New conformable fractional HPT for solving systems of linear and nonlinear conformable fractional PDEs

Maher Jneid
Department of Mathematics and Computer Science
Faculty of Science
Beirut Arab University
Lebanon
m.jneid@bau.edu.lb
jneidmaher@gmail.com

Abstract. In this work, an analytical solution for a linear and nonlinear system of conformable fractional PDEs is obtained by applying a New conformable fractional homotopy perturbation technique (NCFHPT) with the help of fractional series expansion. In fact, the current method is a natural extension of NHPM for partial differential equation of real order in conformable sense. By constructing the homotopy transformation for a given system and gathering the coefficients with the identical power of p, a system of recursive equations is established. In addition, by a convenient assumption on the initial approximate solution. We can consecutively starting with this solution and working upward until getting general term of the intended coefficients to deduce a closed form series solution. The NCFHPT gives an analytical solution without making any linearization, rough conditions or discretization, especially for non-linear problems. This technique shows a powerful and a promise tool to solve linear and nonlinear systems of CFPDEs. The practicable validation and effectiveness of this method are demonstrated by three typical examples.

Keywords: system of conformable fractional PDEs, new conformable fractional HPT, conformable fractional derivative, fractional power series.

1. Introduction

In the two decades before, the researchers have paid close attention to the analytical and numerical solutions of the nonlinear problems which have been arising in the present scientific and technological fields, such as physics, mathematics, biology, engineering, fluid mechanics, fluid dynamic, and viscoelasticity etc.. Such problems are not easy to be solved directly and obtained an exact solution so that suitable methods for finding exact or numerical solutions are required. Thereafter, analytical and numerical methods have been proposed by mathematical, physical and engineering experts, for instance, homotopy technique and its various modifications [1, 2, 3, 4], He variational iteration method [5], He-Laplace Method [6], parameter-expansion method [7], Wavelet-Galerkin method [8], Adomian decomposition method[9], Differential transform method [10] and the cited references.

Many problems that may happen in the real life world can be modelled only by differential equations of fractional order, we refer the readers to [11, 12] and the references therein. Actually, there is no physical meaning to the fractional derivative in order to unify its definition as ordinary derivatives [13]. Therefore, in this study we use the definition of Khalil et al. [14] which looks very similar to the definition of ordinary derivative and it is named a conformable fractional derivative. This novel definition has attracted the awareness of numerous scholars of mathematics and engineering to extend different solution techniques from ordinary differential equations to conformable fractional differential equations [15, 16, 17, 18, 19, 20, 21, 22, 23, 24].

More recently, the HPM in fractional version has showed a vigorous capability in solving conformable fractional PDEs see [25, 26] and references therein. We have motivated by these beneficial results and the conducive outcomes in the works of Biazar and Eslami [27] and Mirzazadeh and Ayati [28], to apply NCFHPT on several systems of CFPDEs with given initial and boundary conditions. In this work, we shall address a system of CFPDEs in the standard form as:

$$(1)\qquad \begin{cases} \partial_t^q u_1 = N_1^q(t, X, U, \partial_{x_i} U, \partial_t^q U) + f_1^q(t, X), \\ \partial_t^q u_2 = N_2^q(t, X, U, \partial_{x_i} U, \partial_t^q U) + f_2^q(t, X), \\ \vdots \\ \partial_t^q u_n = N_n^q(t, X, U, \partial_{x_i} U, \partial_t^q U) + f_n^q(t, X), \end{cases}$$

subject to I.C.

$$(2)\qquad \Big(u_1(t_0, X), u_2(t_0, X), \ldots, u_n(t_0, X)\Big) = \Big(\phi_1(X,), \phi_2(X), \ldots, \phi_n(X)\Big),$$

where $i = 1, 2 \ldots, n$, ∂_t^q stands for the conformable partial fractional derivative of order $0 < q \leq 1$ with respect to t , $\phi_1(X), \phi_2(X), \ldots, \phi_n(X)$ are real-valued functions depend only on the vector $X = (x_1, x_2, \ldots x_{n-1})$, $f_1^q, f_2^q, \ldots f_n^q$ are unknown functions depending on the parameter t and $x_1, x_2, \ldots, x_{n-1}$, and $N_1^q, N_2^q \ldots, N_n^q$ are the nonlinear parts of the equations of the system (1).

The rest of this article is arranged as the following: the appropriate adaptation of NHPM, which is called NCFHPT, for obtaining solutions of a system (1) is presented in the second section. NCFHPT solutions for three special systems of non-linear CFPDEs with given initial and boundary conditions are given in the third section. Finally, In Section 5, we briefly summarize the results in a short conclusion.

2. New conformable fractional homotopy technique

In this section, we present the appropriate adaptation of NHPM, which is called NCFHPT, for obtaining solutions of a system (1).

By the homotopy method [1, 2, 3, 4], we can build a system of homotopies as follows:

$$(3)\quad \begin{cases} (1-p)\big(\partial_t^q v_1 - u_{10}(t,X)\big)+p\Big(-N_1^q(t,X,V,\partial_{x_i}V,\partial_t^q V)-f_1^q(t,X)\Big)=0,\\ (1-p)\big(\partial_t^q v_2 - u_{20}(t,X)\big)+p\Big(-N_2^q(t,X,V,\partial_{x_i}V,\partial_t^q V)-f_2^q(t,X)\Big)=0,\\ \vdots\\ (1-p)\big(\partial_t^q v_n - u_{n0}(t,X)\big)+p\Big(-N_n^q(t,X,V,\partial_{x_i}V,\partial_t^q V)-f_n^q(t,X)\Big)=0 \end{cases}$$

or, equivalently

$$(4)\quad \begin{cases} \partial_t^q v_1 = u_{10}(t,X) - p\Big(u_{10}(t,X) - N_1^q(t,X,V,\partial_{x_i}V,\partial_t^q V) - f_1^q(t,X)\Big),\\ \partial_t^q v_2 = u_{20}(t,X) - p\Big(u_{20}(t,X) - N_2^q(t,X,V,\partial_{x_i}V,\partial_t^q V) - f_2^q(t,X)\Big),\\ \vdots\\ \partial_t^q v_n = u_{n0}(t,X) - p\Big(u_{n0}(t,X) - N_n^q(t,X,V,\partial_{x_i}V,\partial_t^q V) - f_n^q(t,X)\Big), \end{cases}$$

where $V=(v_1,v_2,\dots,v_n)$, X, $f_i^q, i=1\dots n$, are as defined above, $0\le p\le 1$ is an embedding parameter and $u_0=(u_{10}(t,X),u_{20}(t,X),\dots,u_{n0}(t,X))$ is a vector initial approximation of solution for the system (1).

Applying the fractional integral $\int_{t_0}^{t}(t-t_0)^{q-1}dt$ at all equations of the system (4), one can obtain

$$(5)\quad \begin{cases} v_1 = v_1(t_0,X) + \int_{t_0}^t t^{q-1}u_{10}(t,X)dt\\ \qquad -p\int_{t_0}^t t^{q-1}\big(u_{10}(t,X) - N_1^q - f_1^q(t,X)\big)dt,\\ v_2 = v_2(t_0,X) + \int_{t_0}^t t^{q-1}u_{20}(t,X)dt\\ \qquad -p\int_{t_0}^t t^{q-1}\big(u_{20}(t,X) - N_2^q - f_2^q(t,X)\big)dt,\\ \vdots\\ v_n = v_n(t_0,X) + \int_{t_0}^t t^{q-1}u_{n0}(t,X)dt\\ \qquad -p\int_{t_0}^t t^{q-1}\big(u_{n0}(t,X) - N_n^q - f_n^q(t,X)\big)dt, \end{cases}$$

with $V(t_0,X)=U(t_0,X)$.

Suppose that the willful solution of the given system (1) is presented in a power series of p, i.e.

$$(6)\quad \begin{cases} v_1 = v_{10} + pv_{11} + p^2v_{12} + \dots\\ v_2 = v_{20} + pv_{21} + p^2v_{22} + \dots\\ \vdots\\ v_n = v_{n0} + pv_{n1} + p^2v_{n2} + \dots, \end{cases}$$

where, the $v_{ij}, i=1,\dots n, j=1,\dots n$, are the seeking functions which are needed to be calculated. Assume that for $i=1,\dots n$, f_i^q and the initial approximate solution u_{i0} are infinitely q partial differentiable functions with respect to t, around a point t_0, then their fractional power series can be given as:

$$(7)\qquad \begin{cases} u_{10} = \sum_{k=0}^{\infty} \frac{a_{1k}(X)(t-t_0)^{qk}}{q^k k!}, \\ u_{20} = \sum_{k=0}^{\infty} \frac{a_{2k}(X)(t-t_0)^{qk}}{q^k k!}, \\ \vdots \\ u_{n0} = \sum_{k=0}^{\infty} \frac{a_{nk}(X)(t-t_0)^{qk}}{q^k k!} \end{cases}$$

and

$$(8)\qquad \begin{cases} f_1^q = \sum_{k=0}^{\infty} \frac{b_{1k}(X)(t-t_0)^{qk}}{q^k k!}, \\ f_2^q = \sum_{k=0}^{\infty} \frac{b_{2k}(X)(t-t_0)^{qk}}{q^k k!}, \\ \vdots \\ f_n^q = \sum_{k=0}^{\infty} \frac{b_{nk}(X)(t-t_0)^{qk}}{q^k k!}, \end{cases}$$

where $a_{ik'}s$ are unknown functions which must be computed and $b_{ik'}s$ are given ones.

Now, substituting (6) and (7) into (5) and equating the coefficients of terms with identical powers of p, we get

$$p^0 : \begin{cases} v_{10} = \phi_1(X) + \sum_{k=0}^{\infty} \frac{a_{1k}(X)(t-t_0)^{q(k+1)}}{q^{k+1}(k+1)!} \\ v_{20} = \phi_2(X) + \sum_{k=0}^{\infty} \frac{a_{2k}(X)(t-t_0)^{q(k+1)}}{q^{k+1}(k+1)!} \\ \vdots \\ v_{n0} = \phi_n(X) + \sum_{k=0}^{\infty} \frac{a_{nk}(X)(t-t_0)^{q(k+1)}}{q^{k+1}(k+1)!} \end{cases}$$

$$p^1 : \begin{cases} v_{11} = -\sum_{k=0}^{\infty} \frac{a_{1k}(X)(t-t_0)^{q(k+1)}}{q^{k+1}(k+1)!} \\ \qquad + \int_{t_0}^{t} t^{q-1}\big(N_1^q(t, X, v_{.0}, \partial_{x_i} v_{.0}, \partial_t^q v_{.0}) - f_1^q\big)dt \\ v_{21} = -\sum_{k=0}^{\infty} \frac{a_{2k}(X)(t-t_0)^{q(k+1)}}{q^{k+1}(k+1)!} \\ \qquad + \int_{t_0}^{t} t^{q-1}\big(N_2^q(t, X, v_{.0}, \partial_{x_i} v_{.0}, \partial_t^q v_{.0}) - f_2^q\big)dt \\ \vdots \\ v_{n1} = -\sum_{k=0}^{\infty} \frac{a_{nk}(X)(t-t_0)^{q(k+1)}}{q^{k+1}(k+1)!} \\ \qquad + \int_{t_0}^{t} t^{q-1}\big(N_n^q(t, X, v_{.0}, \partial_{x_i} v_{.0}, \partial_t^q v_{.0}) - f_n^q\big)dt \end{cases}$$

$$\vdots$$

$$(9)\qquad p^m:\begin{cases} v_{1m}=\int_{t_0}^{t} t^{q-1}\big(N_1^q(t,X,v_{.0},\dots v_{.m-1},\partial_{x_i}v_{.0},\dots,\partial_{x_i}v_{.m-1},\\ \qquad \partial_t^q v_{.0},\dots,\partial_t^q v_{.m-1}\big)dt,\\ v_{2m}=\int_{t_0}^{t} t^{q-1}\big(N_2^q(t,X,v_{.0},\dots v_{.m-1},\partial_{x_i}v_{.0},\dots,\partial_{x_i}v_{.m-1},\\ \qquad \partial_t^q v_{.0},\dots,\partial_t^q v_{.m-1}\big)dt,\\ \vdots\\ v_{nm}=\int_{t_0}^{t} t^{q-1}\big(N_n^q(t,X,v_{.0},\dots v_{.m-1},\partial_{x_i}v_{.0},\dots,\partial_{x_i}v_{.m-1},\\ \qquad \partial_t^q v_{.0},\dots,\partial_t^q v_{.m-1}\big)dt.\end{cases}$$

$$\vdots$$

Solving these equations under the condition that $v_{i1}=0, i=1,\dots n$, then (9) gives $v_{i2}=v_{i3}=\dots=0, i=1,\dots n$. Thus the approximate solution which constructed from the presented method can be obtained as

$$(10)\qquad \begin{cases} u_1=v_{10}=\phi_1(X)+\sum_{k=0}^{\infty}\frac{a_{1k}(X)(t-t_0)^{q(k+1)}}{q^{k+1}(k+1)!}\\ u_2=v_{20}=\phi_2(X)+\sum_{k=0}^{\infty}\frac{a_{2k}(X)(t-t_0)^{q(k+1)}}{q^{k+1}(k+1)!}\\ \vdots\\ u_n=v_{n0}=\phi_n(X)+\sum_{k=0}^{\infty}\frac{a_{nk}(X)(t-t_0)^{q(k+1)}}{q^{k+1}(k+1)!}.\end{cases}$$

This method can be easily adapted to the system with given boundary conditions. In the following section, one of the examples will be given with boundary conditions.

3. Numerical results

To illustrate the efficiency of NCFHPT, we applied it to three special systems of non-linear CFPDEs with given initial conditions and boundary conditions as well. This method shows high capabilities for obtaining closed form series solutions. These solutions coincide with the exact solutions in the given systems here.

Example 1. Given a system of coupled time conformable fractional Burgers equations

$$(11)\qquad \begin{cases} \partial_t^q u_1=\partial_x^2 u_1-u_1\partial_x u_1-\partial_x(u_1u_2)+x^2-2\frac{t^q}{q}+2x^3\frac{t^{2q}}{q^2}+\frac{t^{2q}}{q^2},\\ \partial_t^q u_2=\partial_x^2 u_2-u_2\partial_x u_2+\partial_x(u_1u_2)+x^{-1}-2x^{-3}\frac{t^q}{q}-2x^{-3}\frac{t^{2q}}{q^2}-\frac{t^{2q}}{q^2}\end{cases}$$

subject to I.C.

$$(12)\qquad \begin{cases} u_1(0,x)=0,\\ u_2(0,x)=0.\end{cases}$$

Thanks to NCFHPT, we have

$$(13)\quad \begin{cases} v_1(t,x) = v_1(0,x) + \int_0^t t^{q-1}u_{10}(t,x)dt - p\int_0^t t^{q-1}u_{10}(t,x)dt \\ \quad -p\int_0^t t^{q-1}\big(-\partial_x^2 v_1 + v_1\partial_x v_1 + \partial_x(v_1v_2) \\ \quad -x^2 + 2\frac{t^q}{q} - 2x^3\frac{t^{2q}}{q^2} - \frac{t^{2q}}{q^2}\big)dt, \\ v_2(t,x) = v_2(0,x) + \int_0^t t^{q-1}u_{20}(t,x)dt - p\int_0^t t^{q-1}u_{20}(t,x)dt \\ \quad -p\int_0^t t^{q-1}\big(-\partial_x^2 v_2 + v_2\partial_x v_2 - \partial_x(v_1v_2) - x^{-1} + 2x^{-3}\frac{t^q}{q} \\ \quad +2x^{-3}\frac{t^{2q}}{q^2} + \frac{t^{2q}}{q^2}\big)dt. \end{cases}$$

Substituting (6) and (7) into (13) and equating the coefficients with the same power of p we obtain

$$p^0 : \begin{cases} v_{10} = v_1(0,x) + \sum_{k=0}^{\infty} \frac{a_{1k}(x)t^{q(k+1)}}{q^{k+1}(k+1)!}, \\ v_{20} = v_2(0,x) + \sum_{k=0}^{\infty} \frac{a_{2k}(x)t^{q(k+1)}}{q^{k+1}(k+1)!}, \end{cases}$$

$$p^1 : \begin{cases} v_{11} = -\sum_{k=0}^{\infty} \frac{a_{1k}(x)t^{q(k+1)}}{q^{k+1}(k+1)!} \\ -\int_0^t t^{q-1}\big(-\partial_x^2 v_{10} + v_{10}\partial_x v_{10} + \partial_x(v_{10}v_{20}) \\ \quad -x^2 + 2\frac{t^q}{q} - 2x^3\frac{t^{2q}}{q^2} - \frac{t^{2q}}{q^2}\big)dt \\ v_{21} = -\sum_{k=0}^{\infty} \frac{a_{2k}(x)t^{q(k+1)}}{q^{k+1}(k+1)!} \\ -\int_0^t t^{q-1}\big(-\partial_x^2 v_{20} + v_{20}\partial_x v_{20} - \partial_x(v_{10}v_{20}) \\ \quad -x^{-1} + 2x^{-3}\frac{t^q}{q} + 2x^{-3}\frac{t^{2q}}{q^2} + \frac{t^{2q}}{q^2}\big)dt \end{cases},$$

$$\vdots$$

Assume that $v_1(0,x) = u_1(0,x), v_2(0,x) = u_2(0,x)$, and we obtain

$$v_{11} = \big(-a_{10} + x^2\big)\frac{t^q}{q} + \big(-a_{11} + a_{10}'' - 2\big)\frac{t^{2q}}{2q^2} + \Big(-2a_{12} + a_{11}'' - 2a_{10}a_{10}' - 2a_{20}a_{10}' - 2a_{10}a_{20}' + 4x^3 + 2\Big)\frac{t^{3q}}{3!q^3} + \dots$$

and

$$v_{21} = \big(-a_{20} + x^{-1}\big)\frac{t^q}{q} + \big(-a_{21} + a_{20}'' - 2x^{-3}\big)\frac{t^{2q}}{2q^2} + \Big(-2a_{22} + a_{11}'' - 2a_{20}a_{20}' + 2a_{20}a_{10}' + 2a_{10}a_{20}' - 2x^{-3} - 2\Big)\frac{t^{3q}}{3!q^3} + \dots$$

Now, setting $v_{11} = v_{21} = 0$ we get:

$$a_{10}(x) = x^2,\ a_{11}(x) = a_{12}(x) =, \dots, = 0,$$
$$a_{20}(x) = x^{-1},\ a_{21}(x) = a_{22}(x) =, \dots, = 0.$$

Therefore, we obtain the new homotopy solution of the given system as

$$\begin{cases} u_1 = v_{10} = \phi_1(x) + \sum_{k=0}^{\infty} \frac{a_{1k}(x)t^{q(k+1)}}{q^{k+1}(k+1)!} = x^2 \frac{t^q}{q}, \\ u_2 = v_{20} = \phi_2(x) + \sum_{k=0}^{\infty} \frac{a_{2k}(x)t^{q(k+1)}}{q^{k+1}(k+1)!} = x^{-1} \frac{t^q}{q}, \end{cases} \tag{14}$$

which gives the exact solution of the system (11).

Example 2. Given a system of two space-conformable fractional partial differential equations

$$\begin{cases} \partial_x^q u_1 = \partial_t(u_1)u_2 - u_1\partial_t u_2 + e^{\frac{x^q}{q}} \sin t - 1, \\ \partial_x^q u_2 = -\partial_t u_1 \partial_x^q u_2 - \partial_x^q u_1 \partial_t u_2 - e^{-\frac{x^q}{q}} \cos t - 1, \end{cases} \tag{15}$$

subject to

$$\begin{cases} u_1(t,0) = \sin t, \\ u_2(t,0) = \cos t. \end{cases} \tag{16}$$

Thanks to NCFHPT, we have

$$\begin{cases} v_1(t,x) = v_1(t,0) + \int_0^x x^{q-1} u_{10}(t,x)dx \\ \qquad -p \int_0^x x^{q-1}\big(u_{10}(t,x) - \partial_t(v_1)v_2 + v_1\partial_t v_2 - e^{\frac{x^q}{q}} \sin t + 1\big)dx, \\ v_2(t,x) = v_2(t,0) + \int_0^x x^{q-1} u_{20}(t,x)dt \\ \qquad -p \int_0^x x^{q-1}\big(u_{20}(t,x) + \partial_t v_{10} \partial_x^q v_{20} + \partial_x^q v_{10} \partial_t v_{20} + e^{-\frac{x^q}{q}} \cos t + 1\big)dx. \end{cases} \tag{17}$$

Substituting (7) and (6) into (17) and equating the terms with the same power of p we obtain

$$p^0 : \begin{cases} v_{10} = v_1(t,0) + \sum_{k=0}^{\infty} \frac{a_{1k}(t)x^{q(k+1)}}{q^{k+1}(k+1)!}, \\ v_{20} = v_2(t,0) + \sum_{k=0}^{\infty} \frac{a_{2k}(t)x^{q(k+1)}}{q^{k+1}(k+1)!}, \end{cases}$$

$$p^1 : \begin{cases} v_{11} = -\sum_{k=0}^{\infty} \frac{a_{1k}(t)x^{q(k+1)}}{q^{k+1}(k+1)!} \\ \qquad - \int_0^x x^{q-1}\big(- \partial_t(v_{10})v_{20} + v_{10}\partial_t v_{20} - e^{\frac{x^q}{q}} \sin t + 1\big)dx, \\ v_{21} = -\sum_{k=0}^{\infty} \frac{a_{2k}(t)x^{q(k+1)}}{q^{k+1}(k+1)!} \\ \qquad - \int_0^x x^{q-1}\big(\partial_t v_1 \partial_x^q v_2 + \partial_x^q v_1 \partial_t v_2 + e^{-\frac{x^q}{q}} \cos t + 1\big)dx, \end{cases}$$

$$\vdots$$

Assume that $v_1(t,0) = u_1(t,0), v_2(t,0) = u_2(t,0)$, and evaluate v_{11} and v_{21} we obtain

$$\begin{aligned} v_{11} &= (\sin t - a_{10}(t))\frac{x^q}{q} + \big(- a_{11}(t) + (a'_{10}(t) \\ &\quad + a_{20}(t))\cos t + (a_{10}(t) - a'_{20}(t))\sin t + \sin t\big)\frac{x^{2q}}{2q^2}\big(- 2a_{12}(t) \\ &\quad + (a'_{11}(t) + a_{21}(t))\cos t + (a_{11}(t) \\ &\quad - a'_{21}(t))\sin t + 2(a_{20}(t)a'_{10}(t) - a'_{20}(t)a_{10}(t)) + \sin t\Big)\frac{x^{3q}}{3!q^3} + \dots \end{aligned}$$

and

$$\begin{aligned} v_{21} &= \Big(a_{20}(t)(-1 - \cos t) + a_{10}(t)\sin t - \cos t - 1\Big)\frac{x^q}{q} \\ &\quad + \Big(a_{21}(t)(-1 - \cos t) - a_{20}(t)a'_{10}(t) \\ &\quad + a_{11}(t)\sin t - a_{10}(t)a'_{20}(t) + \cos t\Big)\frac{x^{2q}}{2q^2} \\ &\quad + \Big(- 2a_{22}(t)(1 + \cos t) - a'_{11}(t)a_{20}(t) - 2a_{21}(t)a'_{10}(t) \\ &\quad + 2a_{12}(t)\sin t - a'_{21}(t)a_{10}(t) - 2a'_{20}(t)a_{11}(t) - \cos t\Big)\frac{x^{3q}}{3!q^3} + \dots \end{aligned}$$

Now, setting $v_{11} = v_{12} = 0$ we get:

$$\begin{aligned} &a_{10}(t) = \sin t,\ a_{11}(t) = \sin t,\ a_{12}(t) = \sin t,\ a_{13}(t) = \sin t, \dots, a_{1n}(t) = \sin t, \dots, \\ &a_{20}(t) = -\cos t,\ a_{21}(t) = \cos t,\ a_{22}(t) = -\cos t, \\ &a_{23}(t) = \cos t, \dots, a_{2n}(t) = (-1)^{n+1}\cos t, \dots. \end{aligned}$$

Therefore, we obtain the new homotopy solution of the given system as

$$\begin{cases} u_1 = v_{10} = \phi_1(x) + \sum_{k=0}^{\infty} \frac{a_{1k}(t)x^{q(k+1)}}{q^{k+1}(k+1)!} = e^{\frac{x^q}{q}}\sin t, \\ u_2 = v_{20} = \phi_2(x) + \sum_{k=0}^{\infty} \frac{a_{2k}(t)x^{q(k+1)}}{q^{k+1}(k+1)!} = e^{-\frac{x^q}{q}}\cos t, \end{cases} \tag{18}$$

which is the exact solution of the system (15).

Example 3. Given a system of two dimensional time-conformable fractional partial differential equations

$$\begin{cases} \partial_t^q u_1 = \partial_x(u_1)u_2 + \partial_t u_2 \partial_y u_1 + 1 - x + y + \frac{t^q}{q}, \\ \partial_t^q u_2 = \partial_x(u_2)u_1 + \partial_t u_1 \partial_y u_2 + 1 - x - y - \frac{t^q}{q}, \end{cases} \tag{19}$$

subject to I.C.

$$\begin{cases} u_1(0,x,y) = x + y - 1, \\ u_2(0,x,y) = x - y + 1. \end{cases} \tag{20}$$

According to NCFHPT, we have

$$\begin{cases} v_1(x,y,t) = v_1(0,x,y) + \int_0^t t^{q-1} u_{10}(x,y,t)dt \\ \qquad -p\int_0^t t^{q-1}\big(u_{10}(x,y,t) - \partial_x(v_1)v_2 \\ \qquad -\partial_t v_2 \partial_y v_1 - 1 + x - y - \frac{t^q}{q}\big)dt, \\ v_2(x,y,t) = v_2(0,x,y) + \int_0^t t^{q-1} u_{20}(x,y,t)dt \\ \qquad -p\int_0^t t^{q-1}\big(u_{20}(x,y,t) - \partial_x(v_2)v_1 \\ \qquad -\partial_t v_1 \partial_y v_2 - 1 + x + y + \frac{t^q}{q}\big)dt. \end{cases} \tag{21}$$

Substituting (7) and (6) into (21) and comparing the equating the terms with the same power of p we have

$$p^0 : \begin{cases} v_{10} = v_1(0,x,y) + \sum_{k=0}^{\infty} \frac{a_{1k}(x,y)t^{q(k+1)}}{q^{k+1}(k+1)!}, \\ v_{20} = v_2(0,x,y) + \sum_{k=0}^{\infty} \frac{a_{2k}(x,y)t^{q(k+1)}}{q^{k+1}(k+1)!}, \end{cases}$$

$$p^1 : \begin{cases} v_{11} = -\sum_{k=0}^{\infty} \frac{a_{1k}((x,y))t^{q(k+1)}}{q^{k+1}(k+1)!} \\ -\int_0^t t^{q-1}\big(-\partial_x(v_{10})v_{20} - \partial_t v_{20}\partial_y v_{10} - 1 + x - y - \frac{t^q}{q}\big)dt, \\ v_{21} = -\sum_{k=0}^{\infty} \frac{a_{2k}(x,y)t^{q(k+1)}}{q^{k+1}(k+1)!} \\ -\int_0^t t^{q-1}\big(-\partial_x(v_{20})v_{10} - \partial_t v_{10}\partial_y v_{20} - 1 + x + y + \frac{t^q}{q}\big)dt, \end{cases}$$

$$p^2 : \begin{cases} v_{12} = \int_0^t t^{q-1}\big(\partial_x(v_{11})v_{20} + v_{21}\partial_x v_{10} + \partial_t v_{20}\partial_y v_{11} + \partial_t v_{21}\partial_y v_{10}\big)dt, \\ v_{22} = \int_0^t t^{q-1}\big(\partial_x(v_{21})v_{10} + v_{11}\partial_x v_{20} + \partial_t v_{10}\partial_y v_{21} + \partial_t v_{11}\partial_y v_{20}\big)dt, \end{cases}$$

$$\vdots$$

Assume that $v_1(0,x,y) = u_1(0,x,y), v_2(0,x,y) = u_2(0,x,y)$, and evaluate v_{11} and v_{21} we obtain

$$\begin{aligned} v_{11} = &\big(-a_{10} + a_{20} + 2\big)\frac{t^q}{q} + \big(-a_{11} \\ &+ a_{20} + (x-y+1)\partial_x a_{10} + a_{21} + a_{20}\partial_y a_{10} + 1\big)\frac{t^{2q}}{2q^2} \\ &+ \Big(-2a_{12} + a_{21} + (x-y+1)\partial_x a_{11} \\ &+ 2a_{20}\partial_x a_{10} + 2a_{22} + 2a_{21}\partial_y a_{10} + a_{20}\partial_y a_{11}\Big)\frac{t^{3q}}{3!q^3} + \dots \end{aligned}$$

and

$$\begin{aligned} v_{21} = &\big(-a_{10} - a_{20}\big)\frac{t^q}{q} + \big(-a_{21} + a_{10} + (x+y-1)\partial_x a_{20} \\ &- a_{11} + a_{10}\partial_y a_{20} - 1\big)\frac{t^{2q}}{2q^2} \end{aligned}$$

$$+\Big(a_{11} - 2a_{21} + (x+y-1)\partial_x a_{21} + 2a_{10}\partial_x a_{20} - 2a_{12} + 2a_{11}\partial_y a_{20} + a_{10}\partial_y a_{21}\Big)\frac{t^{3q}}{3!q^3} + \dots$$

Now, setting $v_{11} = v_{21} = 0$ we get:

$$a_{10}(x,y) = 1,\ a_{11}(x,y) = a_{12}(x,y) = a_{13}(x,y) = \dots = 0,$$
$$a_{20}(x,y) = -1,\ a_{21}(x,y) = a_{22}(x,y) = a_{23}(x,y) = \dots = 0.$$

Therefore, we obtain the new homotopy solution of the given system as

$$\begin{cases} u_1 = v_{10} = \phi_1(x,y) + \sum_{k=0}^{\infty} \frac{a_{1k}(x,y)t^{q(k+1)}}{q^{k+1}(k+1)!} = x+y-1+\frac{t^q}{q}, \\ u_2 = v_{20} = \phi_2(x,y) + \sum_{k=0}^{\infty} \frac{a_{2k}(x,y)t^{q(k+1)}}{q^{k+1}(k+1)!} = x-y+1-\frac{t^q}{q}, \end{cases} \tag{22}$$

which is the exact solution of the system (19).

4. Conclusion

In this paper, we carried out a convenient adaptation of NHPM to obtain a solution for linear and nonlinear systems of conformable fractional PDEs subject to initial and boundary conditions. The proposed method seems to be designed for working without discretization, linearization or any computational complexity. At the end, we successfully apply this new method to three nonlinear systems and it provides exact solutions in direct way and simple computations. By applying NCFHPT on such systems, we obtained a new system of recurrence relations which can be easily solved starting with the given initial condition and then iteratively getting the general form of each required coefficient. As we have seen in the forgoing examples, the NCFHPT solutions of the given three systems were coincide with the exact ones. It is observed that NCFHPT is an effective and promise tool for solving nonlinear systems of CFPDEs, especially when exact solutions of such systems can be expressed as a closed form of a summation.

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Accepted: January 5, 2021