Applications of extended Hadamard K-fractional integral

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Abstract. In this paper, we use the extended Hadamard k-fractional integral to obtain some new fractional integral inequalities by introducing the new parameters s and k. These extended fractional integral inequalities also hold true for usual Hadamard fractional integral when we substitute k is equal to one and s is equal to zero.

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1. Introduction

Fractional calculus has been extensively studied and investigated in the last two decades. Its new results and their applications have emerged as a very effective and powerful tool for many mathematical problems of science and engineering. Recently, fractional derivatives and integrals have employed in many fluid problems to get more accurate and valid results. These fractional operators are used in finance, biophysics, electrochemistry, computed tomography, engineering, control theory, geological surveying, thermodynamics, hydrology, electric conductance of biological systems, statistical mechanics, astrophysics, mathematical physics, and also used for the mathematical modelling of viscoelastic materials.

Diaz and Pariguan [13] give new direction to fractional calculus by introducing k-gamma function and k-beta function which are the extensions of classical gamma and beta functions. So, k-fractional calculus version was introduced. Many results of fractional calculus were extended. Farid and Habibullah [14] defined the Hadamard k-fractional integral. Azam et. al. [3], introduced the extended Hadamard k-fractional integrals of order α.

Let f be continuous on $[0,\infty]$ and α, kϵ , sϵ. Then, $\forall x > a > 0$

$$ {}_k^sI_H^\alpha[f(x)] = \frac{1}{k\Gamma_k(\alpha)}\int_a^x [\log\frac{x}{\tau}]^{\frac{\alpha}{k}-1}(\frac{\tau}{x})^s f(\tau)\frac{d\tau}{\tau}. \tag{1}$$

The objective of this work is to extend some existing fractional integral inequalities by using extended Hadamard k-fractional integral [3]. New parameter s and k are introduced. A few mathematicians have devoted their efforts to generalize and refine the fractional integral inequalities in the recent years due to their applications in different fields of science and technology. We may refer the interested reader to [1, 3, 4, 6, 10, 14, 15, 16, 17, 18].

2. Our some new results and discussions

Now, extension of some fractional integral inequalities using the equation (1) are given below

Theorem 2.1. *Let $(g_i)_{i=1,2,,...n}$ be positive increasing function on $[1,\infty)$, and α, k ,s. Then $\forall z > 1$*

$$ {}_k^sI_H^\alpha({}_{i=1}\prod^n g_i) \geq [{}_k^sI_H^\alpha(I)(z)]^{1-\alpha}{}_{i=1}\prod^n {}_k^sI_H^\alpha(g_i)(z). \tag{2}$$

Proof. We prove this by induction. For $n = 1$

$$ {}_k^sI_H^\alpha g_1(z) \geq_k^s I_H^\alpha g_1(z), \tag{3}$$

which is true. For $n = 2$

$$ {}_k^sI_H^\alpha(g_1g_2)(z) \geq [{}_k^sI_H^\alpha(l)(z)]^{-1}{}_k^sI_H^\alpha g_1(z){}_k^sI_H^\alpha g_2(z), \tag{4} $$

which is also true.

Suppose that the statement is true for $n = k - 1$

$$ {}_k^sI_H^\alpha({}_{i=1}\prod^{n-1} g_i)(z) \geq [{}_k^sI_H^\alpha(l)(z)]^{2-n}{}_{i=1}\prod^{n-1}{}_k^sI_H^\alpha(g_i)(z). \tag{5} $$

Now, $({}_{i=1}\prod^{n-1} g_i)(z)$ is an increasing function on $[1, \infty]$ because of $(g_i)_{i=1,2,\dots,n}$. So, we can get

$$ {}_k^sI_H^\alpha({}_{i=1}\prod^{n} g_i)(z) \geq {}_k^s I_H^\alpha({}_{i=1}\prod^{n} g_ig_n)(z)[{}_k^sI_H^\alpha(l)(z)]^{-1}{}_k^sI_H^\alpha \\ ({}_{i=1}\prod^{n-1} g_i)(z){}_k^sI_H^\alpha(g_n)(z). $$

using (5), we get

$$ {}_k^sI_H^\alpha({}_{i=1}\prod^{n} g_i)(z) \geq [{}_k^sI_H^\alpha(l)(z)]^{-1}[{}_k^sI_H^\alpha(l)(z)]^{2-n}({}_{i=1}\prod^{n-1} g_i)(z){}_k^sI_H^\alpha({}_{i=1}(g_n)(z). $$

Hence, we get

$$ {}_k^sI_H^\alpha({}_{i=1}\prod^{n} g_i)(z) \geq [{}_k^sI_H^\alpha(l)(z)]^{1-n}{}_{i=1}\prod^{n}{}_k^sI_H^\alpha(g_i)(z). \tag{6} $$

□

Theorem 2.2. *For integrable function g on $[1, \infty]$. Assume that:*

A_1, There exist two integrable functions Ψ_1 and Ψ_2 on $[1, \infty]$ such that

$$ \Psi_2(r) \geq g(r) \geq \Psi_1(r), \qquad ,\forall r\epsilon[1, \infty]. \tag{7} $$

Then, for $r > 1$ $s\epsilon$ and $\alpha, \beta, k\epsilon$,

$$ {}_k^sI_H^\beta\Psi_1(r){}_k^sI_H^\alpha g_r + {}_k^sI_H^\alpha\Psi_2(r){}_k^sI_H^\beta g_r \geq {}_k^sI_H^\alpha\Psi_2(r){}_k^sI_H^\beta\Psi_1(r) + {}_k^sI_H^\alpha g(r){}_k^sI_H^\beta g_r. \tag{8} $$

Proof. From A_1, $\forall, p \geq 1, q \geq 1$, we have

$$ [\Psi_2(p) - g(p)][g(q) - \Psi_1(q)] \geq 0. \tag{9} $$

Therefore,

$$ \Psi_2(p)g(q) + g(p)\Psi_1(q) \geq \Psi_1(q)\Psi_2(p) + g(p)g(q). \tag{10} $$

Multiplying (10), by

$$ \frac{1}{k\Gamma_k(\alpha)}[\log\frac{r}{p}]^{\frac{\alpha}{k}-1}[\frac{p}{r}]^s\frac{1}{p}, \tag{11} $$

and integrating w.r.t. p on $[1,\infty]$

$$g(q)\frac{1}{k\Gamma_k(\alpha)}\int_1^r[[\log\frac{r}{p}]^{\frac{\alpha}{k}-1}[\frac{p}{r}]^s\Psi_2(p)\frac{dp}{p}] + \Psi_1(q)\frac{1}{k\Gamma_k(\alpha)}\int_1^r[[\log\frac{r}{p}]^{\frac{\alpha}{k}-1}[\frac{p}{r}]^s g(p)\frac{dp}{p}] \geq \Psi_2(w)\frac{1}{k\Gamma_k(\alpha)}\int_1^r[[\log\frac{r}{p}]^{\frac{\alpha}{k}-1}[\frac{p}{r}]^s\Psi_2(p)\frac{dp}{p}] + g(w)\frac{1}{k\Gamma_k(\alpha)}\int_1^r[[\log\frac{r}{p}]^{\frac{\alpha}{k}-1}[\frac{p}{r}]^s g(p)\frac{dp}{p}]. \tag{12}$$

Using the result (1),

$$g(q)_k^sI_H^\alpha\Psi_2(r)+\Psi_1(q)_k^sI_H^\alpha g_r \geq \Psi_1(q)_k^sI_H^\alpha\Psi_2(r)+g(q)_k^sI_H^\alpha g(r). \tag{13}$$

Multiplying (13) by

$$\frac{1}{k\Gamma_k(\beta)}[\log\frac{r}{p}]^{\frac{\beta}{k}-1}[\frac{q}{r}]^s\frac{1}{q}, \tag{14}$$

and integrating w.r.t. q on $[1,\infty]$,

$${}_k^sI_H^\alpha\Psi_2(r)\frac{1}{k\Gamma_k(\beta)}\int_1^r[[\log\frac{r}{q}]^{\frac{\beta}{k}-1}[\frac{q}{r}]^s g(q)\frac{dq}{q}] + {}_k^sI_H^\alpha g(r)\frac{1}{k\Gamma_k(\beta)}\int_1^r[[\log\frac{r}{q}]^{\frac{\beta}{k}-1}[\frac{q}{r}]^s\Psi_1(q)\frac{dq}{q}] \geq {}_k^sI_H^\alpha\Psi_2(r)\frac{1}{k\Gamma_k(\beta)}\int_1^r[[\log\frac{r}{q}]^{\frac{\beta}{k}-1}[\frac{q}{r}]^s\Psi_1(q)\frac{dq}{q}] + {}_k^sI_H^\alpha g(r)\frac{1}{k\Gamma_k(\beta)}\int_1^r[[\log\frac{r}{q}]^{\frac{\beta}{k}-1}[\frac{q}{r}]^s\Psi_1(q)\frac{dq}{q}] + {}_k^sI_H^\alpha g(r)\frac{1}{k\Gamma_k(\beta)}\int_1^r[[\log\frac{r}{q}]^{\frac{\beta}{k}-1}[\frac{q}{r}]^s g(q)\frac{dq}{q}]. \tag{15}$$

Using the equation (1), we get (8). □

Theorem 2.3. *Let f,g and h are positive valued and continuous functions on $[0.\infty]$ such that*

$$[g(v)-g(w)](\frac{f(w)}{h(w)}-\frac{f(v)}{h(v)})\geq 0, \tag{16}$$

for all $v,w\epsilon(0,z)$. Then $\forall z>0, s\epsilon,\alpha,\beta,k\epsilon,$

$$\frac{{}_k^sI_H^\alpha[gh](z){}_k^sI_H^\beta[f](z)+{}_k^sI_H^\alpha[f](z){}_k^sI_H^\beta[gh](z)}{{}_k^sI_H^\alpha[fg](z){}_k^sI_H^\beta[h](z)+{}_k^sI_H^\alpha[h](z){}_k^sI_H^\beta[fg](z)}\geq 1. \tag{17}$$

Proof. Multiplying

$$[g(v)-g(w)](\frac{f(w)}{h(w)}-\frac{f(v)}{h(v)})\geq 0, \tag{18}$$

by $h(v)h(w)$, we can get

$$g(v)f(w)h(v)-g(v)f(v)h(w)-g(w)f(w)h(w)+g(w)f(v)h(w)\geq 0. \tag{19}$$

Multiplying by $\frac{1}{k\Gamma_k(\alpha)}[\log\frac{z}{v}]^{\frac{\alpha}{k}-1}[\frac{v}{z}]^s\frac{1}{v}$ and integrating w.r.t. v on $[1,\infty]$

$$\begin{aligned}\frac{1}{k\Gamma_k(\alpha)}\int_1^z[\log\frac{z}{v}]^{\frac{\alpha}{k}-1}[\frac{v}{z}]^s g(v)f(w)h(v)\frac{dv}{v} \\ -\frac{1}{k\Gamma_k(\alpha)}\int_1^z[\log\frac{z}{v}]^{\frac{\alpha}{k}-1}[\frac{v}{z}]^s g(v)f(v)h(w)\frac{dv}{v} \\ -\frac{1}{k\Gamma_k(\alpha)}\int_1^z[\log\frac{z}{v}]^{\frac{\alpha}{k}-1}[\frac{v}{z}]^s g(w)f(w)h(v)\frac{dv}{v} \\ +\frac{1}{k\Gamma_k(\alpha)}\int_1^z[\log\frac{z}{v}]^{\frac{\alpha}{k}-1}[\frac{v}{z}]^s g(w)f(v)h(w)\frac{dv}{v}\geq 0.\end{aligned} \tag{20}$$

Using the equation (1),

$$\begin{aligned}&f(w){}_k^sI_H^\alpha[gh](z)-h(w){}_k^sI_H^\alpha[fg](z)-f(w){}_k^sI_H^\alpha[h](z)\\ &+h(w)g(w){}_k^sI_H^\alpha[f](z)\geq 0.\end{aligned} \tag{21}$$

Multiplying by $\frac{1}{k\Gamma_k(\beta)}[\log\frac{z}{w}]^{\frac{\beta}{k}-1}[\frac{w}{z}]^s\frac{1}{w}$, and integrating w.r.t. w on $[1,\infty]$

$$\begin{aligned}&{}_k^sI_H^\alpha[gh](z)\frac{1}{k\Gamma_k(\beta)}\int_1^z[\log\frac{z}{w}]^{\frac{\beta}{k}-1}[\frac{w}{z}]^s f(w)\frac{dw}{w}\\ &-{}_k^sI_H^\alpha[fg](z)\frac{1}{k\Gamma_k(\beta)}\int_1^z[\log\frac{z}{w}]^{\frac{\beta}{k}-1}[\frac{w}{z}]^s h(w)\frac{dw}{w}\\ &-{}_k^sI_H^\alpha[h](z)\frac{1}{k\Gamma_k(\beta)}\int_1^z[\log\frac{z}{w}]^{\frac{\beta}{k}-1}[\frac{w}{z}]^s f(w)g(w)\frac{dw}{w}\\ &+{}_k^sI_H^\alpha[f](z)\frac{1}{k\Gamma_k(\beta)}\int_1^z[\log\frac{z}{w}]^{\frac{\beta}{k}-1}[\frac{w}{z}]^s h(w)g(w)\frac{dw}{w}\geq 0.\end{aligned} \tag{22}$$

Using the equation (1), we get

$$\begin{aligned}&{}_k^sI_H^\alpha[gh](z){}_k^sI_H^\beta[f](z)-{}_k^sI_H^\alpha[fg](z){}_k^sI_H^\beta[h](z)\\ &-{}_k^sI_H^\alpha[h](z){}_k^sI_H^\beta[fg](z)+{}_k^sI_H^\alpha[f](z){}_k^sI_H^\beta[gh](z)\geq 0.\end{aligned} \tag{23}$$

Which gives (17). □

Corollary 2.1. *Let f, g and h are positive valued and continuous functions on* $[0.\infty]$ *such that*

$$[g(v)-g(w)](\frac{f(w)}{h(w)}-\frac{f(v)}{h(v)})\geq 0, \tag{24}$$

for all $v, w\epsilon(0,z)$. *Then* $\forall z>0, s\epsilon, \alpha, k\epsilon$,

$$\frac{{}^s_kI^{\alpha}_H[f](z)}{{}^s_kI^{\beta}_H[h](z)} \geq \frac{{}^s_kI^{\alpha}_H[fg](z)}{{}^s_kI^{\beta}_H[gh](z)}. \tag{25}$$

Proof. By substituting $\beta=\alpha$ in (17), we get

$$\frac{{}^s_kI^{\alpha}_H[f](z){}^s_kI^{\alpha}_H[gh](z)}{{}^s_kI^{\beta}_H[h](z){}^s_kI^{\alpha}_H[fg](z)} \geq 1. \tag{26}$$

Which gives (25). □

Theorem 2.4. *For integrable function* g *on* $[1,\infty]$ *and constants* $l \geq m \geq 0$, $l \neq 0$. *Let* (A_1) *holds. Then, for any* $z>1$, $s\epsilon$ *and* $\alpha, \beta, k, q\epsilon$, *we have*

$$\begin{aligned} &{}^s_kI^{\alpha}_H(\Psi_2-g)^{\frac{m}{l}}(z)+\frac{m}{l}q^{\frac{m-l}{l}}\,{}^s_kI^{\alpha}_Hg(z) \\ &\leq \frac{m}{l}q^{\frac{m-l}{l}}\,{}^s_kI^{\alpha}_H\Psi_2(z)+\frac{m-l}{l}q^{\frac{m}{l}}\,{}^s_kI^{\alpha}_HI(z), \end{aligned} \tag{27}$$

$$\begin{aligned} &{}^s_kI^{\alpha}_H(g-\Psi_1)^{\frac{m}{l}}(z)+\frac{m}{l}q^{\frac{m-l}{l}}\,{}^s_kI^{\alpha}_H\Psi_1(z) \\ &\leq \frac{m}{l}q^{\frac{m-l}{l}}\,{}^s_kI^{\alpha}_Hg(z)+\frac{l-m}{l}q^{\frac{m}{l}}\,{}^s_kI^{\alpha}_HI(z). \end{aligned} \tag{28}$$

Proof. By the condition (A_1) holds and for $l \geq m \geq 0$, $l \neq 0$, we have

$$[\Psi_2(w)-g(w)]^{\frac{m}{l}} \leq \frac{m}{l}q^{\frac{m-l}{l}}[\Psi_2(w)-g(w)]+\frac{l-m}{l}q^{\frac{m}{l}}, \tag{29}$$

multiplying this by $\frac{1}{k\Gamma_k(\alpha)}[\log\frac{z}{w}]^{\frac{\alpha}{k}-1}[\frac{w}{z}]^s\frac{1}{w}$, and integrating w.r.t. w on $[1,z]$, we have

$$\begin{aligned} &\frac{1}{k\Gamma_k(\alpha)}\int_1^z(\log\frac{z}{w})^{\frac{\alpha}{k}-1}(\frac{w}{z})^s[\Psi_2(w)-g(w)]^{\frac{m}{l}}\frac{dw}{w} \\ &\leq \frac{m}{l}q^{\frac{m-l}{l}}\frac{1}{k\Gamma_k(\alpha)}\int_1^z(\log\frac{z}{w})^{\frac{\alpha}{k}-1}(\frac{w}{z})^s[\Psi_2(w)-g(w)]\frac{dw}{w} \\ &+\frac{l-m}{l}q^{\frac{m}{l}}\frac{1}{k\Gamma_k(\alpha)}\int_1^z(\log\frac{z}{w})^{\frac{\alpha}{k}-1}(\frac{w}{z})^s\frac{dw}{w}. \end{aligned} \tag{30}$$

Using equation (1), we get

$$\begin{aligned} &{}^s_kI^{\alpha}_H[\Psi_2-g]^{\frac{m}{l}}(z)+\frac{m}{l}q^{\frac{m-l}{l}}\,{}^s_kI^{\alpha}_Hg(z) \leq \frac{m}{l}q^{\frac{m-l}{l}}\,{}^s_kI^{\alpha}_H\Psi_2(z) \\ &+\frac{l-m}{l}q^{\frac{m}{l}}\,{}^s_kI^{\alpha}_H(I)(z). \end{aligned} \tag{31}$$

For (14), we can use similar steps. □

Theorem 2.5. *For integrable functions g and h on $[1,\infty]$. Let (A_1) holds and also suppose the following:*

(A_2) There exist the integrable functions ϕ_1 and ϕ_2 on $[1,\infty]$ such that

$$\phi_1(r) \leq h(r) \leq \phi_2(r), \qquad \forall r\epsilon[1,\infty]. \tag{32}$$

Then, for any $r>1$, $s\epsilon$ and $\lambda,\gamma,\kappa\epsilon$,

$$\begin{aligned} &{}^s_kI^\alpha_H\phi_1(r){}^s_kI^\alpha_Hg(r) + {}^s_kI^\alpha_H\Psi_2(r){}^s_kI^\alpha_Hh(r) \\ &\quad \geq {}^s_kI^\alpha_H\Psi_2(r){}^s_kI^\alpha_H\phi_1(r) + {}^s_kI^\alpha_Hg(r){}^s_kI^\alpha_Hh(r), \end{aligned} \tag{33}$$

$$\begin{aligned} &{}^s_kI^\alpha_H\Psi_1(r){}^s_kI^\alpha_Hh(r) + {}^s_kI^\alpha_H\phi_2(r){}^s_kI^\alpha_Hg(r) \\ &\quad \geq {}^s_kI^\alpha_H\phi_2(r){}^s_kI^\alpha_H\Psi_1(r) + {}^s_kI^\alpha_Hh(r){}^s_kI^\alpha_Hg(r), \end{aligned} \tag{34}$$

$$\begin{aligned} &{}^s_kI^\alpha_H\phi_2(r){}^s_kI^\alpha_H\Psi_2(r) + {}^s_kI^\alpha_Hg(r){}^s_kI^\alpha_Hh(r) \\ &\quad \geq {}^s_kI^\alpha_H\Psi_2(r){}^s_kI^\alpha_Hh(r) + {}^s_kI^\alpha_Hg(r){}^s_kI^\alpha_H\phi_2(r), \end{aligned} \tag{35}$$

$$\begin{aligned} &{}^s_kI^\alpha_H\Psi_1(r){}^s_kI^\alpha_H\phi_1(r) + {}^s_kI^\alpha_Hg(r){}^s_kI^\alpha_Hh(r) \\ &\quad \geq {}^s_kI^\alpha_H\Psi_1(r){}^s_kI^\alpha_Hh(r) + {}^s_kI^\alpha_Hg(r){}^s_kI^\alpha_H\phi_1(r). \end{aligned} \tag{36}$$

Proof. From (A_1) and (A_2), $\forall p \geq 1, q \geq 1$, we have

$$[\Psi_2(p) - g(p)][h(q) - \phi_1(q)] \geq 0. \tag{37}$$

Therefore,

$$\Psi_2(p)h(q) + \phi_1(q)g(p) \geq \phi_1(p)\Psi_2(p) + g(q)h(q). \tag{38}$$

Multiplying by $\frac{1}{k\Gamma_k(\lambda)}[\log\frac{r}{p}]^{\frac{\lambda}{k}-1}[\frac{p}{r}]^s\frac{1}{p}$ and integrating w.r.t. p on $[1,z]$, we have

$$\begin{aligned} &h(q)\frac{1}{k\Gamma_k(\lambda)}\int_1^r(\log\frac{r}{p})^{\frac{\lambda}{k}-1}(\frac{p}{r})^s\Psi_2(p)\frac{dp}{p} \\ &+ \phi_1(q)\frac{1}{k\Gamma_k(\lambda)}\int_1^r(\log\frac{r}{p})^{\frac{\lambda}{k}-1}(\frac{p}{r})^s g(p)\frac{dp}{p} \\ &\geq \phi_1(q)\frac{1}{k\Gamma_k(\lambda)}\int_1^r(\log\frac{r}{p})^{\frac{\lambda}{k}-1}(\frac{p}{r})^s\Psi_2(p)\frac{dp}{p} \\ &+ h(p)\frac{1}{k\Gamma_k(\lambda)}\int_1^r(\log\frac{r}{p})^{\frac{\lambda}{k}-1}(\frac{p}{r})^s g(p)\frac{dp}{p}. \end{aligned} \tag{39}$$

Using equation (1)

$$h(q){}^s_kI^\lambda_H\Psi_2(r) + \phi_1(q){}^s_kI^\lambda_Hg(r) \geq \phi_1(p){}^s_kI^\lambda_H\Psi_2(r) + h(q){}^s_kI^\lambda_Hh(r). \tag{40}$$

Multiplying by $\frac{1}{k\Gamma_k(\gamma)}[\log\frac{r}{q}]^{\frac{\gamma}{k}-1}[\frac{q}{z}]^s\frac{1}{q}$ and integrating w.r.t. q on $[1,z]$, using the equation (1), we get

$$
\begin{aligned}
&{}^s_kI^\lambda_H\phi_1(r){}^s_kI^\lambda_Hg(r)+{}^s_kI^\lambda_H\Psi_2(r){}^s_kI^\lambda_Hh(r)\\
&\geq {}^s_kI^\lambda_H\Psi_2(r){}^s_kI^\lambda_H\phi_1(r)+{}^s_kI^\lambda_Hg(r){}^s_kI^\lambda_Hh(r). \qquad (41)
\end{aligned}
$$

For (34), (35) and (36), we can use similar steps. □

Corollary 2.2. *For integrable functions g and h on* $[1,\infty]$ *and constants* $l\geq m\geq 0$, $l\neq 0$. *Let* (A_1) *holds. Then for any* $z>1$, $s\epsilon$ *and* $\alpha,\beta,k,q\epsilon$,

$$
\begin{aligned}
&{}^s_kI^\alpha_H(\Psi_2-g)^{\frac{m}{l}}(z)+\frac{m}{l}q^{\frac{m-l}{l}}{}^s_kI^\alpha_H\Psi_2h(z)+\frac{m}{l}q^{\frac{m-l}{l}}{}^s_kI^\alpha_Hg\phi_2(z)\\
&\leq \frac{m}{l}q^{\frac{m-l}{l}}{}^s_kI^\alpha_H\Psi_2\phi_2(z)+\frac{m}{l}q^{\frac{m-l}{l}}{}^s_kI^\alpha_Hgh(z)+\frac{l-m}{l}q^{\frac{m}{l}}{}^s_kI^\alpha_HI(z). \qquad (42)
\end{aligned}
$$

$$
\begin{aligned}
&{}^s_kI^\alpha_H(g-\Psi_1)^{\frac{m}{l}}(z)+\frac{m}{l}q^{\frac{m-l}{l}}{}^s_kI^\alpha_H\Psi_1h(z)+\frac{m}{l}q^{\frac{m-l}{l}}{}^s_kI^\alpha_Hg\phi_1(z)\\
&\leq \frac{m}{l}q^{\frac{m-l}{l}}{}^s_kI^\alpha_H\Psi_1\phi_1(z)+\frac{m}{l}q^{\frac{m-l}{l}}{}^s_kI^\alpha_Hgh(z)+\frac{l-m}{l}q^{\frac{m}{l}}{}^s_kI^\alpha_HI(z). \qquad (43)
\end{aligned}
$$

$$
\begin{aligned}
&{}^s_kI^\alpha_H(g-\Psi_1)^{\frac{m}{l}}(z){}^s_kI^\beta_H(h-\phi_1)^{\frac{m}{l}}(z)+\frac{m}{l}q^{\frac{m-l}{l}}{}^s_kI^\alpha_Hg(z){}^s_kI^\beta_H\phi_1(z)\\
&\leq \frac{m}{l}q^{\frac{m-l}{l}}{}^s_kI^\alpha_H\Psi_1(z){}^s_kI^\alpha_H\phi_1(z)+\frac{m}{l}q^{\frac{m-l}{l}}{}^s_kI^\alpha_Hg(z){}^s_kI^\beta_Hh(z) \qquad (44)\\
&+\frac{l-m}{l}q^{\frac{m}{l}}{}^s_kI^{\alpha+\beta}_HI(z).
\end{aligned}
$$

Theorem 2.6. *Let g and h are positive valued and continuous functions on* $[0.\infty]$ *such that* $g\leq h$. *If g is increasing and* $\frac{g}{h}$ *is decreasing on* $[0,\infty]$, *then for any* $q\geq 0$, $s\epsilon$, $\alpha,\beta,k,z\epsilon$,

$$
\frac{{}^s_kI^\alpha_H[g](z){}^s_kI^\beta_H[h^q](z)+{}^s_kI^\alpha_H[H^q](z){}^s_kI^\beta_H[g](z)}{{}^s_kI^\alpha_H[h](z){}^s_kI^\beta_H[g^q](z)+{}^s_kI^\alpha_H[g^q](z){}^s_kI^\beta_H[h](z)}\geq 1 \qquad (45)
$$

Proof. Using $g\leq h$, we can get

$$
hg^{q-1}(z)\leq h^q(z). \qquad (46)
$$

Multiplying (46) by $\frac{1}{k\Gamma_k(\alpha)}[\log\frac{z}{v}]^{\frac{\alpha}{k}-1}[\frac{v}{z}]^s\frac{1}{v}$ and integrating w.r.t. v on $[1,z]$,

$$
\begin{aligned}
&\frac{1}{k\Gamma_k(\alpha)}\int_1^z(\log\frac{z}{v})^{\frac{\alpha}{k}-1}(\frac{v}{z})^shg^{q-1}(v)\frac{dv}{v}\\
&\leq \frac{1}{k\Gamma_k(\alpha)}\int_1^z(\log\frac{z}{v})^{\frac{\alpha}{k}-1}(\frac{v}{z})^sh^q(v)\frac{dv}{v}. \qquad (47)
\end{aligned}
$$

Using the equation (1)

$$ {}^{s}_{k}I^{\alpha}_{H} hg^{q-1}(z) \leq {}^{s}_{k}I^{\alpha}_{H} h^{q}(z). \tag{48} $$

Multiplying by ${}^{s}_{k}I^{\beta}_{H}[g](z)$

$$ {}^{s}_{k}I^{\alpha}_{H} hg^{q-1}(z) {}^{s}_{k}I^{\beta}_{H}[g](z) \leq {}^{s}_{k}I^{\beta}_{H}[g](z) {}^{s}_{k}I^{\alpha}_{H} h^{q}(z). \tag{49} $$

Multiplying (46) by $\frac{1}{k\Gamma_k(\beta)}[\log \frac{z}{w}]^{\frac{\beta}{k}-1}[\frac{w}{z}]^{s}\frac{1}{w}$ and integrating w.r.t. w on $[1, z]$,

$$ \begin{aligned} &\frac{1}{k\Gamma_k(\beta)} \int_1^z (\log \frac{z}{w})^{\frac{\beta}{k}-1} (\frac{w}{z})^{s} hg^{q-1}(w) \frac{dw}{w} \\ &\leq \frac{1}{k\Gamma_k(\beta)} \int_1^z (\log \frac{z}{w})^{\frac{\beta}{k}-1} (\frac{w}{z})^{s} h^{q}(w) \frac{dw}{w}. \end{aligned} \tag{50} $$

Using the equation (1), we get

$$ {}^{s}_{k}I^{\beta}_{H} hg^{q-1}(z) \leq {}^{s}_{k}I^{\alpha}_{H} h^{q}(z). \tag{51} $$

Multiplying by ${}^{s}_{k}I^{\alpha}_{H}[g](z)$

$$ {}^{s}_{k}I^{\beta}_{H} hg^{q-1}(z) {}^{s}_{k}I^{\alpha}_{H}[g](z) \leq {}^{s}_{k}I^{\alpha}_{H}[g](z) {}^{s}_{k}I^{\beta}_{H} h^{q}(z). \tag{52} $$

Adding (49) and (52), then simplifying we get

$$ \frac{{}^{s}_{k}I^{\alpha}_{H}[g](z) {}^{s}_{k}I^{\beta}_{H}[h^{q}](z) + {}^{s}_{k}I^{\beta}_{H}[g](z) {}^{s}_{k}I^{\alpha}_{H}[H^{q}](z)}{{}^{s}_{k}I^{\beta}_{H}[hg^{q-1}](z) {}^{s}_{k}I^{\alpha}_{H}[g](z) + {}^{s}_{k}I^{\alpha}_{H}[hg^{q-1}](z) {}^{s}_{k}I^{\beta}_{H}[g](z)} \geq 1. \tag{53} $$

Substituting $g = g^{q-1}$ and $f = g$ in Theorem 2.3, we get

$$ \frac{{}^{s}_{k}I^{\alpha}_{H}[hg^{q-1}](z) {}^{s}_{k}I^{\beta}_{H}[g](z) + {}^{s}_{k}I^{\alpha}_{H}[g](z) {}^{s}_{k}I^{\beta}_{H}[hg^{q-1}](z)}{{}^{s}_{k}I^{\alpha}_{H}[g^{q}](z) {}^{s}_{k}I^{\beta}_{H}[h](z) + {}^{s}_{k}I^{\alpha}_{H}[h](z) {}^{s}_{k}I^{\beta}_{H}[g^{q}](z)} \geq 1. \tag{54} $$

(27) and (28) give (23). □

Corollary 2.3. *Let g and h are positive valued and continuous functions on $[0.\infty]$ such that $g \leq h$. If g is increasing and $\frac{g}{h}$ is decreasing on $[0, \infty]$, then for any $q \geq 0$, $s\epsilon$, $\alpha, k, z\epsilon$,*

$$ \frac{{}^{s}_{k}I^{\alpha}_{H}[g](z)}{{}^{s}_{k}I^{\alpha}_{H}[h](z)} \geq \frac{{}^{s}_{k}I^{\alpha}_{H}[g^{q}](z)}{{}^{s}_{k}I^{\alpha}_{H}[h]^{q}(z)}. \tag{55} $$

Proof. By substituting $\beta = \alpha$ (45), we get

$$ \frac{{}^{s}_{k}I^{\alpha}_{H}[g](z) {}^{s}_{k}I^{\alpha}_{H}[h]^{q}(z)}{{}^{s}_{k}I^{\alpha}_{H}[h](z) {}^{s}_{k}I^{\alpha}_{H}[g^{q}](z)} \geq 1, \tag{56} $$

which gives (55). □

Theorem 2.7. *For integrable functions g and h on $[0.\infty]$ satisfying $\frac{1}{\theta_1}+\frac{1}{\theta_2}=1$, $\theta_1,\theta_2\epsilon(0,\infty)$. Let$(A_1$ and $(A_2$ holds. Then, for $z>1$, $s\epsilon$, $\alpha,\beta,k\epsilon$, we have*

$$\frac{1}{\theta_1}{}_k^sI_H^{\beta}(I)(z){}_k^sI_H^{\alpha}(\Psi_2-g)^{\theta_1}(z) + \frac{1}{\theta_2}{}_k^sI_H^{\alpha}(I)(z){}_k^sI_H^{\alpha}(\phi_2-h)^{\theta_2}(z) \geq {}_k^sI_H^{\alpha}(\Psi_2-g)(z){}_k^sI_H^{\beta}(\phi_2-h)(z). \tag{57}$$

$$\frac{1}{\theta_1}{}_k^sI_H^{\beta}(I)(z){}_k^sI_H^{\alpha}(g-\Psi_1)^{\theta_1}(z)+\frac{1}{\theta_2}{}_k^sI_H^{\alpha}(I)(z){}_k^sI_H^{\beta}(h-\phi_1)^{\theta_2}(z) \geq {}_k^sI_H^{\alpha}(g-\Psi_1)(z){}_k^sI_H^{\beta}(h-\phi_1)(z), \tag{58}$$

$$\frac{1}{\theta_1}{}_k^sI_H^{\beta}(\phi_2-h)^{\theta_1}(z){}_k^sI_H^{\alpha}(\Psi_2-g)^{\theta_1}(z) + \frac{1}{\theta_2}{}_k^sI_H^{\alpha}(\phi_2-g)^{\theta_2}(z){}_k^sI_H^{\beta}(\Psi_2-g)^{\theta_2}(z) \geq {}_k^sI_H^{\alpha}(\Psi_2-g)(\phi_2-h)(z){}_k^sI_H^{\beta}(\Psi_2-g)(\phi_2-h)(z), \tag{59}$$

$$\frac{1}{\theta_1}{}_k^sI_H^{\beta}(h-\phi_1)^{\theta_1}(z){}_k^sI_H^{\alpha}(g-\Psi_1)^{\theta_1}(z) + \frac{1}{\theta_2}{}_k^sI_H^{\alpha}(h-\phi_2)^{\theta_2}(z){}_k^sI_H^{\beta}(g-\Psi_1)^{\theta_2}(z) \geq {}_k^sI_H^{\alpha}(g-\Psi_1)(h-\phi_1)(z){}_k^sI_H^{\beta}(g-\Psi_1)(h-\phi_1)(z). \tag{60}$$

Proof. By Young's inequality

$$\frac{1}{\theta_1}(x)^{\theta_1}+\frac{1}{\theta_2}(y)^{\theta_2}\geq xy, \qquad \forall x,y\epsilon[1,\infty],\theta_1,\theta_2\epsilon(0,\infty), \tag{61}$$

Also,

$$\frac{1}{\theta_1}+\frac{1}{\theta_2}=1, \qquad \theta_1,\theta_2\epsilon(0,\infty). \tag{62}$$

Let $x=\Psi_2(v)-g(v($ and $y=\phi_2(w)-h(w)$, $v,w\epsilon(0,\infty)$, we get

$$\frac{1}{\theta_1}[\Psi_2(v)-g(v)]^{\theta_1}+\frac{1}{\theta_2}(\phi_2(w)-h(w))^{\theta_2}\geq[\Psi_2(v)-g(v)][\phi_2(w)-h(w)]. \tag{63}$$

Multiplying by $\frac{1}{k\Gamma_k(\alpha)}[\log\frac{z}{v}]^{\frac{\alpha}{k}-1}[\frac{v}{z}]^s\frac{1}{v}$ and integrating w.r.t. v on $[1,z]$,

$$\frac{1}{\theta_1}\frac{1}{k\Gamma_k(\alpha)}\int_1^z(\log\frac{z}{v})^{\frac{\alpha}{k}-1}(\frac{v}{z})^s[\Psi_2(v)-g(v)]^{\theta_1}\frac{dv}{v} + \frac{1}{\theta_2}[\phi_2(w)-h(w)]^{\theta_2}\frac{1}{k\Gamma_k(\alpha)}\int_1^z(\log\frac{z}{v})^{\frac{\alpha}{k}-1}(\frac{v}{z})^s\frac{dv}{v} \geq[\phi_2(w)-h(w)]\frac{1}{k\Gamma_k(\alpha)}\int_1^z(\log\frac{z}{v})^{\frac{\alpha}{k}-1}(\frac{v}{z})^s[\Psi_2(v)-g(v)]\frac{dv}{v}. \tag{64}$$

Using the equation (1)

$$\frac{1}{\theta_1}{}^s_kI^\alpha_H[\Psi_2-g]^{\theta_1(z)}+\frac{1}{\theta_2}(\phi_2(w)-h(w))^{\theta_2}{}^s_kI^\alpha_H[I](z) \geq [\phi_2(w)-h(w)]^s_kI^\alpha_H[\Psi_2-g](z). \tag{65}$$

Multiplying by $\frac{1}{k\Gamma_k(\beta)}[\log\frac{z}{w}]^{\frac{\beta}{k}-1}[\frac{w}{z}]^s\frac{1}{w}$ and integrating w.r.t. w on $[1,\infty]$, and usingthe definition (1), we have

$$\frac{1}{\theta_1}{}^s_kI^\beta_H(I)(z){}^s_kI^\alpha_H(\Psi_2-g)^{\theta_1}(z)+\frac{1}{\theta_2}{}^s_kI^\alpha_H(I)(z){}^s_kI^\beta_H(\phi_2-h)^{\theta_2}(z) \geq {}^s_kI^\alpha_H(\Psi_2-g)(z){}^s_kI^\beta_H(\phi_2-h)(z). \tag{66}$$

For (58), (59) and (60), we can use similar steps. □

Theorem 2.8. *For integrable function g on $[1.\infty]$ satisfying $\theta_1+\theta_2=1$, $\theta_1,\theta_2\epsilon(0,\infty)$. Let$(A_1$ holds. Then for $z>1$, $s\epsilon$, and $\alpha,\beta,k\epsilon$,*

$$\begin{aligned}\theta_1{}^s_kI^\beta_H(I)(z){}^s_kI^\alpha_H(\Psi_2(z))+\theta_2{}^s_kI^\alpha_H(I)(z){}^s_kI^\beta_Hg(z) &\geq {}^s_kI^\alpha_H(\Psi_2-g)^{\theta_1}(z){}^s_kI^\beta_H(g-\Psi_1)^{\theta_1}\\ &+\theta_1{}^s_kI^\beta_H(I)(z){}^s_kI^\alpha_Hg(z)+\theta_2{}^s_kI^\alpha_H(I)(z){}^s_kI^\beta_H\Psi_1(z).\end{aligned} \tag{67}$$

Proof. By the weighted AM-GM inequality,

$$\theta_1(x)^{\theta_1}+\theta_2(y)^{\theta_2}\geq x^{\theta_1}y^{\theta_2}, \qquad \forall x,y\epsilon[0,\infty],\theta_1,\theta_2\epsilon(0,\infty). \tag{68}$$

Also

$$\theta_1+\theta_2=1, \qquad \theta_1,\theta_2\epsilon(0,\infty). \tag{69}$$

Let $x=\Psi_2(p)-g(p)$ and $y=g(q)-\Psi_2(q)$, $p,q\epsilon(1,\infty)$, we get

$$\theta_1[\Psi_2(p)-g(p)]+\theta_2(g(q)-\Psi_1(q))\geq[\Psi_2(p)-g(p)]^{\theta_1}[g(q)-\Psi_1(q)]^{\theta_2}. \tag{70}$$

Multiplying by $\frac{1}{k\Gamma_k(\alpha)}[\log\frac{z}{p}]^{\frac{\alpha}{k}-1}[\frac{p}{z}]^s\frac{1}{p}$ and integrating w.r.t. p on $[1,\infty]$,

$$\begin{aligned}&\frac{\theta_1}{k\Gamma_k(\alpha)}\int_1^z(\log\frac{z}{p})^{\frac{\alpha}{k}-1}(\frac{p}{z})^s[\Psi_2(p)-g(p)]\frac{dp}{p}\\ &+\theta_2[g(q)-\Psi_1(q)]\frac{1}{k\Gamma_k(\alpha)}\int_1^z(\log\frac{z}{p})^{\frac{\alpha}{k}-1}(\frac{p}{z})^s\frac{dp}{p}\\ &\geq[g(q)-\Psi_1(q)]^{\theta_1}\frac{1}{k\Gamma_k(\alpha)}\int_1^z(\log\frac{z}{p})^{\frac{\alpha}{k}-1}(\frac{p}{z})^s[\Psi_2(p)-g(p)]^{\theta_2}\frac{dp}{p}.\end{aligned} \tag{71}$$

Using the equation (1)

$$\begin{aligned}&\theta_1{}^s_kI^\alpha_H[\Psi_2-g](z)+\theta_2(g(q)-\phi_1(q)){}^s_kI^\alpha_H(I)(z)\\ &\geq[g(q)-\Psi_1(q)]^{\theta_2}{}^s_kI^\alpha_H[\Psi_2-h]^{\theta_1}(z).\end{aligned} \tag{72}$$

Multiplying by $\frac{1}{k\Gamma_k(\beta)}[\log\frac{z}{q}]^{\frac{\beta}{k}-1}[\frac{q}{z}]^s\frac{1}{q}$ and integrating w.r.t. q on $[1,\infty]$,

$$\begin{aligned}
&\theta_1{}^s_k IH^{\alpha}(\Psi_2-g)(z)\frac{1}{k\Gamma_k(\beta)}\int_1^z(\log\frac{z}{q})^{\frac{\beta}{k}-1}(\frac{q}{z})^s\frac{dq}{q}\\
&+\theta_2{}^s_k I_H^{\alpha}(I)(z)\frac{1}{k\Gamma_k(\beta)}\int_1^z(\log\frac{z}{q})^{\frac{\beta}{k}-1}(\frac{q}{z})^s\frac{dq}{q}\\
&\geq {}^s_k IH^{\alpha}(\Psi_2-g)(z)\frac{1}{k\Gamma_k(\beta)}\int_1^z(\log\frac{z}{q})^{\frac{\beta}{k}-1}(\frac{q}{z})^s[g(q)-\Psi_1(q)]^{\theta_2}\frac{dq}{q}.
\end{aligned}\tag{73}$$

Using the equation (1)

$$\begin{aligned}
&\theta_1{}^s_k I_H^{\beta}(I)(z){}^s_k I_H^{\alpha}(\Psi_2-g(z))(z)+\theta_2{}^s_k I_H^{\alpha}(I)(z){}^s_k I_H^{\beta}[g-\Psi_1](z)\\
&\qquad\geq {}^s_k I_H^{\alpha}(\Psi_2-g)^{\theta_1}(z){}^s_k I_H^{\beta}(g-\Psi_1)^{\theta_1}(z).
\end{aligned}\tag{74}$$

Due to linearity of integrals we get (67). □

Theorem 2.9. *For integrable functions g and h on $[1,\infty]$ satisfying $\theta_1+\theta_2=1$, $\theta_1,\theta_2\epsilon(0,\infty)$. Let$(A_1$. Let (A_1) and (A_2) hold. Then, for $r>1$, $s\epsilon$, $\lambda,\gamma,k\epsilon$,*

$$\begin{aligned}
&\theta_1{}^s_k I_H^{\gamma}(I)(r){}^s_k I_H^{\lambda}(\Psi_2)(r)+\theta_2{}^s_k I_H^{\lambda}(I)(r){}^s_k I_H^{\gamma}[\phi_2](r)\\
&\geq {}^s_k I_H^{\lambda}(\Psi_2-g)^{\theta_1}(r){}^s_k I_H^{\gamma}(\phi_2-h)^{\theta_2}(r)+\theta_1{}^s_k I_H^{\gamma}(I)(r){}^s_k I_H^{\lambda}(g)(r)\\
&+\theta_2{}^s_k I_H^{\lambda}(I)(r){}^s_k I_H^{\gamma}(h)(r),
\end{aligned}\tag{75}$$

$$\begin{aligned}
&\theta_1{}^s_k I_H^{\gamma}(I)(r){}^s_k I_H^{\lambda}(g)(r)+\theta_2{}^s_k I_H^{\lambda}(I)(r){}^s_k I_H^{\gamma}(h)(r)\\
&\geq {}^s_k I_H^{\lambda}(g-\Psi_1)^{\theta_1}(r){}^s_k I_H^{\gamma}(h-\phi_1)^{\theta_2}(r)+\theta_1{}^s_k I_H^{\gamma}(I)(r){}^s_k I_H^{\lambda}(\Psi_2)(r)\\
&+\theta_1{}^s_k I_H^{\gamma}(I)(r){}^s_k I_H^{\gamma}(\phi_1)(r),
\end{aligned}\tag{76}$$

$$\begin{aligned}
&\theta_1{}^s_k I_H^{\gamma}(\phi 1)(r){}^s_k I_H^{\lambda}(\Psi_2)(r)+\theta_1{}^s_k I_H^{\gamma}(h)(r){}^s_k I_H^{\gamma}g(r)\\
&+\theta_2{}^s_k I_H^{\lambda}(\phi 2)(r){}^s_k I_H^{\gamma}(\Psi_2)(r)+\theta_2{}^s_k I_H^{\gamma}(g)(r){}^s_k I_H^{\gamma}h(r)\\
&\geq {}^s_k I_H^{\lambda}(\Psi_2-g)^{\theta_1}(\phi_2-h)^{\theta_1}(r){}^s_k I_H^{\gamma}(\phi_2-h)^{\theta_1}(\Psi_2-g)^{\theta_2}(r)\\
&+\theta_1{}^s_k I_H^{\gamma}(h)(r){}^s_k I_H^{\lambda}(\Psi_2)(r)+\theta_2{}^s_k I_H^{\lambda}(\phi_2)(r){}^s_k I_H^{\gamma}(g)(r)\\
&+\theta_2{}^s_k I_H^{\lambda}h(r){}^s_k I_H^{\gamma}(\Psi_2)(r),
\end{aligned}\tag{77}$$

$$\begin{aligned}
&\theta_1{}^s_k I_H^{\gamma}(h)(r){}^s_k I_H^{\lambda}(g)(r)+\theta_1{}^s_k I_H^{\gamma}(\phi_1)(r){}^s_k I_H^{\lambda}\Psi_2(r)\\
&+\theta_2{}^s_k I_H^{\lambda}(\phi 1)(r){}^s_k I_H^{\gamma}(\Psi_1)(r)+\theta_2{}^s_k I_H^{\gamma}(g)(r){}^s_k I_H^{\lambda}h(r)\\
&\geq {}^s_k I_H^{\lambda}(g-\Psi_1)^{\theta_1}(h-\phi_2)^{\theta_2}(r){}^s_k I_H^{\gamma}(h-\phi_1)^{\theta_1}(g-\Psi_1)^{\theta_2}(r)\\
&+\theta_1{}^s_k I_H^{\gamma}(\phi_1)(r){}^s_k I_H^{\lambda}(g)(r)+\theta_1{}^s_k I_H^{\gamma}(h)(r){}^s_k I_H^{\lambda}(\Psi_1)(r)\\
&+\theta_2{}^s_k I_H^{\lambda}h(r){}^s_k I_H^{\gamma}(\Psi_1)(r)+\theta_2{}^s_k I_H^{\lambda}\phi_1(r){}^s_k I_H^{\gamma}(g)(r).
\end{aligned}\tag{78}$$

Theorem 2.10. *Let g, Ψ_1 and Ψ_2 are integrable functions on $[1,\infty]$, assume that condition (A_1) holds. Then for $r>1$, $s\epsilon$, $\lambda, k\epsilon$,*

$$\begin{aligned}
&{}^s_kI^\lambda_H(I)(r){}^s_kI^\lambda_H g^2(r)-[{}^s_kI^\lambda_H g(r)]^2\\
&=[{}^s_kI^\lambda_H\Psi_2(r)-{}^s_kI^\lambda_H g(r)][{}^s_kI^\lambda_H g(r)-{}^s_kI^\lambda_H\Psi_1(r)]\\
&-{}^s_kI^\lambda_H(I)(r){}^s_kI^\lambda_H[\Psi_2-g](r)[g-\Psi_1(r)]+{}^s_kI^\lambda_H(I)(r){}^s_kI^\lambda_H\Psi_1 g(r)\\
&-{}^s_kI^\lambda_H(\Psi_1(r)){}^s_kI^\lambda_H g(r)+{}^s_kI^\lambda_H(\Psi_2)g(r){}^s_kI^\lambda_H(I)(r)-{}^s_kI^\lambda_H(\Psi_2)(r){}^s_kI^\lambda_H g(r)\\
&+{}^s_kI^\lambda_H(\Psi_1)(r){}^s_kI^\lambda_H\Psi_2(r)-{}^s_kI^\lambda_H(\Psi_1\Psi_2(r)){}^s_kI^\lambda_H(I)(r).
\end{aligned}\tag{79}$$

Proof. For $p>1, q>1$, we have

$$\begin{aligned}
&[\Psi_2(q)-g(q)][g(p)-\Psi_1(p)]\\
&+[\Psi_2(p)-g(p)][g(q)-\Psi_1(q)]-[\Psi_2(p)-g(p)],\\
&[g(p)-\Psi_1(p)]-[\Psi_2(q)-g(q)][g(q)-\Psi_1(q)]=g^2(p)+g^2(q)-2g(p)g(q)\\
&+\Psi_2(q)g(p)+\Psi_1(p)g(q)-\Psi_1(p)\Psi_2(q)+\Psi_2(p)g(q)+\Psi_1(q)g(p)\\
&-\Psi_2(p)g(p)-\Psi_1(p)\Psi_2(p)+\Psi_1(q)\Psi_2(q){}^s_kI^\lambda_H(I)(r)-\Psi_1(q)g(q){}^s_kI^\lambda_H(I)(r).
\end{aligned}\tag{80}$$

Multiplying by $\frac{1}{k\Gamma_k(\lambda)}[\log\frac{z}{q}]^{\frac{\lambda}{k}-1}[\frac{q}{z}]^s\frac{1}{q}$ and integrating w.r.t. q on $[1,\infty]$, and using equation (1), we can get (79). □

Theorem 2.11. *Let g, h be two positive functions on $[0,\infty]$, such that $\forall z>1$, $s\epsilon$, $\alpha, k\epsilon$, $q\ge 1$ and ${}^s_kI^\alpha_H g^p(z)<\infty$, ${}^s_kI^\alpha_H h^p(z)<\infty$. If $0<m\le\frac{g(v)}{h(v)}\le M$, m, M,ϵ, $\forall v\epsilon(0,z)$. Then, we have*

$$[{}^s_kI^\alpha_H g^q(z)]^{\frac{1}{q}}+[{}^s_kI^\alpha_H[h]^q(z)]^{\frac{1}{q}}\le\frac{M(m+2)+1}{(M+1)(m+1)}[{}^s_kI^\alpha_H[g+h]^q(z)]^{\frac{1}{q}},\tag{81}$$

$$[{}^s_kI^\alpha_H g^q(z)]^{\frac{2}{q}}+[{}^s_kI^\alpha_H[h]^q(z)]^{\frac{2}{q}}\ge\frac{(M+1)(m+1)}{(M)}[{}^s_kI^\alpha_H[g]^q(z)]^{\frac{1}{q}}[{}^s_kI^\alpha_H[h]^q(z)]^{\frac{1}{q}}.\tag{82}$$

Proof. Using $\frac{g(v)}{h(v)}\le M, \forall v\epsilon(0,z)$, we can get $\frac{[g+h](v)}{h(v)}\le M+1$ and hence

$$(M+1)^q g^q\le M^q[g+h]^q(v).\tag{83}$$

Multiplying by $\frac{1}{k\Gamma_k(\alpha)}[\log\frac{z}{v}]^{\frac{\alpha}{k}-1}[\frac{v}{z}]^s\frac{1}{v}$ and integrating w.r.t. v on $[1,\infty]$,

$$\frac{(M+1)}{k\Gamma_k(\alpha)}[\int_1^z\log\frac{z}{v}]^{\frac{\alpha}{k}-1}[\frac{v}{z}]^s\frac{1}{v}\le\frac{(M)^q}{k\Gamma_k(\alpha)}[\int_1^z\log\frac{z}{v}]^{\frac{\alpha}{k}-1}[\frac{v}{z}]^s\frac{1}{v}.\tag{84}$$

Using the equation (1)

$$(M+1)^q{}^s_kI^\alpha_H g^q(z)\le(M)^q{}^s_kI^\alpha_H[g+h]^q(z).\tag{85}$$

Which gives

$$[{}^s_k I^\alpha_H g^q(z)]^{\frac{1}{q}} \leq \frac{M}{M+1}[{}^s_k I^\alpha_H [g+h]^q(z)]^{\frac{1}{q}}. \tag{86}$$

Using $m \leq \frac{g(v)}{h(v)}$, $\forall v \epsilon (0,\infty)$, we get

$$(1+\frac{1}{m})^q h^q(v) \leq (\frac{1}{m})^q [g+h]^q(v), \tag{87}$$

Multiplying by $\frac{1}{k\Gamma_k(\alpha)}[\log \frac{z}{v}]^{\frac{\alpha}{k}-1}[\frac{v}{z}]^s \frac{1}{v}$ and integrating w.r.t. v on $[1,\infty]$,

$$\begin{aligned} &\frac{(1+\frac{1}{m})}{k\Gamma_k(\alpha)}[\int_1^z \log \frac{z}{v}]^{\frac{\alpha}{k}-1}[\frac{v}{z}]^s h^q \frac{dv}{v} \\ &\leq \frac{(\frac{1}{m})^q}{k\Gamma_k(\alpha)}[\int_1^z \log \frac{z}{v}]^{\frac{\alpha}{k}-1}[\frac{v}{z}]^s [g+h]^q(v)\frac{dv}{v}. \end{aligned} \tag{88}$$

Using the equation (1)

$$(1+\frac{1}{m})^q {}^s_k I^\alpha_H h^q(v) \leq (\frac{1}{m})^q {}^s_k I^\alpha_H [g+h]^q(v). \tag{89}$$

Which gives,

$$[{}^s_k I^\alpha_H h^q(v)]^{\frac{1}{q}} \leq (\frac{1}{m+1})[{}^s_k I^\alpha_H [g+h]^q(z)]^{\frac{1}{q}}. \tag{90}$$

Adding (42) and (43), we get (40).

Multiplying (42) and (43)

$$\frac{(M+1)(m+1)}{M}[{}^s_k I^\alpha_H g^q(z)]^{\frac{1}{q}}[{}^s_k I^\alpha_H h^q(z)]^{\frac{1}{q}} \leq [{}^s_k I^\alpha_H [g+h]^q(z)]^{\frac{1}{q}}. \tag{91}$$

Using Minkowski inequalities on R.H.S, we can get (41). □

Applications

There are many applications of the fractional integral inequalities. Some of them are as under. In boundary value problems, we can use fractional integral inequalities to establish uniqueness of the solutions. They are also used in finding the unique solutions in fractional partial differential equations.

Conclusions

Results of fractional integral inequalities are determined by using extended Hadamard k-fractional integral. With these inequalities the uniqueness and continuous dependence of the solution of the nonlinear fractional differential equations can also be established. Furthermore, we can also extend these inequalities for α by analytical continuation.

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