Strongly regular relation and $n$-Bell groups derived from it

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Abstract. A new strongly regular relation $\theta_{n}^{*}$ is defined on polygroup $P$ such that the quotient $P/\theta_{n}^{*}$, the set of all equivalence classes, is a Bell group for $n \in \{2, 3\}$.

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1. Introduction

Hyperstructure theory was first initiated by Marty [15] in 1934. Let $H$ be a non-empty set and $o : H \times H \rightarrow P^{*}(H)$ be a hyperoperation where $P^{*}(H)$ is the family of non-empty subset of $H$. The couple $(H,o)$ is called a hypergroupoid. For any two non-empty subset $A$ and $B$ of $H$ and $x \in H$, we define $A \circ B = \bigcup_{a \in A, b \in B} a \circ b, A \circ x = A \circ \{x\}$ and $B \circ x = B \circ \{x\}$. A hypergroupoid $(H,o)$ is called semihypergroup if for all $a, b, c \in H$, we define $a \circ b, a \circ c = a \circ (b \circ c)$ which means that $\bigcup_{u \in A \circ b} u \circ c = \bigcup_{v \in b \circ c} a \circ v$ and hypergroupoid $(H,o)$ is called quasihypergroup if for all $a \in H$, we have $a \circ H = H \circ a = H$, which is called reproduction axiom. This axiom means that for any $x, y \in H$, there exist $u, v \in H$ such that $y \in x \circ u, y \in v \circ x$. A hypergroupoid $(H,o)$ which is both a semihypergroup and a quasihypergroup is called hypergroup.
Definition 1.1 ([6]). A polygroup is a hypergroup \( (P, \cdot, e^{-1}) \) where \( e \in P, e^{-1} \) is a unitary operation on \( P \), and the following axiom hold for all \( x, y, z \in P \):

(i) \( e \cdot x = x \cdot e = x \);

(ii) \( x \in y \cdot z \Longleftrightarrow y \in x \cdot z^{-1} \Longrightarrow z \in y^{-1} \cdot x \).

Definition 1.2 ([5]). Let \((H, \cdot)\) be a hypergroup and \( \rho \subseteq H \times H \) be an equivalence relation. For non-empty subset \( A \) and \( B \) of \( H \), we define \( A \tilde\rho B \) if and only if \( a \rho b \), for all \( a \in A \) and \( b \in B \). The relation \( \rho \) is called strongly regular on the left (on the right) if \( x \rho y \), then \( a \circ x \tilde\rho a \circ y \) \((x \circ a \tilde\rho y \circ b, \text{respectively})\), for all \( x, y, a \in H \).

Moreover, \( \rho \) is called strongly regular if it is strongly regular on the left and on the right.

Theorem 1.3 ([4]). If \((H, \cdot)\) is a hypergroup and \( \rho \) is a strongly regular relation on \( H \), then \( H/\rho \) is a group under the operation:

\[ \rho(x) \otimes \rho(y) = \rho(z), \ \forall z \in x \cdot y. \]

For all \( n \geq 1 \), we define the relation \( \beta_n \) on a semihipergrup \( H \), as follows, \( \beta_n \) \( b \), if and only if there exists \((x_1, \ldots, x_n)\) in \( H^n \) such that \( \{a, b\} \subseteq \prod_{i=1}^{n} x_i \) and \( \beta = \bigcup_{n \geq 1} \beta_n \), where \( \beta_1 = \{(x, x); x \in H\} \), is the diagonal relation on \( H \).

This relation was introduced by Koskas [14]. Suppose that \( \beta^* \) is the transitive closure of \( \beta \), the relation \( \beta^* \) is a strongly regular relation [4].

In [11], \( \gamma = \bigcup_{n \geq 1} \gamma_n \), where \( \gamma_1 \) is the diagonal relation and for every integer \( n > 1 \), \( \gamma_n \) is the relation defined as follows, \( x \gamma_n y \) if and only if there exists \((z_1, \ldots, z_n)\) in \( H^n \) and \( \tau \in S_n \) such that \( x \in \prod_{i=1}^{n} z_i \) and \( y \in \prod_{i=1}^{n} z_{\tau(i)} \), where \( S_n \) is the symmetric group of order \( n \). Suppose that \( \gamma^* \) is the transitive closure of \( \gamma \). The relation \( \gamma^* \) is a strongly regular relation [11].

The relation \( \beta^* \) is the least equivalence relation on hypergroup \( H \) such that the quotient \( H/\beta^* \) is a group, while \( \gamma^* \) is the least equivalence relation on hypergroup \( H \), such that the quotient \( H/\gamma^* \) is an abelian group.

In [12], \( \tau_n = \bigcup_{m \geq 1} \tau_{m,n} \), where \( \tau_{1,n} \) is the diagonal relation and for every integer \( m > 1 \), \( \tau_{m,n} \) is the relation defined as follows, \( x \tau_{m,n} y \) if and only if there exists \((z_1, \ldots, z_m)\) in \( H^m \), and \( \sigma \in S_m \) such that \( \sigma(i) = i, \text{if} \ z_i \notin H^{(n)} \) such that \( x \in \prod_{i=1}^{m} z_i \) and \( y \in \prod_{i=1}^{m} z_{\sigma(i)} \), where

1. \( H^{(0)} = H; \)
2. \( H^{(k+1)} = \{ h \in H^{(k)} \mid xy \cap hyx \neq \emptyset \ ; \ x, y \in H^{(k)} \}. \)

Clearly, for every integer \( n \geq 1 \), the relation \( \tau_n \) is reflexive and symmetric.

Now, suppose that \( \tau_n^* \) is the transitive closure of \( \tau_n \). The relation \( \tau_n^* \) is strongly regular such that the quotient \( H/\tau_n^* \) is a solubale group of the class at most \( n + 1 \).
In [1], \( \nu_n = \bigcup_{m \geq 1} \nu_{m,n} \), where \( \nu_{1,n} \) is the diagonal relation and for every integer \( m > 1 \), \( \nu_{m,n} \) is the relation defined as follows, \( x \nu_{m,n} y \) if and only if, there exists \( (z_1, \ldots, z_m) \) in \( H^m \) and \( \sigma \in S_m \) such that \( \sigma(i) = i \), if \( z_i \notin L_{n}(H) \) such that \( x \in \prod_{i=1}^{m} z_i \) and \( y \in \prod_{i=1}^{m} z_{\sigma(i)} \), where

1. \( L_0(H) = H \);
2. \( L_{k+1}(H) = \{ h \mid xy \cap hyx \neq \emptyset ; x \in L_k(H), y \in H \} \).

Clearly, for every integer \( n \geq 1 \), the relation \( \nu_n \) is reflexive and symmetric.

Now, suppose that \( \nu_n^* \) is the transitive closure of \( \nu_n \). The relation \( \nu_n^* \) is strongly regular such that the quotient \( H/\nu_n^* \) is a nilpotent group of the class at most \( n+1 \).

In [2], \( \xi_{n,s} = \bigcup_{m \geq 1} \xi_{m,n,s} \), where \( \xi_{1,n,s} \) is the diagonal relation and for every integer \( m \geq 1 \), \( \xi_{m,n,s} \) is the relation defined as follows:

\( x \xi_{m,n,s} y \) if and only if, there exists \( (z_1, \ldots, z_m) \) in \( H^m \) and \( \delta \in S_m \) such that \( \delta(i) = i \) if \( z_i \notin L_{n,s}(H) \) such that \( x \in \prod_{i=1}^{m} z_i \) and \( y \in \prod_{i=1}^{m} z_{\delta(i)} \), where

1. \( L_{0,s}(H) = H \);
2. \( L_{k+1,s}(H) = \{ h \mid xs \cap hsx \neq \emptyset ; x \in L_{k,s}(H) \}, \forall k \geq 0 \),

for fix element \( s \in H \).

Obviously, for every \( n \geq 1 \), the relation \( \xi_{n,s} \) is reflexive and symmetric. Now let \( \xi_{n,s}^* \) be the transitive closure of \( \xi_{n,s} \).

In [2], the authors proved that the relation \( \xi_{n,s}^* \) is strongly regular such that the quotient \( H/\xi_{n,s}^* \) is an \( n \)-Engel group.

Let \( n \neq 0, 1 \) be an integer. A group \( G \) is said to be \( n \)-Bell if \( [x^n, y] = [x, y^n] \) for all \( x \) and \( y \) in \( G \), where \( [x, y] \) is the commutator of \( x \) and \( y \). The study of \( n \)-Bell groups was introduced by Kappe and Brandl in [3], [13] and it was also the subject of several papers, see for instance [8], [9], [10] and [18]. For example all of groups of finite exponent dividing \( n \), groups of finite exponent dividing \( n - 1 \), 2-Engel groups and \( n \)-Levi groups, are \( n \)-Bell groups (see, [9]).

In this paper, we define a new relation \( \theta_n \) on a polygroup and then we show that \( \theta_n^* \) is a strongly regular relation. In continue, we bring some results related to \( \theta_n^* \) and one of the main result of this paper is about the relation of \( \theta_n^* \) and \( n \)-Bell groups for \( n = 2 \) and \( 3 \). Also, if we set \( \theta^* = \bigcap_{n \geq 1} \theta_n^* \), then we show that \( P/\theta^* \) is a Bell group for any finite polygroup \( P \).

In a polygroup \( P \), the commutator of two elements \( x, y \) in \( P \) is defined by \( [x, y] = \{ t \mid t \in xyx^{-1}y^{-1} \} \). If \( A \subseteq P \), then \( [A, y] = \{ t \mid t \in AyA^{-1}y^{-1} \} \).

**Theorem 1.4** ([2], Theorem 2.2). Let \( P \) be a polygroup. Then, for all \( x, y, h, \in P \), \( \{ h \mid xy \cap hyx \neq \emptyset \} = \{ h \mid h \in xyx^{-1}y^{-1} \} \).

**Remark 1.5.** Let \( P \) be a polygroup. Then, for all \( x, y, h, \in P \) and \( n \in N \), \( \{ h \mid x^n y \cap hyx^n \neq \emptyset \} = \{ h \mid h \in x^n y x^{-n} y^{-1} \} \).

**Theorem 1.6** ([2], Theorem 2.10). \( H/\xi_{n,s}^* \) is an \( n \)-Engel group.
Theorem 1.7 ([1], Theorem 2.9). $H/\nu^*_n$ is a nilpotent group of the class at most $n+1$.

2. New strongly regular relation $\theta_n^*$

Now, we introduce a new strongly regular relation $\theta_n^*$ on a polygroup $P$.

In the whole of this paper, $P$ is a polygroup and $S_n$ is symmetric group.

Definition 2.1. Let $P$ be a polygroup. For fix elements $x, y \in P$, we define:

1. $L_{0,x,y}(P) = P$;
2. $L_{n+1,x,y}(P) = \{h \mid h \in L_{n,x,y}(P), x^{n+1}y \cap hyx^{n+1} \neq \emptyset\}$.

Let $\theta_n = \bigcup_{m \geq 1} \theta_{m,n}$ where $\theta_{1,n}$ is diagonal relation and for every integer $m \geq 1$, $\theta_{m,n}$ is relation defined as follows:

$x \theta_{m,n} y$ if and only if, there exists $(z_1, \cdots, z_m)$ in $P^m$ and $\zeta \in S_m$ if, $z_i \notin L_{n,x,y}(P)$ and $z_i^{-1} \notin L_{n,x,y}(P)$, then $\zeta(i) = i$ and $x \in \prod_{i=1}^{m} z_i$ and $y \in \prod_{i=1}^{m} z_{\zeta(i)}$.

Clearly, $\theta_n$ is reflexive and symmetric. Let $\theta_n^*$ be the transitive closure of $\theta_n$.

Theorem 2.2. For every $n \in \mathbb{N}$, the relation $\theta_n^*$ is strongly regular relation.

Proof. Suppose that $n \in \mathbb{N}$. Clearly, $\theta_n^*$ is an equivalence relation. In order to prove that it is strongly regular. First we have to show that if $x \theta_n y$, then $x \cdot z \theta_n y \cdot z$, $z \cdot x \theta_n z \cdot y$, for every $z \in P$. Suppose that $x \theta_n y$. Then, there exists $m \in \mathbb{N}$ such that $x \theta_{m,n} y$. Hence, there exists $(z_1, \cdots, z_m) \in P^m$, $\zeta \in S_m$ with $\zeta(i) = i$ if $z_i \notin L_{n,x,y}(P)$ and $z_i^{-1} \notin L_{n,y,x}(P)$ such that $x \in \prod_{i=1}^{m} z_i$ and $y \in \prod_{i=1}^{m} z_{\zeta(i)}$. Suppose that $z \in P$. We have $x \cdot z \subseteq (\prod_{i=1}^{m} z_i) \cdot z, y \cdot z \subseteq (\prod_{i=1}^{m} z_{\zeta(i)}) \cdot z$. Now, suppose that $z_{m+1} = z$ and we define the permutation $\zeta' \in S_{m+1}$ as follows:

\[
\begin{cases}
\zeta'(i) = \zeta(i), & \text{for all } 1 \leq i \leq m, \\
\zeta'(m+1) = m+1.
\end{cases}
\]

Thus, $x \cdot z \subseteq \prod_{i=1}^{m+1} z_i, y \cdot z \subseteq \prod_{i=1}^{m+1} z_{\zeta'(i)}$, such that $\zeta'(i) = i$ if $z_i \notin L_{n,x,y}(P)$ and $z_i^{-1} \notin L_{n,y,x}(P)$. Therefore, $x \cdot z \theta_{n} y \cdot z$. Similarly, we have $z \cdot x \theta_n z \cdot y$. Now, if $x \theta_n^* y$, then, there exists $k \in \mathbb{N}$ and $(x = u_0, u_1, \cdots, u_k = y) \in P^{k+1}$ such that $x = u_0 \theta_n u_1 \theta_n \cdots \theta_n u_k \theta_n = y$. Hence, we obtain $x \cdot z = u_0 \cdot z \theta_n^* u_1 \cdot z \theta_n^* u_2 \cdot z \theta_n^* \cdots \theta_n^* u_k \theta_n \cdot z = y \cdot z$ and so $x \cdot z \theta_n^* y \cdot z$. Similarly, we can prove that $z \cdot x \theta_n^* z \cdot y$, therefore $\theta_n^*$ is strongly regular relation on $P$.

Proposition 2.3. For every $n \in \mathbb{N}$, we have $\theta_n^* \subseteq \theta_n^*$. 

Proof. Let $x \theta_{n+1} y$, so, there exists $m \in \mathbb{N}$ and $(z_1, \cdots, z_m) \in P^m$ and $\zeta \in S_m$ such that $\zeta(i) = i$ if $z_i \notin L_{n+1,x,y}(P)$ and $z_i^{-1} \notin L_{n+1,y,x}(P)$, such that $x \in \prod_{i=1}^{m} z_i$ and $y \in \prod_{i=1}^{m} z_{\zeta(i)}$. Now, let $\zeta_1 = \zeta$, since $L_{n+1,x,y}(P) \subseteq L_{n,x,y}(P)$ and $L_{n+1,y,x}(P) \subseteq L_{n,y,x}(P)$, we have $x \theta_n y$. 

\[\square\]
Corollary 2.4. If $P$ is a commutative hypergroup, then $\beta^* = \theta^*_n = \xi^*_n = \nu^*_n = \gamma^*$.

Definition 2.5 ([13]). Let $G$ be a group and $n$ be an integer. The $n$-Bell center of $G$ denoted by $B_n$ and defined as follows:

$$B_n = B(G, n) = \{ x \in G \mid [x^n, y] = [x, y^n] ; \forall y \in G \}.$$

Clearly, $B(G, 0) = B(G, 1) = G$, and easy to see that $B(G, 2)$ and $B(G, 3)$ are subgroup of $G$.

Remark 2.6. For every integer $n$, a group is $n$-Bell if $B(G, n) = G$.

Theorem 2.7. If $P$ is a polygroup and $\rho$ is a strongly regular relation on $P$, then for fix elements $x, y \in P$;

$$L_{n+1, x, y}(P) = \{ [\bar{x}^{n+1}, \bar{y}] \},$$

where $\bar{x}, \bar{y}$ are the classes of $x, y$ with respect to $\rho$.

Proof. The proof follows from definition of commutator of two elements in a polygroup, Theorem 1.4 and Remark 1.5.

3. $n$-Bell groups derived from polygroups for $n \in \{2, 3\}$

In this section, we obtain an $n$-Bell group derived from polygroup for $n = 2, 3$, and then we propose an open problem related to $n$-Bell groups.

Theorem 3.1. Let $P$ be a polygroup. Then, for $n \in \{2, 3\}$, $P/\theta^*_n$ is an $n$-Bell group.

Proof. Let $G = P/\theta^*_n$. For $n \in \{2, 3\}$, we have $B(G, n) \leq G$. By Remark 2.6, it is enough to prove that $G \leq B(G, n)$. For this we should show that for every $\bar{h} \in L_{n, x, y}(G)$ we have $\bar{h}^{-1} \in L_{n, \bar{x}, \bar{y}}(G)$ and is obvious by Theorem 2.7.

Definition 3.2 ([16]). A group $G$ is called an Engel group if, for each ordered pair $(x, y)$ of elements in $G$, there is a positive integer $n = n(x, y)$, such that $[x, y]^n = 1$.

Theorem 3.3 ([17]). Let $G$ be a group. Then

(a) $G$ is a $n$-Bell group if and only if $G$ is a $(1 - n)$-Bell group;
(b) $G$ is a 2-Bell group if and only if $G$ is a 2-Engel group;
(c) $G$ is a 3-Bell group if and only if $G$ is a 3-Engel group satisfying the identity $[x, y, y]^3 = 1$, for all $x, y \in G$. In addition $G$ has nilpotent of class at most 4.
**Definition 3.4.** Let $H_1$ and $H_2$ be two hypergroups (polygroups), and $\rho_1$ and $\rho_2$ be two strongly regular relations. If $H_1/\rho_1$ and $H_2/\rho_2$ are isomorphism groups, then we say that $\rho_1$ is “the same” property to $\rho_2$.

**Remark 3.5.** According to the above definition, $\theta_2^*$ is the same property to $\xi_{2,s}^*$, by Theorem 3.1, 3.3 and apply Theorem 1.6, and $\theta_3^*$ is the same property to $\xi_{3,s}^*$ and $\nu_3^*$, by Theorem 3.1, 3.3 and apply Theorem 1.6 and 1.7.

**Example 3.6** ([2]). Let $H$ be $\{e, a, b, c, d, f, g\}$. Consider the non-commutative polygroup $(H, \cdot)$, defined on $H$ as follows: It is easy to see that $H/\beta^* \cong S_3$ (for more details, see [7]). Since $S_3$ is not nilpotent, we conclude that $\beta^* \neq \nu_n^*$, hence $H/\nu_n^*$ is an abelian group of order less than 6 and the class of nilpotency of $H/\nu_n^*$ is one for all $n \in \mathbb{N}$ [1], besides, $S_3$ is not Engel and $H/\xi_{n,s}^* \subseteq H/\beta^* \cong S_3$, then it concluded $H/\xi_{n,s}^*$ is an abelian group of order less than 6 and $H/\xi^*$ is 1-Engel group. Then, $H/\theta_2^*$ is not 2-Bell or 3-Bell group, by apply the Remark 3.5.

**Remark 3.7.** We know that $B(G, n)$ is called the $n$-Bell center of $G$. It is open problem whether the $n$-Bell center always forms a subgroup. But, it is shown that $B(G, 2)$ is characteristic subgroup of all right 2-Engel elements and $B(G, 3)$ is characteristic subgroup of $G$ which is nilpotent of class at most 4 (see, [13]).

Hence, according to above remark, we can put the following open problem:

**Open Problem 3.8.** Let $H$ be non-commutative polygroup, for all $n \geq 4$, is $H/\theta_n^*$ a $n$-Bell group?

4. On Bell groups derived from finite polygroup

In this section, we introduce a strongly regular relation $\theta^*$ on finite polygroup $P$ such that $P/\theta^*$ is a Bell group.

**Definition 4.1.** Let $P$ be a finite polygroup. Then, we define the relation $\theta^*$ on $P$ by $\theta^* = \bigcap_{n \geq 1} \theta_n^*$.

**Definition 4.2.** An equivalence relation $\rho$ on a finite polygroup $P$, is called Bell if and only if its derived group $P/\rho$ is a Bell group.
Example 4.3. \(\theta_2^*\) and \(\theta_3^*\) are Bell relations. By using the Remark 3.5, and Example 3.3 in [2], Bell relations \(\theta_2^*\) and \(\theta_3^*\) are the same with Engel relations \(\xi_{2,s}^*\) and \(\xi_{3,s}^*\).

**Theorem 4.4.** (a) The relation \(\theta^*\) is a strongly regular relation on a finite polygroup \(P\).

(b) \(P/\theta^*\) is a Bell group.

**Proof.** (a) Since \(\theta^* = \bigcap_{n \geq 1} \theta_n^*\), it is easy to see that \(\theta^*\) is strongly regular relation on \(P\).

(b) By using Proposition 2.3, we conclude that there exists \(k \in \mathbb{N}\) \((k \geq 1)\) such that \(\theta_{k+1}^* = \theta_k^*\) and so \(\theta^* = \theta_m^*\) for some \(m \in \mathbb{N}\).

5. Transitivity of \(\theta^*

**Definition 5.1.** Let \(X\) be a non-empty subset of \(P\) and \(x, y\) are fix elements of \(P\). Then, we say that \(X\) is a \(\theta\)-part of \(P\) if for every \(t \in \mathbb{N}\), \((z_1, \ldots, z_t) \in P^t\) and for every \(\zeta \in S_t\) if \(z_i \notin \bigcup_{n \geq 1} L_{n,x,y}(P)\), \(z_i^{-1} \notin \bigcup_{n \geq 1} L_{n,y,x}(P)\), then \(\zeta(i) = i\), then

\[
\prod_{i=1}^{t} z_i \cap X \neq \emptyset \implies \prod_{i=1}^{t} z_{\zeta(i)} \subseteq X.
\]

**Theorem 5.2.** Let \(X\) be a non-empty subset of a polygroup \(P\). Then the following conditions are equivalent:

1. \(X\) is a \(\theta\)-part of \(P\).
2. \(x \in X, x \theta y \implies y \in X\).
3. \(x \in X, x \theta^* y \implies y \in X\).

**Proof.** (1) \(\implies\) (2): If \((x, y) \in P^2\) is a pair, such that \(x \in X, x \theta y\), then there exist \((z_1, \ldots, z_t) \in P^t\) such that \(x \in \prod_{i=1}^{t} z_i \cap X, y \in \prod_{i=1}^{t} z_{\zeta(i)}\) and \(\zeta(i) = i\) if \(z_i \notin \bigcup_{n \geq 1} L_{n,x,y}(P)\), \(z_i^{-1} \notin \bigcup_{n \geq 1} L_{n,y,x}(P)\). Since \(X\) is a \(\theta\)-part of \(P\), we have \(\prod_{i=1}^{t} z_{\zeta(i)} \subseteq X\) and so \(y \in X\).

(2) \(\implies\) (3): Suppose that \((x, y) \in P^2\) is a pair, such that \(x \in X, x \theta^* y\), then there exist \((z_1, \ldots, z_t) \in P^t\) such that \(x = z_0 \theta z_1 \theta \cdots \theta z_t = y\). Now, by using (2) “\(t\)” times iterated then, we obtain \(y \in X\).

(3) \(\implies\) (1): Suppose that \(x \in \prod_{i=1}^{t} z_i \cap X\) and \(\zeta \in S_t\), such that \(\zeta(i) = i\) if \(z_i \notin \bigcup_{n \geq 1} L_{n,x,y}(P)\), \(z_i^{-1} \notin \bigcup_{n \geq 1} L_{n,y,x}(P)\). Let \(y \in \prod_{i=1}^{t} z_{\zeta(i)}\). Since \(x \theta y\) by (3), we have \(y \in X\). Consequently, \(\prod_{i=1}^{t} z_{\zeta(i)} \subseteq X\) and so \(X\) is a \(\theta\)-part.

**Theorem 5.3.** The following conditions are equivalent:

1. For every \(a \in P\), \(\theta(a)\) is a \(\theta\)-part of \(P\).
(2) $\theta$ is transitive.

Proof. (1) $\implies$ (2): Suppose that $x \theta^* y$. Then, there is $(z_1, \ldots, z_t) \in P^t$ such that $x = z_0 \theta z_1 \theta \cdots \theta z_t = y$. Since $\theta(z_i)$, for all $0 \leq i \leq t$, is a $\theta$-part, we have $z_i \in \theta(z_{i-1})$, for all $0 \leq i \leq t$, thus $y \in \theta(x)$, which means that $x \theta y$.

(2) $\implies$ (1): Suppose that $x \in P$, $z \in \theta(x)$ and $z \theta y$. By transitivity of $\theta$, we have $y \in \theta(x)$. Now, according to Theorem 5.2, $\theta(x)$ is a $\theta$-part of $P$. \hfill $\square$

**Definition 5.4.** Let $A$ be a non-empty subset of a polygroup $P$. The intersection of all $\theta$-part, which contain $A$ is called $\theta$-closure of $A$ in $P$ and it will be denoted by $K(A)$. 

In follows, we determine the set $Z(A)$.

Assume that $Z_1(A) = A$ and $Z_{n+1}(A) = \{x \in P | \exists (z_1, \ldots, z_t) \in P^t, x \in \prod_{i=1}^t z_i, \exists \zeta \in S_t$ if $z_i \notin \bigcup_{s \geq 1} L_{s,x,y}(P)$ and $z_i^{-1} \notin \bigcup_{s \geq 1} L_{s,y,x}(P)$ then, $\zeta(i) = i$ and $\prod_{i=1}^t \zeta(i) \cap Z_n(A) \neq \emptyset\}$. 

We denote $Z(A) = \bigcup_{n \geq 1} Z_n(A)$.

**Theorem 5.5.** For any non-empty subset $A$ of $P$, the following statements hold:

(1) $Z(A) = K(A)$;

(2) $K(A) = \bigcup_{a \in A} K(a)$.

Proof. (1) It is enough to prove that:

(a) $Z(A)$ is a $\theta$-part.

(b) If $A \subseteq B$ and $B$ is a $\theta$-part, then $Z(A) \subseteq B$.

In order to (a), suppose that $\prod_{i=1}^t z_i \cap Z(A) \neq \emptyset$ and $\zeta \in S_t$ such that $\zeta(i) = i$ if $z_i \notin \bigcup_{s \geq 1} L_{n,x,y}(P)$ and $z_i^{-1} \notin \bigcup_{s \geq 1} L_{n,y,x}(P)$. Therefore, there exists $n \in \mathbb{N}$ such that $\prod_{i=1}^t z_i \cap Z(A) \neq \emptyset$, where it follows that $\prod_{i=1}^t \zeta(i) \subseteq Z_{n+1}(A) \subseteq Z(A)$.

Now, we prove (b) by induction on $n$. We have $Z_1(A) = A \subseteq B$.

Suppose that $Z_n(A) \subseteq B$. We prove that $Z_{n+1}(A) \subseteq B$. If $z \in Z_{n+1}(A)$, then $z \in \prod_{i=1}^t z_i$ and there exists $\zeta \in S_t$ such that $\zeta(i) = i$, if $z_i \notin \bigcup_{s \geq 1} L_{s,x,y}(P)$ and $z_i^{-1} \notin \bigcup_{s \geq 1} L_{s,y,x}(P)$, and also $\prod_{i=1}^t \zeta(i) \cap Z_n(A) \neq \emptyset$. Therefore, $\prod_{i=1}^t \zeta(i) \cap B \neq \emptyset$ and hence $z \in \prod_{i=1}^t z_i \subseteq B$.

(2) It is clear that for all $a \in A$, $K(a) \subseteq K(A)$. By part (1), we have $K(A) = \bigcup_{n \geq 1} Z_n(A)$ and $Z_1(A) = A = \bigcup_{a \in A} a$. It is enough to prove that $Z_n(A) = \bigcup_{a \in A} Z_n(a)$, for all $n \in \mathbb{N}$. We follow by induction on $n$. Suppose it is true for $n$. We prove that $Z_{n+1}(A) = \bigcup_{a \in A} Z_{n+1}(a)$. If $z \in Z_{n+1}(A)$, then $z \in \prod_{i=1}^t z_i$ and there exists $\zeta \in S_t$ such that $\zeta(i) = i$, if $z_i \notin \bigcup_{s \geq 1} L_{s,x,y}(P)$ and $z_i^{-1} \notin \bigcup_{s \geq 1} L_{s,y,x}(P)$ and also $\prod_{i=1}^t \zeta(i) \cap Z_n(A) \neq \emptyset$. By the hypothesis of induction $\prod_{i=1}^t \zeta(i) \cap Z_n(a') \neq \emptyset$, for some $a' \in A$.

Therefore, $z \in Z_{n+1}(a')$, and so $Z_{n+1}(A) \subseteq \bigcup_{a \in A} Z_{n+1}(a)$. Hence, $K(A) = \bigcup_{a \in A} K(a)$.
The following relation is equivalence relation on $P$,

$$x Z y \iff x \in Z(y),$$

for every $(x, y) \in P^2$, where $Z(y) = Z(\{y\})$.

**Proof.** It is easy to see that $Z$ is reflexive and transitive. For the proof of symmetric of relation $Z$, it is enough that we prove the following statements:

1. For all $n \geq 2$ and $x \in H$, $Z_n(Z_2(x)) = Z_{n+1}(x)$.
2. $x \in Z_n(y)$ if and only if $y \in Z_n(x)$.

We prove (1) by induction on $n$. Suppose that $z \in Z_2(Z_2(x))$. Then, $z \in \prod_{i=1}^t z_i$ and there is $\zeta \in S_t$ such that $\zeta(i) = i$, if $z_i \notin \bigcup_{s \geq 1} L_{s,x,y}(P)$, $z_i^{-1} \notin \bigcup_{s \geq 1} L_{s,y,x}(P)$ and also $\prod_{i=1}^t z_{\zeta(i)} \cap Z_2(x) \neq \emptyset$, thus $z \in Z_3(x)$. if $z \in Z_{n+1}(Z_2(x))$, then $z \in \prod_{i=1}^t z_i$ and there exists $\zeta \in S_t$ such that $\zeta(i) = i$, if $z_i \notin \bigcup_{s \geq 1} L_{s,x,y}(P)$, $z_i^{-1} \notin \bigcup_{s \geq 1} L_{s,y,x}(P)$ and also $\prod_{i=1}^t z_{\zeta(i)} \cap Z_n(Z_2(x)) \neq \emptyset$. By hypothis of induction, we have $\prod_{i=1}^t z_{\zeta(i)} \cap Z_{n+1}(x) \neq \emptyset$ and so $z \in Z_{n+2}(x)$.

Now, we prove (2) by induction on $n$, too. It is clear that $x \in Z_2(y)$ if and only if $y \in Z_2(x)$. Then $x \in \prod_{i=1}^t z_i$ and there exists $\zeta \in S_t$ such that $\zeta(i) = i$, if $z_i \notin \bigcup_{s \geq 1} L_{s,x,y}(P)$, $z_i^{-1} \notin \bigcup_{s \geq 1} L_{s,y,x}(P)$ and also $\prod_{i=1}^t z_{\zeta(i)} \cap Z_n(y) \neq \emptyset$. Suppose that $b \in \prod_{i=1}^t z_{\zeta(i)} \cap Z_n(y)$, then, we have $y \in Z_n(b)$. From $x \in \prod_{i=1}^t z_i \cap Z_1(x)$ and $b \in \prod_{i=1}^t Z_{\zeta(i)}$ we conclude that $b \in Z_2(x)$. Therefore, $y \in Z_n(Z_2(x)) = Z_{n+1}(x)$.

6. Conclusion

In this paper, we have introduced a new strongly regular relation $\theta_n^*$ on a polygroup $P$ and we have shown that $P/\theta_n^*$ is a $n$-Bell group for $n = 2, 3$.

We defined the same relation structure between two strongly regular relations on a hypergroup (polygroup), and we bring an open problem relate to $n$-Bell group of $P/\theta_n^*$. In continue, we obtained some results related to $\theta_n^*$ We try to answer the mention open problem and in this regard, for the other research work.

**References**


