Wrapped Aradhana distribution

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Abstract. In this paper, we introduce a new circular distribution generated from Aradhana distribution. We obtained the probability densities and characteristic functions of wrapped Aradhana distribution. The maximum likelihood estimator of the new distribution parameter is also obtained. Then, some properties of the distribution are studied such as means, skewness, and kurtosis.

Keywords: Aradhana distribution, wrapped distribution, maximum likelihood estimation, circular variance, circular standard deviation, trigonometric moments, mean direction, mean resultant length, skewness, kurtosis.

1. Introduction

R. Shanker in [7] proposed a new lifetime distribution which is a combination of three component mixture of exponential distributions, an exponential λ distribution with mixing proportion $\lambda^2/(\lambda^2+2\lambda+2)$, a gamma $(2,\lambda)$ distribution with mixing proportion $2\lambda/(\lambda^2+2\lambda+2)$ and a gamma $(3,\lambda)$ distribution with mixing proportion $2/(\lambda^2+2\lambda+2)$. The new distribution entitled Aradhana distribution (AD) is more flexible for modeling lifetime data than other well known distributions. Also, R. Shanker in [9] obtained the generalized distribution of AD of which AD is a special case. The two parameter quasi AD was proposed by R. Shanker in [12]. A two parameter power AD was studied by R. Shanker in [10]. R. Ganaie, V. Rajagopalan, A. Rather in [1] proposed a generalization of AD called weighted AD, and M. Gharaibieh in [2] used the quadratic rank transmutation map to introduce the transmuted AD as a new generalization of AD. Moreover, R. Shanker in [8] proposed the discrete Poisson-AD by compounding the Poisson distribution with AD and studied its properties, and the size-biased Poisson-AD was proposed by R. Shanker in [13]. Recently, R. Shanker in [11] proposed a quasi Poisson-AD by compounding a Poisson distribution with a quasi AD.

The probability density function (pdf) of AD is given by

$$f(x;\lambda)=\frac{\lambda^3}{\lambda^2+2\lambda+2}(1+x)^2e^{-x\lambda};\quad x>0,\quad \lambda>0, \tag{1}$$

with cumulative distribution function (cdf)

$$F(x;\lambda) = 1 - \left[1 + \frac{x\lambda(x\lambda + 2\lambda + 2)}{\lambda^2 + 2\lambda + 2}\right] e^{-x\lambda}; \quad x > 0, \quad \lambda > 0. \tag{2}$$

The characteristic function AD is defined as

$$\varphi_X(t) = \frac{\lambda^3}{\lambda^2 + 2\lambda + 2}\left[\frac{1}{\lambda - it} + \frac{2}{(\lambda - it)^2} + \frac{2}{(\lambda - it)^3}\right],; \quad i = \sqrt{-1}. \tag{3}$$

Direction data are data measured by two or three dimensions and they are used in the natural and physical sciences. Researchers use circular distributions to analyze and model directional data, and many of these circular distributions are generated from known probability distribution using different methods, such as, wrapping the distribution around the unite circle (see, for example, [6, 3]).

In this paper, wrapped Aradhana distribution (WAD) has been proposed and its statistical properties including probability densities functions for varying values of the parameters, characteristic function, moments trigonometric moments, means, skewness and kurtosis has been studied. The maximum likelihood estimation has been discussed for estimating its parameters.

2. Wrapped Aradhana Distribution (WAD)

Let X be an Aradhana random variable and θ be the circular random variable defined by $\theta = X(mod\ 2\pi)$. The probability density function for θ is given by

$$g(\theta) = \sum_{m=-\infty}^{\infty} f(\theta + 2m\pi), \qquad 0 \le \theta < 2\pi.$$

The probability density function $g(\theta)$ satisfies the following conditions:

(1) $g(\theta) \ge 0$, for all θ,

(2) $\int_0^{2\pi} g(\theta)d\theta = 1$,

(3) $g(\theta)$ is periodic function, such that, $g(\theta) = g(\theta + 2\pi m)$ for $m \in \mathbb{Z}$.

Hence, we can say that θ is a wrapped Aradhana random variable with probability density function (pdf) $g(\theta;\lambda)$ given by

$$\begin{aligned} g(\theta;\lambda) &= \sum_{m=0}^{\infty} f(\theta + 2m\pi) \\ &= \sum_{m=0}^{\infty} \frac{\lambda^3}{\lambda^2 + 2\lambda + 2}(1 + \theta + 2m\pi)^2 e^{-(\theta+2m\pi)\lambda} \\ &= \frac{\lambda^3\ e^{-\theta\lambda}}{\lambda^2 + 2\lambda + 2}\left[\frac{(1+\theta)^2}{1 - e^{-2\pi\lambda}} + \frac{4(\theta+1)\pi e^{-2\pi\lambda}}{(1 - e^{-2\pi\lambda})^2} + \frac{4\pi^2 e^{-2\pi\lambda}(1 + e^{-2\pi\lambda})}{(1 - e^{-2\pi\lambda})^3}\right]. \end{aligned} \tag{4}$$

Moreover, the cumulative distribution function $G(\theta;\lambda)$, for $\theta \in [0, 2\pi)$ and $\lambda > 0$ is given by

$$
\begin{aligned}
G(\theta;\lambda) &= \sum_{m=0}^{\infty} [F(\theta+2m\pi) - F(2m\pi)] \\
&= \sum_{m=0}^{\infty} \Bigg[1 - \left(1 + \frac{(\theta+2m\pi)\lambda[(\theta+2m\pi)\lambda+2\lambda+2]}{\lambda^2+2\lambda+2}\right) e^{-(\theta+2m\pi)\lambda} \\
&\qquad - 1 + \left(1 + \frac{2m\pi\lambda[2m\pi\lambda+2\lambda+2]}{\lambda^2+2\lambda+2}\right) e^{-(2m\pi)\lambda}\Bigg] \\
&= \frac{1-e^{-\theta\lambda}}{1-e^{-2\pi\lambda}} - \frac{\theta\lambda(\theta\lambda+2\lambda+2)e^{-\theta\lambda}}{(\lambda^2+2\lambda+2)(1-e^{-2\pi\lambda})} \\
&\quad + \frac{4\pi\lambda\left[(1+\lambda)(1-e^{-\theta\lambda}) - \theta\lambda e^{-\theta\lambda}\right] e^{-2\pi\lambda}}{(\lambda^2+2\lambda+2)(1-e^{-2\pi\lambda})^2} \\
&\qquad + \frac{4\pi^2\lambda^2(1-e^{-\theta\lambda})(1+e^{-2\pi\lambda})e^{-2\pi\lambda}}{(\lambda^2+2\lambda+2)(1-e^{-2\pi\lambda})^3}.
\end{aligned}
\tag{5}
$$

Figure 1 shows the pdf and cdf of WAD for different values of λ.

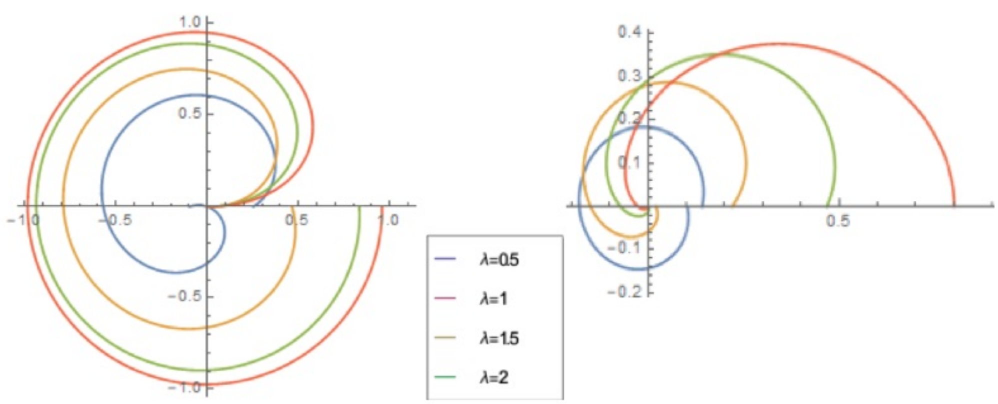

Figure 1: The Circular representation for WAD for $\lambda = 0.5, 1, 1.5, 2$.

3. Characteristic function and trigonometric moments

The p^{th} trigonometric moment of WAD is given by

$$
\begin{aligned}
\varphi_\theta(p) &= \varphi_X(p) \\
&= \frac{\lambda^3}{\lambda^2+2\lambda+2}\left[\frac{1}{\lambda-ip} + \frac{2}{(\lambda-ip)^2} + \frac{2}{(\lambda-ip)^3}\right]; \\
& i = \sqrt{-1}, p = \mp 1, \mp 2, \ldots
\end{aligned}
\tag{6}
$$

$$= \frac{\lambda^3}{\lambda^2+2}\left[\frac{(\lambda-ip)^2+2}{(\lambda-ip)^3}\right]$$
$$= \frac{\lambda^3}{\lambda^2+2}\left[(\lambda^2+2\lambda-p^2+2)-2\ ip(1+\lambda)\right](\lambda-ip)^{-3},$$

using the fact that $(a-ib)^{-r}=(a^2+b^2)^{\frac{-r}{2}}e^{ir\arctan(b/a)}$, for any $a,b,r\in\Re$, we can write the p^{th} trigonometric moment for WAD $\varphi_\theta(p)$ as

$$\varphi_\theta(p)=\rho_p e^{i\mu_p};\quad i=\sqrt{-1},\ p=\mp1,\mp2,\ldots, \tag{7}$$

where

$$\rho_p=\frac{\lambda^3}{\lambda^2+2\lambda+2}\sqrt{\frac{(\lambda^2+2\lambda-p^2+2)^2+4p^2(1+\lambda^2)}{(\lambda^2+p^2)^3}},$$

and

$$\mu_p=3\arctan\left(\frac{p}{\lambda}\right)-\arctan\left(\frac{2p(1+\lambda)}{\lambda^2+2\lambda-p^2+2}\right).$$

Since $\varphi_\theta(p)=\alpha_p+i\beta_p$ with $\alpha_p=E(\cos\ p\theta)$ and $\beta_p=E(\sin\ p\theta)$ the non-central trigonometric moments for WAD are given by

$$\begin{aligned}\alpha_p&=\rho_p\cos\ \mu_p\\&=\frac{\lambda^3}{\lambda^2+2\lambda+2}\sqrt{\frac{(\lambda^2+2\lambda-p^2+2)^2+4p^2(1+\lambda^2)}{(\lambda^2+p^2)^3}}\cos\Bigg[3\arctan\left(\frac{p}{\lambda}\right)\\&\qquad-\arctan\left(\frac{2p(1+\lambda)}{\lambda^2+2\lambda-p^2+2}\right)\Bigg],\end{aligned} \tag{8}$$

and

$$\begin{aligned}\beta_p&=\rho_p\sin\ \mu_p\\&=\frac{\lambda^3}{\lambda^2+2\lambda+2}\sqrt{\frac{(\lambda^2+2\lambda-p^2+2)^2+4p^2(1+\lambda^2)}{(\lambda^2+p^2)^3}}\sin\Bigg[3\arctan\left(\frac{p}{\lambda}\right)\\&\qquad-\arctan\left(\frac{2p(1+\lambda)}{\lambda^2+2\lambda-p^2+2}\right)\Bigg].\end{aligned} \tag{9}$$

The central trigonometric moments for WAD are given by

$$\begin{aligned}\bar{\alpha}_p&=\rho_p\cos(\mu_p-p\mu_1)\\&=\frac{\lambda^3}{\lambda^2+2\lambda+2}\sqrt{\frac{(\lambda^2+2\lambda-p^2+2)^2+4p^2(1+\lambda)^2}{(\lambda^2+p^2)^3}}\cos\Bigg[3\arctan\left(\frac{p}{\lambda}\right)\\&-\arctan\left(\frac{2p(1+\lambda)}{\lambda^2+2\lambda-p^2+2}\right)-3p\arctan\left(\frac{1}{\lambda}\right)+p\arctan\left(\frac{2(1+\lambda)}{\lambda^2+2\lambda+1}\right)\Bigg],\end{aligned} \tag{10}$$

and

$$\begin{aligned}\bar{\beta}_p &= \rho_p \sin(\mu_p - p\mu_1) \\ &= \frac{\lambda^3}{\lambda^2+2\lambda+2}\sqrt{\frac{(\lambda^2+2\lambda-p^2+2)^2+4p^2(1+\lambda)^2}{(\lambda^2+p^2)^3}}\sin\left[3\arctan\left(\frac{p}{\lambda}\right)\right. \\ &\left. -\arctan\left(\frac{2p(1+\lambda)}{\lambda^2+2\lambda-p^2+2}\right) -3p\arctan\left(\frac{1}{\lambda}\right) +p\arctan\left(\frac{2(1+\lambda)}{\lambda^2+2\lambda+1}\right)\right]. \end{aligned} \tag{11}$$

Table 1 shows the values of non-central and central trigonometric moments for $p = 1, 2$ and $\lambda = 0.5, 1, 1.5, 2, 4$.

Parameter	Non-central Trigonometric Moments				Central Trigonometric Moments			
λ	α_1	α_2	β_1	β_2	$\bar{\alpha}_1$	$\bar{\alpha}_2$	$\bar{\beta}_1$	$\bar{\beta}_2$
0.5	-0.064	0.013	0.059	-0.015	0.087	0.016	0	0.012
1	0	-0.031	0.346	0.098	0.346	0.031	0	-0.098
1.5	0.266	0.011	0.508	0.235	0.573	0.187	0	-0.143
2	0.499	0.120	0.517	0.361	0.719	0.357	0	-0.133
4	0.867	0.601	0.321	0.460	0.924	0.756	0	-0.042

Table 1: Trigonometric Moments for WAD for $p = 1, 2$ and $\lambda = 0.5, 1, 1.5, 2, 4$.

4. Means and related measures

The mean direction and resultant length for WAD are given respectively by

$$\mu = \mu_1 = 3\arctan\left(\frac{1}{\lambda}\right) - \arctan\left(\frac{2(1+\lambda)}{\lambda^2+2\lambda+1}\right), \tag{12}$$

and

$$\rho = \rho_1 = \frac{\lambda^3}{\lambda^2+2\lambda+2}\sqrt{\frac{(\lambda^2+2\lambda+1)^2+4(1+\lambda^2)}{(\lambda^2+1)^3}}. \tag{13}$$

Table 2 shows the values of mean direction and mean resultant for $\lambda = 0.5, 1, 1.5, 2, 4$.

Means	Parameter λ				
	0.5	1	1.5	2	4
Mean Direction μ	2.394	1.571	1.089	0.803	0.354
Mean Resultant Length ρ	0.087	0.346	0.573	0.719	0.924

Table 2: Mean Direction and Mean Resultant Length of WAD for $\lambda = 0.5, 1, 1.5, 2, 4$.

While, the circular variance and standard deviation for WAD are given respectively by

$$
\begin{aligned}
V &= 1-\rho \\
&= 1-\frac{\lambda^3}{\lambda^2+2\lambda+2}\sqrt{\frac{(\lambda^2+2\lambda+1)^2+4(1+\lambda^2)}{(\lambda^2+1)^3}},
\end{aligned}
\tag{14}
$$

and

$$
\begin{aligned}
\sigma &= \sqrt{-2\ln\rho} \\
&= \sqrt{-2\ln\left(\frac{\lambda^3}{\lambda^2+2\lambda+2}\sqrt{\frac{(\lambda^2+2\lambda+1)^2+4(1+\lambda^2)}{(\lambda^2+1)^3}}\right)}.
\end{aligned}
\tag{15}
$$

Table 3 shows the values of circular variance and circular standard deviation for $\lambda = 0.5, 1, 1.5, 2, 4$.

Measures of Variation	**Parameter λ**				
	0.5	1	1.5	2	4
Circular Variance V	0.913	0.654	0.427	0.281	0.076
Circular Standard Deviation σ	2.208	1.456	1.055	0.812	0.396

Table 3: Circular Variance and Standard Deviation for WAD for $\lambda = 0.5, 1, 1.5, 2, 4$.

5. Skewness and kurtosis

The skewness coefficient ζ_1 and the kurtosis coefficient ζ_2 for WAD are given respectively by

$$
\begin{aligned}
\zeta_1 &= \bar{\beta}_2 V^{-3/2} \\
&= \frac{\lambda^3}{\lambda^2+2\lambda+2}\sqrt{\frac{(\lambda^2+2\lambda-2)^2+16(1+\lambda)^2}{(\lambda^2+4)^3}}\sin\left[3\arctan\left(\frac{2}{\lambda}\right)\right. \\
&\left.-\arctan\left(\frac{4(1+\lambda)}{\lambda^2+2\lambda-2}\right)-6\arctan\left(\frac{1}{\lambda}\right)+2\arctan\left(\frac{4\lambda}{\lambda^2+2\lambda+1}\right)\right] \\
&\left(1-\frac{\lambda^3}{\lambda^2+2\lambda+2}\sqrt{\frac{(\lambda^2+2\lambda+1)^2+4(1+\lambda^2)}{(\lambda^2+1)^3}}\right)^{-3/2},
\end{aligned}
\tag{16}
$$

and

$$
\begin{aligned}
\zeta_2 &= \left(\bar{\alpha}_2-(1-V)^4\right)V^{-2} \\
&= \left(1-\frac{\lambda^3}{\lambda^2+2\lambda+2}\sqrt{\frac{(\lambda^2+2\lambda+1)^2+4(1+\lambda^2)}{(\lambda^2+1)^3}}\right)^{-2}
\end{aligned}
\tag{17}
$$

$$\left[\frac{\lambda^3}{\lambda^2+2\lambda+2}\sqrt{\frac{(\lambda^2+2\lambda-2)^2+16(1+\lambda)^2}{(\lambda^2+4)^3}}\cos\left[3\arctan\left(\frac{2}{\lambda}\right)\right.\right.$$
$$\left.-\arctan\left(\frac{4(1+\lambda)}{\lambda^2+2\lambda-2}\right)-6\arctan\left(\frac{1}{\lambda}\right)+2\arctan\left(\frac{4\lambda}{\lambda^2+2\lambda+1}\right)\right]$$
$$\left.\left(\frac{\lambda^3}{\lambda^2+2\lambda+2}\sqrt{\frac{(\lambda^2+2\lambda+1)^2+4(1+\lambda^2)}{(\lambda^2+1)^3}}\right)^4\right].$$

Table 4 shows the values of skewness and kurtosis coefficients for $\lambda = 0.5, 1, 1.5, 2, 4$.

Coefficients	**Parameter λ**				
	0.5	1	1.5	2	4
Skewness ζ_1	0.014	-0.186	-0.514	-0.893	-2.031
Kurtosis ζ_2	0.019	0.038	0.435	1.133	4.394

Table 4: Skewness and Kurtosis Coefficients for WAD for $\lambda = 0.5, 1, 1.5, 2, 4$.

Conclusions

We introduced a new circular distribution generated from wrapping Aradhana distribution around the unit circle. The explicit expressions for the probability density function and the cumulative density function for WAD were obtained. Moreover, the trigonometric moments, measures of variation, and some special characteristics for the new distribution were studied.

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Accepted: January 10, 2021