Strongly regular relation and n-Bell groups derived from it

Aazam Nokhbeh
Phd Student of Mathematics
Tarbiat Modares University
Tehran
Iran
a.nokhbeh@modares.ac.ir

Ali Iranmanesh*
Department of Mathematics
Faculty of Mathematical Sciences
Tarbiat Modares University
P. O. Box 14115-137
Tehran
Iran
iranmanesh@modares.ac.ir

M.M Zahedi
Graduate University of Advanced Technology
Kerman
Iran
zahedi_ mm@kgut.ac.ir

Abstract. A new strongly regular relation θ_n^* is defined on polygroup P such that the quotient P/θ_n^*, the set of all equivalence classes, is a Bell group for $n \in \{2, 3\}$.
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1. Introduction

Hyperstructure theory was first initiated by Marty [15] in 1934. Let H be a non-empty set and $o : H \times H \longrightarrow P^*(H)$ be a hyperopration where $P^*(H)$ is the family of non-empty subset of H. The couple (H,o) is called a hypergroupoid. For any two non-empty subset A and B of H and $x \in H$, we define $A \circ B = \bigcup_{a\in A,b\in B} a \circ b, A \circ x = A \circ \{x\}$ and $B\ o\ x = B\ o\ \{x\}$. A hypergroupoid (H,o) is called semihypergroup if for all $a, b, c \in H$, we have $(a \circ b) \circ c = a \circ (b \circ c)$ which means that $\bigcup_{u\in a\circ b} u \circ c = \bigcup_{v\in b\circ c} a \circ v$ and hypergrupoid (H,o) is called qusihypergroup if for all a of H, we have $a \circ H = H \circ a = H$, which is called reproduction axiom. This axiom means that for any $x, y \in H$, there exist $u, v \in H$ such that $y \in x \circ u, y \in v \circ x$. A hypergroupoid $(H, \circ)$ which is both a semihypergroup and a qusihypergroup is called hypergroup.

∗. Corresponding author

Definition 1.1 ([6]). *A polygroup is a hypergroup $\langle P, \cdot, e, ^{-1} \rangle$ where $e \in P, ^{-1}$ is a unitary operation on P, and the following axiom hold for all $x, y, z \in P$*

(*i*) $e \cdot x = x \cdot e = x$;

(*ii*) $x \in y \cdot z \Longrightarrow y \in x \cdot z^{-1} \Longrightarrow z \in y^{-1} \cdot x$.

Definition 1.2 ([5]). *Let $(H, \cdot)$ be a hypergroup and $\rho \subseteq H \times H$ be an equivalence relation. For non-empty subset A and B of H, we define $A \ \bar{\bar{\rho}} \ B$ if and only if $a \ \rho \ b$, for all $a \in A$ and $b \in B$. The relation ρ is called strongly regular on the left (on the right) if $x \ \rho \ y$, then $a \circ x \ \bar{\bar{\rho}} \ a \circ y$ ($x \circ a \ \bar{\bar{\rho}} \ y \circ b$, respectively), for all $x, y, a \in H$.*

Moreover, ρ is called strongly regular if it is strongly regular on the left and on the right.

Theorem 1.3 ([4]). *If $(H, \cdot)$ is a hypergroup and ρ is a strongly regular relation on H, then H/ρ is a group under the operation:*

$$\rho(x) \otimes \rho(y) = \rho(z), \quad \forall z \in x \cdot y.$$

For all $n \geq 1$, we define the relation β_n on a semihypergroup H, as follows, $a \ \beta_n \ b$, if and only if there exists $(x_1, \ldots, x_n)$ in H^n such that $\{a, b\} \subseteq \prod_{i=1}^{n} x_i$ and $\beta = \bigcup_{n \geq 1} \beta_n$, where $\beta_1 = \{(x, x); x \in H\}$, is the diagonal relation on H. This relation was introduced by Koskas [14]. Suppose that β^* is the transitive closure of β, the relation β^* is a strongly regular relation [4].

In [11], $\gamma = \bigcup_{n \geq 1} \gamma_n$, where γ_1 is the diagonal relation and for every integer $n > 1$, γ_n is the relation defined as follows, $x \ \gamma_n \ y$ if and only if there exists $(z_1, \cdots, z_n)$ in H^n and $\tau \in S_n$ such that $x \in \prod_{i=1}^{n} z_i$ and $y \in \prod_{i=1}^{n} z_{\tau(i)}$, where S_n is the symmetric group of order n. Suppose that γ^* is the transitive closure of γ. The relation γ^* is a strongly regular relation [11].

The relation β^* is the least equivalence relation on hypergroup H such that the quotient H/β^* is a group, while γ^* is the least equivalence relation on hypergroup H, such that the quotient H/γ^* is an abelian group.

In [12], $\tau_n = \bigcup_{m \geq 1} \tau_{m,n}$, where $\tau_{1,n}$ is the diagonal relation and for every integer $m > 1$, $\tau_{m,n}$ is the relation defined as follows, $x \ \tau_{m,n} \ y$ if and only if there exists $(z_1, \cdots, z_m)$ in H^m, and $\sigma \in S_m$ such that $\sigma(i) = i$, if $z_i \notin H^{(n)}$ such that $x \in \prod_{i=1}^{m} z_i$ and $y \in \prod_{i=1}^{m} z_{\sigma(i)}$, where

(1) $H^{(0)} = H$;

(2) $H^{(k+1)} = \{h \in H^{(k)} \mid xy \cap hyx \neq \emptyset \ ; \ x, y \in H^{(k)}\}$.

Clearly, for every integer $n \geq 1$, the relation τ_n is reflexive and symmetric.

Now, suppose that τ_n^* is the transitive closure of τ_n. The relation τ_n^* is strongly regular such that the quotient H/τ_n^* is a solubale group of the class at most $n + 1$.

In [1], $\nu_n = \bigcup_{m\geq 1} \nu_{m,n}$, where $\nu_{1,n}$ is the diagonal relation and for every integer $m > 1$, $\nu_{m,n}$ is the relation defined as follows, $x\ \nu_{m,n}\ y$ if and only if, there exists $(z_1, \cdots, z_m)$ in H^m and $\sigma \in S_m$ such that $\sigma(i) = i$, if $z_i \notin L_n(H)$ such that $x \in \prod_{i=1}^m z_i$ and $y \in \prod_{i=1}^m z_{\sigma(i)}$, where

(1) $L_0(H) = H$;

(2) $L_{k+1}(H) = \{h \mid xy \cap hyx \neq \emptyset\ ;\ x \in L_k(H),\ y \in H\}$.

Clearly, for every integer $n \geq 1$, the relation ν_n is reflexive and symmetric.

Now, suppose that ν_n^* is the transitive closure of ν_n. The relation ν_n^* is strongly regular such that the quotient H/ν_n^* is a nilpotent group of the class at most $n + 1$.

In [2], $\xi_{n,s} = \bigcup_{m\geq 1} \xi_{m,n,s}$, where $\xi_{1,n,s}$ is the diagonal relation and for every integer $m \geq 1$, $\xi_{m,n,s}$ is the relation defined as follows:

$x\ \xi_{m,n,s}\ y$ if and only if, there exists $(z_1, \cdots, z_m)$ in H^m and $\delta \in S_m$ such that $\delta(i) = i$ if $z_i \notin L_{n,s}(H)$ such that $x \in \prod_{i=1}^m z_i$ and $y \in \prod_{i=1}^m z_{\delta(i)}$, where

(1) $L_{0,s}(H) = H$;

(2) $L_{k+1,s}(H) = \{h \mid xs \cap hsx \neq \emptyset\ ;\ x \in L_{k,s}(H)\}, \forall k \geq 0$,

for fix element $s \in H$.

Obviously, for every $n \geq 1$, the relation $\xi_{n,s}$ is reflexive and symmetric. Now let $\xi_{n,s}^*$ be the transitive closure of $\xi_{n,s}$.

In [2], the authors proved that the relation $\xi_{n,s}^*$ is strongly regular such that the quotient $H/\xi_{n,s}^*$ is an n-Engel group.

Let $n \neq 0, 1$ be an integer. A group G is said to be n-Bell if $[x^n, y] = [x, y^n]$ for all x and y in G, where $[x, y]$ is the commutator of x and y. The study of n-Bell groups was introduced by Kappe and Brandl in [3], [13] and it was also the subject of several papers, see for instance [8], [9], [10] and [18]. For example all of groups of finite exponent dividing n, groups of finite exponent dividing $n - 1$, 2-Engel groups and n-Levi groups, are n-Bell groups (see, [9]).

In this paper, we define a new relation θ_n on a polygroup and then we show that θ_n^* is a strongly regular relation. In continue, we bring some results related to θ_n^* and one of the main result of this paper is about the relation of θ_n^* and n-Bell groups for $n = 2$ and 3. Also, if we set $\theta^* = \bigcap_{n\geq 1} \theta_n^*$, then we show that P/θ^* is a Bell group for any finite polygoup P.

In a polygroup P, the commutator of two elements x, y in P is defined by $[x, y] = \{t \mid t \in xyx^{-1}y^{-1}\}$. If $A \subseteq P$, then $[A, y] = \{t \mid t \in AyA^{-1}y^{-1}\}$.

Theorem 1.4 ([2], Theorem 2.2)**.** *Let P be a polygroup. Then, for all $x, y, h, \in P$, $\{h \mid xy \cap hyx \neq \emptyset\} = \{h \mid h \in xyx^{-1}y^{-1}\}$.*

Remark 1.5. Let P be a polygroup. Then, for all $x, y, h, \in P$ and $n \in N$, $\{h \mid x^n y \cap hyx^n \neq \emptyset\} = \{h \mid h \in x^n yx^{-n}y^{-1}\}$.

Theorem 1.6 ([2], Theorem 2.10)**.** *$H/\xi_{n,s}^*$ is an n-Engel group.*

Theorem 1.7 ([1], Theorem 2.9)**.** *H/ν_n^* is a nilpotent group of the class at most $n+1$.*

2. New strongly regular relation θ_n^*

Now, we introduce a new strongly regular relation θ_n^* on a polygroup P.

In the whole of this paper, P is a polygroup and S_n is symmetric group.

Definition 2.1. *Let P be a polygroup. For fix elements $x, y \in P$, we define:*

(1) $L_{0,x,y}(P) = P$;

(2) $L_{n+1,x,y}(P) = \{h \mid h \in L_{n,x,y}(P), x^{n+1}y \cap hyx^{n+1} \neq \emptyset\}$.

Let $\theta_n = \bigcup_{m\geq 1} \theta_{m,n}$ where $\theta_{1,n}$ is diagonal relation and for every integer $m \geq 1, \theta_{m,n}$ is relation defined as follows:

$x\ \theta_{m,n}\ y$ if and only if, there exists $(z_1, \cdots, z_m)$ in P^m and $\zeta \in S_m$ if, $z_i \notin L_{n,x,y}(P)$ and $z_i^{-1} \notin L_{n,y,x}(P)$, then $\zeta(i) = i$ and $x \in \prod_{i=1}^m z_i$ and $y \in \prod_{i=1}^m z_{\zeta(i)}$.

Clearly, θ_n is reflexive and symmetric. Let θ_n^* be the transitive closure of θ_n.

Theorem 2.2. *For every $n \in \mathbb{N}$, the relation θ_n^* is strongly regular relation.*

Proof. Suppose that $n \in \mathbb{N}$. Clearly, θ_n^* is an equivalence relation. In order to prove that it is strongly regular. First we have to show that if $x\ \theta_n\ y$, then $x \cdot z\ \overline{\overline{\theta}}_n\ y \cdot z$, $z \cdot x\ \overline{\overline{\theta}}_n\ z \cdot y$, for every $z \in P$. Suppose that $x\theta_n y$. Then, there exists $m \in \mathbb{N}$ such that $x\ \theta_{m,n}\ y$. Hence, there exists $(z_1, \cdots, z_m) \in P^m$, $\zeta \in S_m$ with $\zeta(i) = i$ if $z_i \notin L_{n,x,y}(P)$ and $z_i^{-1} \notin L_{n,y,x}(P)$ such that $x \in \prod_{i=1}^m z_i$ and $y \in \prod_{i=1}^m z_{\zeta(i)}$. Suppose that $z \in P$. We have $x \cdot z \subseteq (\prod_{i=1}^m z_i) \cdot z, y \cdot z \subseteq (\prod_{i=1}^m z_{\zeta(i)}) \cdot z$. Now, suppose that $z_{m+1} = z$ and we define the permutation $\zeta' \in S_{m+1}$ as follows:

$$\begin{cases} \zeta'(i) = \zeta(i), & \text{for all } 1 \leq i \leq m, \\ \zeta'(m+1) = m+1. \end{cases}$$

Thus, $x \cdot z \subseteq \prod_{i=1}^{m+1} z_i, y \cdot z \subseteq \prod_{i=1}^{m+1} z_{\zeta'(i)}$. such that $\zeta'(i) = i$ if $z_i \notin L_{n,x,y}(P)$ and $z_i^{-1} \notin L_{n,y,x}(P)$. Therefore, $x \cdot z\ \overline{\overline{\theta}}_n\ y \cdot z$. Similary, we have $z \cdot x\ \overline{\overline{\theta}}_n\ z \cdot y$. Now, if $x\ \theta_n^*\ y$, then, there exists $k \in \mathbb{N}$ and $(x = u_0, u_1, \cdots, u_k = y) \in P^{k+1}$ such that $x = u_0\ \theta_n\ u_1\ \theta_n \cdots \theta_n\ u_{k-1}\ \theta_n\ u_k = y$. Hence, we obtain $x \cdot z = u_0 \cdot z\ \overline{\overline{\theta}}{}_n^*\ u_1 \cdot z\ \overline{\overline{\theta}}{}_n^*\ u_2 \cdot z\ \overline{\overline{\theta}}{}_n^* \cdots \overline{\overline{\theta}}{}_n^*\ u_{k-1} \cdot z\ \overline{\overline{\theta}}{}_n^*\ u_k \cdot z = y \cdot z$ and so $x \cdot z\ \overline{\overline{\theta}}{}_n^*\ y \cdot z$. Similarly, we can prove that $z \cdot x\ \overline{\overline{\theta}}{}_n^*\ z \cdot y$, therefore θ_n^* is strongly regular relation on P. □

Proposition 2.3. *For every $n \in \mathbb{N}$, we have $\theta_{n+1}^* \subseteq \theta_n^*$.*

Proof. Let $x\ \theta_{n+1}\ y$, so, there exists $m \in \mathbb{N}$ and $(z_1, \cdots, z_m) \in P^m$ and $\zeta \in S_m$ such that $\zeta(i) = i$ if $z_i \notin L_{n+1,x,y}(P)$ and $z_i^{-1} \notin L_{n+1,y,x}(P)$, such that $x \in \prod_{i=1}^m z_i$ and $y \in \prod_{i=1}^m z_{\zeta(i)}$. Now, let $\zeta_1 = \zeta$, since $L_{n+1,x,y}(P) \subseteq L_{n,x,y}(P)$ and $L_{n+1,y,x}(P) \subseteq L_{n,y,x}(P)$, we have $x\ \theta_n\ y$. □

Corollary 2.4. *If P is a commutative hypergroup, then $\beta^* = \theta_n^* = \xi_n^* = \nu_n^* = \gamma^*$.*

Definition 2.5 ([13])**.** *Let G be a group and n be an integer. The n-Bell center of G denoted by B_n and defined as follows:*

$$B_n = B(G,n) = \{x \in G \mid [x^n, y] = [x, y^n], ;\ \forall y \in G\}.$$

Clearly, $B(G,0) = B(G,1) = G$, and easy to see that $B(G,2)$ and $B(G,3)$ are subgroup of G.

Remark 2.6. For every integer n, a group is n-Bell if $B(G,n) = G$.

Theorem 2.7. *If P is a polygroup and ρ is a strongly regular relation on P, then for fix elements $x, y \in P$;*

$$L_{n+1,\bar{x},\bar{y}}(\frac{P}{\rho}) = \{[\bar{x}^{n+1}, \bar{y}]\},$$

where $\bar{x}, \bar{y}$ are the classes of x,y with respect to ρ.

Proof. The proof follows from definition of commutator of two elements in a polygroup, Theorem 1.4 and Remark 1.5. □

3. n-Bell groups derived from polygroups for $n \in \{2,3\}$

In this section, we obtain an n-Bell group derived from polygroup for $n = 2,3$, and then we propose an open problem related to n-Bell groups.

Theorem 3.1. *Let P be a polygroup. Then, for $n \in \{2,3\}$, P/θ_n^* is an n-Bell group.*

Proof. Let $G = P/\theta_n^*$. For $n \in \{2,3\}$, we have $B(G,n) \leq G$. By Remark 2.6, it is enough to prove that $G \leq B(G,n)$. For this we should show that for every $\bar{h} \in L_{n,\bar{x},\bar{y}}(G)$ we have $\bar{h}^{-1} \in L_{n,\bar{y},\bar{x}}(G)$ and is obvious by Theorem 2.7. □

Definition 3.2 ([16])**.** *A group G is called an Engel group if, for each ordered pair (x,y) of elements in G, there is a positive integer $n = n(x,y)$, such that $[x,_n y] = 1$.*

Theorem 3.3 ([17])**.** *Let G be a group. Then*

(a) *G is a n-Bell group if and only if G is a $(1-n)$-Bell group;*

(b) *G is a 2-Bell group if and only if G is a 2-Engel group;*

(c) *G is a 3-Bell group if and only if G is a 3-Engel group satisfying the identity $[x,y,y]^3 = 1$, for all $x, y \in G$. In addition G has nilpotent of class at most 4.*

Definition 3.4. *Let H_1 and H_2 be two hypergroups (polygroups), and ρ_1 and ρ_2 be two strongly regular relations. If H_1/ρ_1 and H_2/ρ_2 are isommorphism groups, then we say that ρ_1 is "the same" property to ρ_2.*

Remark 3.5. According to the above definition, θ_2^* is the same property to $\xi_{2,s}^*$, by Theorem 3.1, 3.3 and apply Theorem 1.6, and θ_3^* is the same property to $\xi_{3,s}^*$ and ν_3^*, by Theorem 3.1, 3.3 and apply Theorem 1.6 and 1.7.

Example 3.6 ([2])**.** Let H be $\{e, a, b, c, d, f, g\}$. Consider the non-commutative polygroup $(H, \cdot)$, defined on H as follows: It is easy to see that $H/\beta^* \cong S_3$ (for

.	e	a	b	c	d	f	g
e	e	a	b	c	d	f,g	f,g
a	a	e	d	f,g	b	c	c
b	b	f,g	e	d	c	a	a
c	c	d	f,g	e	a	b	b
d	d	c	a	b	f,g	e	e
f	f,g	b	c	a	e	d	d
g	f,g	b	c	a	e	d	d

more details, see [7]). Since S_3 is not nilpotent, we conclude that $\beta^* \neq \nu_n^*$, hence H/ν_n^* is an abelian group of order less than 6 and the class of nilpotency of H/ν_n^* is one for all $n \in \mathbb{N}$ [1], besides, S_3 is not Engel and $H/\xi_{n,s}^* \subseteq H/\beta^* \cong S_3$, then it concluded $H/\xi_{n,s}^*$ is an abelian group of order less than 6 and H/ξ^* is 1-Engel group. Then, H/θ_2^* is not 2-Bell or 3-Bell group, by apply the Remark 3.5.

Remark 3.7. We know that $B(G, n)$ is called the n-Bell center of G. It is open problem whether the n-Bell center always forms a subgroup. But, it is shown that $B(G, 2)$ is characteristic subgroup of all right 2-Engel elements and $B(G, 3)$ is characteristic subgroup of G which is nilpotent of class at most 4 (see, [13]).

Hence, according to above remark, we can put the following open problem:

Open Problem 3.8. *Let H be non-commutative polygroup, for all $n \geq 4$, is H/θ_n^* a n-Bell group?*

4. On Bell groups derived from finite polygroup

In this section, we introduce a strongly regular relation θ^* on finite polygroup P such that P/θ^* is a Bell group.

Definition 4.1. *Let P be a finite polygroup. Then, we define the relation θ^* on P by $\theta^* = \bigcap_{n\geq 1} \theta_n^*$.*

Definition 4.2. *An equivalence relation ρ on a finite polygroup P, is called Bell if and only if its derived group P/ρ is a Bell group.*

Example 4.3. θ_2^* and θ_3^* are Bell relations. By using the Remark 3.5, and Example 3.3 in [2], Bell relations θ_2^* and θ_3^* are the same with Engel relations $\xi_{2,s}^*$ and $\xi_{3,s}^*$.

Theorem 4.4. *(a) The relation θ^* is a strongly regular relation on a finite polygroup P.*

(b) P/θ^* is a Bell group.

Proof. (a) Since $\theta^* = \bigcap_{n\geq 1} \theta_n^*$, it is easy to see that θ^* is strongly regular relation on P.

(b) By using Proposition 2.3, we conclude that there exists $k \in \mathbb{N}$ $(k \geq 1)$ such that $\theta_{k+1}^* = \theta_k^*$ and so $\theta^* = \theta_m^*$ for some $m \in \mathbb{N}$. □

5. Transitivity of θ^*

Definition 5.1. *Let X be a non-empty subset of P and x, y are fix elements of P. Then, we say that X is a θ-part of P if for every $t \in \mathbb{N}$, $(z_1, \cdots, z_t) \in P^t$ and for every $\zeta \in S_t$ if $z_i \notin \bigcup_{n\geq 1} L_{n,x,y}(P)$, $z_i^{-1} \notin \bigcup_{n\geq 1} L_{n,y,x}(P)$, then $\zeta(i) = i$, then*

$$\prod_{i=1}^{t} z_i \cap X \neq \emptyset \Longrightarrow \prod_{i=1}^{t} z_{\zeta(i)} \subseteq X.$$

Theorem 5.2. *Let X be a non-empty subset of a polygroup P. Then the following conditions are equivalent:*

(1) *X is a θ-part of P.*

(2) *$x \in X,\ x\ \theta\ y \Longrightarrow y \in X$.*

(3) *$x \in X,\ x\ \theta^*\ y \Longrightarrow y \in X$.*

Proof. (1) $\Longrightarrow$ (2) : If $(x, y) \in P^2$ is a pair, such that $x \in X,\ x\ \theta\ y$, then there exist $(z_1, \cdots, z_t) \in P^t$ such that $x \in \prod_{i=1}^{t} z_i \cap X,\ y \in \prod_{i=1}^{t} z_{\zeta(i)}$ and $\zeta(i) = i$ if $z_i \notin \bigcup_{n\geq 1} L_{n,x,y}(P)$, $z_i^{-1} \notin \bigcup_{n\geq 1} L_{n,y,x}(P)$. Since X is a θ-part of P, we have $\prod_{i=1}^{t} z_{\zeta(i)} \subseteq X$ and so $y \in X$.

(2) $\Longrightarrow$ (3) : Suppose that $(x, y) \in P^2$ is a pair, such that $x \in X,\ x\ \theta^*\ y$, then there exist $(z_1, \cdots, z_t) \in P^t$ such that $x = z_0\ \theta\ z_1\ \theta\ \cdots\ \theta\ z_t = y$. Now, by using (2) "t" times iterated then, we obtain $y \in X$.

(3) $\Longrightarrow$ (1) : Suppose that $x \in \prod_{i=1}^{t} z_i \cap X$ and $\zeta \in S_t$ such that $\zeta(i) = i$ if $z_i \notin \bigcup_{n\geq 1} L_{n,x,y}(P)$, $z_i^{-1} \notin \bigcup_{n\geq 1} L_{n,y,x}(P)$. Let $y \in \prod_{i=1}^{t} z_{\zeta(i)}$. Since $x\ \theta\ y$ by (3), we have $y \in X$. Consequently, $\prod_{i=1}^{t} z_{\zeta(i)} \subseteq X$ and so X is a θ-part. □

Theorem 5.3. *The following conditions are equivalent:*

(1) *For every $a \in P$, $\theta(a)$ is a θ-part of P.*

(2) *θ is transitive.*

Proof. (1) $\Longrightarrow$ (2) : Suppose that $x\ \theta^*\ y$. Then, there is $(z_1,\cdots,z_t) \in P^t$ such that $x = z_0\ \theta\ z_1\ \theta\ \cdots\ \theta\ z_t = y$. Since $\theta(z_i)$, for all $0 \le i \le t$, is a θ-part, we have $z_i \in \theta(z_{i-1})$, for all $0 \le i \le t$, thus $y \in \theta(x)$, which means that $x\ \theta\ y$.

(2) $\Longrightarrow$ (1) : Suppose that $x \in P,\ z \in \theta(x)$ and $z\ \theta\ y$. By transitivity of θ, we have $y \in \theta(x)$. Now, according to Theorem 5.2, $\theta(x)$ is a θ-part of P. $\square$

Definition 5.4. *Let A be a non-empty subset of a polygroup P. The intersection of all θ-part, which contain A is called θ-closure of A in P and it will be denoted by $K(A)$.*

In follows, we determine the set $Z(A)$.

Assume that $Z_1(A) = A$ and $Z_{n+1}(A) = \{x \in P | \exists(z_1,\cdots,z_t) \in P^t,\ x \in \prod_{i=1}^t z_i,\ \exists\zeta \in S_t$ if $z_i \notin \bigcup_{s\ge1} L_{s,x,y}(P),\ \&\ z_i^{-1} \notin \bigcup_{s\ge1} L_{s,y,x}(P)$ then, $\zeta(i) = i$ and $\prod_{i=1}^t z_{\zeta(i)} \cap Z_n(A) \neq \emptyset\}$.

We denote $Z(A) = \bigcup_{n\ge1} Z_n(A)$.

Theorem 5.5. *For any non-empty subset A of P, the following statements hold:*

(1) *$Z(A) = K(A)$;*

(2) *$K(A) = \bigcup_{a\in A} K(a)$.*

Proof. (1) It is enough to prove that:

(a) $Z(A)$ is a θ-part.

(b) If $A \subseteq B$ and B is a θ-part, then $Z(A) \subseteq B$.

In order to (a), suppose that $\prod_{i=1}^t z_i \cap Z(A) \neq \emptyset$ and $\zeta \in S_t$ such that $\zeta(i) = i$ if $z_i \notin \bigcup_{n\ge1} L_{n,x,y}(P)$ and $z_i^{-1} \notin \bigcup_{n\ge1} L_{n,y,x}(P)$. Therefore, there exists $n \in \mathbb{N}$ such that $\prod_{i=1}^t z_i \cap Z(A) \neq \emptyset$, where it follows that $\prod_{i=1}^t z_{\zeta(i)} \subseteq Z_{n+1}(A) \subseteq Z(A)$.

Now, we prove (b) by induction on n. We have $Z_1(A) = A \subseteq B$.

Suppose that $Z_n(A) \subseteq B$. We prove that $Z_{n+1}(A) \subseteq B$. If $z \in Z_{n+1}(A)$, then $z \in \prod_{i=1}^t z_i$ and there exists $\zeta \in S_t$ such that $\zeta(i) = i$, if $z_i \notin \bigcup_{s\ge1} L_{s,x,y}(P),\ z_i^{-1} \notin \bigcup_{s\ge1} L_{s,y,x}(P)$, and also $\prod_{i=1}^t z_{\zeta(i)} \cap Z_n(A) \neq \emptyset$. Therefore, $\prod_{i=1}^t z_{\zeta(i)} \cap B \neq \emptyset$ and hence $z \in \prod_{i=1}^t z_i \subseteq B$.

(2) It is clear that for all $a \in A,\ K(a) \subseteq K(A)$. By part (1), we have $K(A) = \bigcup_{n\ge1} Z_n(A)$ and $Z_1(A) = A = \bigcup_{a\in A} a$. It is enough to prove that $Z_n(A) = \bigcup_{a\in A} Z_n(a)$, for all $n \in \mathbb{N}$. We follow by induction on n. Suppose it is true for n. We prove that $Z_{n+1}(A) = \bigcup_{a\in A} Z_{n+1}(a)$. If $z \in Z_{n+1}(A)$, then $z \in \prod_{i=1}^t z_i$ and there exists $\zeta \in S_t$ such that $\zeta(i) = i$, if $z_i \notin \bigcup_{s\ge1} L_{s,x,y}(P)$ and $z_i^{-1} \notin \bigcup_{s\ge1} L_{s,y,x}(P)$ and also $\prod_{i=1}^t z_{\zeta(i)} \cap Z_n(A) \neq \emptyset$. By the hypothesis of induction $\prod_{i=1}^t z_{\zeta(i)} \cap Z_n(a') \neq \emptyset$, for some $a' \in A$. therefore, $z \in Z_{n+1}(a')$, and so $Z_{n+1}(A) \subseteq \bigcup_{a\in A} Z_{n+1}(a)$. Hence, $K(A) = \bigcup_{a\in A} K(a)$. $\square$

Theorem 5.6. *The following relation is equivalence relation on P,*

$$x \ Z \ y \Longleftrightarrow x \in Z(y),$$

for every $(x, y) \in P^2$, where $Z(y) = Z(\{y\})$.

Proof. It is easy to see that Z is reflexive and transitive. For the proof of symmetric of relation Z, it is enough that we prove the following statements:

(1) For all $n \geq 2$ and $x \in H$, $Z_n(Z_2(x)) = Z_{n+1}(x)$.

(2) $x \in Z_n(y)$ if and only if $y \in Z_n(x)$.

We prove (1) by induction on n. Suppose that $z \in Z_2(Z_2(x))$. Then, $z \in \prod_{i=1}^{t} z_i$ and there is $\zeta \in S_t$ such that $\zeta(i) = i$, if $z_i \notin \bigcup_{s\geq 1} L_{s,x,y}(P)$, $z_i^{-1} \notin \bigcup_{s\geq 1} L_{s,y,x}(P)$ and also $\prod_{i=1}^{t} z_{\zeta(i)} \cap Z_2(x) \neq \emptyset$, thus $z \in Z_3(x)$. if $z \in Z_{n+1}(Z_2(x))$, then $z \in \prod_{i=1}^{t} z_i$ and there exists $\zeta \in S_t$ such that $\zeta(i) = i$, if $z_i \notin \bigcup_{s\geq 1} L_{s,x,y}(P)$, $z_i^{-1} \notin \bigcup_{s\geq 1} L_{s,y,x}(P)$ and also $\prod_{i=1}^{t} z_{\zeta(i)} \cap Z_n(Z_2(x)) \neq \emptyset$. By hypothesis of induction, we have $\prod_{i=1}^{t} z_{\zeta(i)} \cap Z_{n+1}(x) \neq \emptyset$ and so $z \in Z_{n+2}(x)$.

Now, we prove (2) by induction on n, too. It is clear that $x \in Z_2(y)$ if and only if $y \in Z_2(x)$. Then $x \in \prod_{i=1}^{t} z_i$ and there exists $\zeta \in S_t$ such that $\zeta(i) = i$, if $z_i \notin \bigcup_{s\geq 1} L_{s,x,y}(P)$, $z_i^{-1} \notin \bigcup_{s\geq 1} L_{s,y,x}(P)$ and also $\prod_{i=1}^{t} z_{\zeta(i)} \cap Z_n(y) \neq \emptyset$. Suppose that $b \in \prod_{i=1}^{t} z_{\zeta(i)} \cap Z_n(y)$, then, we have $y \in Z_n(b)$. From $x \in \prod_{i=1}^{t} z_i \cap Z_1(x)$ and $b \in \prod_{i=1}^{t} Z_{\zeta(i)}$ we conclude that $b \in Z_2(x)$. Therefore, $y \in Z_n(Z_2(x)) = Z_{n+1}(x)$. □

6. Conclusion

In this paper, we have introduced a new strongly regular relation θ_n^* on a polygroup P and we have shown that P/θ_n^* is a n-Bell group for $n = 2, 3$.

We defined the same relation structure between two strongly regular relations on a hypergroup (polygroup), and we bring an open problem relate to n-Bell group of P/θ_n^*. In continue, we obtained some results related to θ_n^*. We try to answer the mention open problem and in this regard, for the other research work.

References

[1] H. Aghabozorgi, B. Davvaz, M. Jafarpour, *Nilpotent groups derived from hypergroups,* J. Algebra, 382 (2013), 177-184.

[2] R. Ameri, E. Mohammadzadeh, *Engel groups derived from hypergroups*, European Journal of Combinatorics, 44 (2015), 191-197.

[3] R. Brandl, L.C. Kappe, *On n-Bell groups,* Comm. Algebra, 17 (1989), 787-807.

[4] P. Corsini, *Prolegomena of hypergroup theory*, Aviani Editore, Tricesimo, 1993.

[5] P. Corsini, V. Leoreanu, *Applications of hyperstructure theory*, Kluwer Academic Publishers, Dordrecht, 2003.

[6] B. Davvaz, *Applications of the γ^*-relation to polygroups,* Comm. Algebra, 35 (2007), 2698-2706.

[7] B. Davvaz, *Polygroup theory and related systems*, World Scientific, 2013.

[8] C. Delizia, P. Moraves, C. Nicotera, *Locally graded Bell groups*, Publ. Math. Debrecen, 71 (2007), 1-9.

[9] C. Delizia, M.R.R. Moghadam, A.H. Rhemtulla, *The structure of Bell groups*, J. Group Theory, 9 (2006), 117-125.

[10] C. Delizia, A. Tortora, *Locally graded groups with a Bell condition on infinite subset*, J. Group Theory, 12 (2009), 753-759.

[11] D. Freni, *A new characterization of the derived hypergroup via strongly regular equivalences,* Comm. Algebra, 30 (2002), 3977-3989.

[12] M. Jafarpour, H. Aghabozorgi, B. Davvaz, *Solvable groups derived from hypergroups,* J. Algebra. Appl., 15 (2016), 9.

[13] L.C. Kappe, M.L. Newell, *On the Levi and Bell centers of a group,* Prossedings of the second international group theory conference, Bressanone, Brixen, 1989, 11-17.

[14] M. Koskas, *Groupoides, demi-hypergroups et hypergroups,* J. Math. Pures et Appl., 49 (1970), 155-192.

[15] F. Marty, *Sur une generalization de la notion de groupe,* 8th Congress Math. Scandenaves, Stockholm, Sweden, (1934), 45-49.

[16] D.J.S. Rabinson, *A course in the theory of groups*, Springer-Verlag, 80 (1993).

[17] B. Taeri, *A combinatorial condition on variety of groups*, Arch. Math. (Basel), 77 (2001), 456-460.

[18] A. Tortora, *Some properties of Bell groups*, Comm. Algebra, 37 (2009), 431-438.

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