x_i^2+y_i^2\right)\left[\left(\frac{2\theta_j(2\theta_j-1)}{R^4}\tau^{2\theta_j-2}\right)\psi'(\rho) + \left(\frac{4\theta_j^2}{R^8}\tau^{4\theta_j-2}\right)\psi''(\rho)\right]$$

finally

$$\begin{aligned}\Delta_{\mathbb{H}}\psi(\rho) = {} & \frac{4\theta_j}{R^4}[(N+(4\theta_j-2))(|x|^{4\theta_j-2}+|y|^{4\theta_j-2}) \\ & + (4\theta_j-2)\tau^{2\theta_j-2}(|x|^2+|y|^2)]\psi'(\rho) \\ & + \frac{16\theta_j^2}{R^8}[|x|^{8\theta_j-2}+|y|^{8\theta_j-2}+\frac{1}{2}\tau^{2\theta_j-1}\langle x,y\rangle(|x|^{4\theta_j-2}-|y|^{4\theta_j-2}) \\ & + \tau^{4\theta_j-2}(|x|^2+|y|^2)]\psi''(\rho)\end{aligned}$$

and we apply the change of next variables in the form

$$\eta = (x,y,\tau) \longrightarrow \tilde{\eta} = (\tilde{x},\tilde{y},\tilde{\tau}),$$

where

$$\tilde{x} = R^{\frac{-1}{\theta_j}}x \;\; \tilde{y} = R^{\frac{-1}{\theta_j}}y \;\; \tilde{\tau} = R^{\frac{-2}{\theta_j}}\tau, \quad \tilde{t}_1 = R^{-1}t_1, \quad \tilde{t}_2 = R^{-1}t_2$$

we put

$$\Omega_1^j = \left\{\tilde{\eta}\in\mathbb{H}:\ \tilde{\rho} = \tilde{\tau}^{2\theta_j} + |\tilde{x}|^{4\theta_j} + |\tilde{y}|^{4\theta_j} \le 2\right\},$$

for Ω_2 and Ω_3, see the equality (3.7). Then

$$\begin{aligned}\Delta_{\mathbb{H}}\psi(\rho) = {} & \frac{4\theta_j}{R^{\frac{2}{\theta_j}}}[(N+(4\theta_j-2))(|\tilde{x}|^{4\theta_j-2}+|\tilde{y}|^{4\theta_j-2}) \\ & + (4\theta_j-2)\tilde{\tau}^{2\theta_j-2}(|\tilde{x}|^2+|\tilde{y}|^2)]\psi'(\tilde{\rho}) \\ & + \frac{16\theta_j^2}{R^{\frac{2}{\theta_j}}}[|\tilde{x}|^{8\theta_j-2}+|\tilde{y}|^{8\theta_j-2}+\frac{1}{2}\tilde{\tau}^{2\theta_j-1}\langle\tilde{x},\tilde{y}\rangle(|\tilde{x}|^{4\theta_j-2}-|\tilde{y}|^{4\theta_j-2}) \\ & + \tilde{\tau}^{4\theta_j-2}(|\tilde{x}|^2+|\tilde{y}|^2)]\psi''(\tilde{\rho})\end{aligned}$$

this means

$$\Delta_{\mathbb{H}}\psi\left(\rho\right)=\frac{1}{R^{\frac{2}{\theta_j}}}\Delta_{\mathbb{H}}\psi\left(\tilde{\rho}\right),\ \forall\tilde{\eta}\in\Omega_1^j,$$
$$\left(-\Delta_{\mathbb{H}}\right)^{\alpha/2}\psi\left(\rho\right)=R^{\frac{-\alpha}{\theta_j}}\left(-\Delta_{\mathbb{H}}\right)^{\alpha/2}\psi\left(\tilde{\rho}\right),$$
$$\left(-\Delta_{\mathbb{H}}\right)^{\beta/2}\psi\left(\rho\right)=R^{\frac{-\beta}{\theta_j}}\left(-\Delta_{\mathbb{H}}\right)^{\beta/2}\psi\left(\tilde{\rho}\right)$$

as

$$d\eta=R^{\frac{N}{\theta_j}+\frac{N}{\theta_j}+\frac{2}{\theta_j}}d\tilde{\eta}=R^{\frac{2N+2}{\theta_j}}d\tilde{\eta},$$

we make the following estimates:

- For $j=1$, we choose $\theta_1=\frac{\alpha}{\alpha_1}$ and as $\alpha_1<\alpha_2$ we obtain

$$A\left(q,m\right)=\mathrm{C}_1R^{-\alpha_1+\frac{(q-m)}{q}\left(\frac{(2N+2)\alpha_1}{\alpha}+2\right)}$$

where

$$\mathrm{C}_1=\mu\left(\int_{\Omega_1^1}\left|(-\Delta_{\mathbb{H}})^{\alpha/2}\psi(\tilde{\rho})\right|^{\frac{q}{q-m}}\psi^{\mu-\frac{q}{q-m}}(\tilde{\rho})d\tilde{\eta}\int_{\Omega_2}\psi^{\mu}(\tilde{t}_1)d\tilde{t}_1\int_{\Omega_3}\psi^{\mu}(\tilde{t}_2)d\tilde{t}_2\right)^{\frac{q-m}{q}}$$

and

$$B\left(q\right)=\mathrm{C}_2R^{-\alpha_1+\frac{(q-1)}{q}\left(\frac{(2N+2)\alpha_1}{\alpha}+2\right)}$$

where

$$\mathrm{C}_2=\left(\int_{\Omega_1^1}\psi^{\mu}\left(\tilde{\eta}\right)d\tilde{\eta}\int_{\Omega_2}\left|D^{\alpha_1}_{\tilde{t}_1|R^{-1}T}\psi^{\mu}\left(\tilde{t}_1\right)\right|^{\frac{q}{q-1}}\psi^{\frac{-\mu}{q-1}}\left(\tilde{t}_1\right)d\tilde{t}_1\int_{\Omega_3}\psi^{\mu}\left(\tilde{t}_2\right)d\tilde{t}_2\right)^{\frac{q-1}{q}}$$

and

$$C\left(q\right)=\mathrm{C}_3R^{-\alpha_2+\frac{(q-1)}{q}\left(\frac{(2N+2)\alpha_1}{\alpha}+2\right)}$$

where

$$\mathrm{C}_3=\left(\int_{\Omega_1^1}\psi^{\mu}\left(\tilde{\eta}\right)d\tilde{\eta}\int_{\Omega_2}\psi^{\mu}\left(\tilde{t}_1\right)d\tilde{t}_1\int_{\Omega_3}\left|D^{\alpha_2}_{\tilde{t}_2|R^{-1}T}\psi^{\mu}\left(\tilde{t}_2\right)\right|^{\frac{q}{q-1}}\psi^{\frac{-\mu}{q-1}}\left(\tilde{t}_2\right)d\tilde{t}_2\right)^{\frac{q-1}{q}}.$$

- For $j=2$, we choose $\theta_2=\frac{\beta}{\beta_1}$ and as $\beta_1<\beta_2$ we obtain

$$A\left(p,n\right)=\mathrm{C}_4R^{\frac{-\beta}{\theta_2}+\frac{(p-n)}{p}\left(\frac{(2N+2)\beta_1}{\beta}+2\right)}$$

where

$$\mathrm{C}_4=\mu\left(\int_{\Omega_1^2}\left|(-\Delta_{\mathbb{H}})^{\beta/2}\psi(\tilde{\rho})\right|^{\frac{p}{p-n}}\psi^{\mu-\frac{p}{p-n}}(\tilde{\rho})d\tilde{\eta}\int_{\Omega_2}\psi^{\mu}(\tilde{t}_1)d\tilde{t}_1\int_{\Omega_3}\psi^{\mu}(\tilde{t}_2)d\tilde{t}_2\right)^{\frac{p-n}{p}}$$

and

$$B(p) = C_5 R^{-\beta_1 + \frac{(p-1)}{p}\left(\frac{(2N+2)\beta_1}{\beta}+2\right)},$$

where

$$C_5 = \left(\int_{\Omega_1^1} \psi^{\mu}(\tilde{\eta})\, d\tilde{\eta} \int_{\Omega_2} \left| D^{\beta_1}_{\tilde{t}_1|R^{-1}T} \psi^{\mu}(\tilde{t}_1)\right|^{\frac{p}{p-1}} \psi^{\frac{-\mu}{p-1}}(\tilde{t}_1)\, d\tilde{t}_1 \int_{\Omega_3} \psi^{\mu}(\tilde{t}_2)\, d\tilde{t}_2\right)^{\frac{p-1}{p}}$$

and

$$C(p) = C_6 R^{-\beta_2 + \frac{(p-1)}{p}\left(\frac{(2N+2)\beta_1}{\beta}+2\right)}$$

where

$$C_6 = \left(\int_{\Omega_1^1} \psi^{\mu}(\tilde{\eta})\, d\tilde{\eta} \int_{\Omega_2} \psi^{\mu}(\tilde{t}_1)\, d\tilde{t}_1 \int_{\Omega_3} \left| D^{\beta_2}_{\tilde{t}_2|R^{-1}T} \psi^{\mu}(\tilde{t}_2)\right|^{\frac{p}{p-1}} \psi^{\frac{-\mu}{p-1}}(\tilde{t}_2)\, d\tilde{t}_2\right)^{\frac{p-1}{p}},$$

for some positive constant $K\widehat{C}$, where

$$\widehat{C} = \max\left\{(C_5^{\frac{1}{q}} C_2)^{\frac{pq}{pq-1}}, (C_6^{\frac{1}{q}} C_2)^{\frac{pq}{pq-1}}, (C_4^{\frac{1}{q}} C_2)^{\frac{pq}{pq-n}}, (C_5^{\frac{1}{q}} C_3)^{\frac{pq}{pq-1}},\right.$$
$$\left.(C_6^{\frac{1}{q}} C_3)^{\frac{pq}{pq-1}}, (C_4^{\frac{1}{q}} C_3)^{\frac{pq}{pq-n}}, (C_5^{\frac{m}{q}} C_1)^{\frac{pq}{pq-m}}, (C_6^{\frac{m}{q}} C_1)^{\frac{pq}{pq-m}}, (C_4^{\frac{m}{q}} C_1)^{\frac{pq}{pq-nm}}\right\}.$$

Hence, we obtain

$$I_v + I_0^{\mu} \leq K\widehat{C}\left\{R^{\sigma_1} + R^{\sigma_2} + \ldots + R^{\sigma_9}\right\}. \tag{4.10}$$

Similarly, we obtain for I_u the estimate

$$I_u + J_0^{\mu} \leq K\widehat{\widehat{C}}\left\{R^{\delta_1} + R^{\delta_2} + \ldots + R^{\delta_9}\right\}, \tag{4.11}$$

where the value $\widehat{\widehat{C}}$ is set as the value setting $\widehat{C}$. Finally, by tending $R \to \infty$, we observe that:

Either $\max\{\sigma_1, \ldots, \sigma_9, \delta_1, \ldots, \delta_9\} < 0$ and in this case, the right hand side tends to zero while the left hand side is strictly positive. Hence, we obtain a contradiction. Or, $\max\{\sigma_1, \ldots, \sigma_9, \delta_1, \ldots, \delta_9\} = 0$ and in this case, following the analysis similar as in one equation, we prove a contradiction. □

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