Synchronization of chaotic systems via periodically piecewise feedback control

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Abstract. In this paper, the synchronization of chaotic systems via periodically piecewise feedback control is investigated. An exponential synchronization criterion is given by means of Lyapunov function and inequalities. The effectiveness of theoretical result is confirmed via an example based on Chua's circuit.

Keywords: chaos, synchronization, periodically piecewise feedback control.

1. Introduction

The synchronization of chaotic systems was presented by Fujisaka and Yamada [1]. Since then, chaos synchronization has many applications in biological chemical reactions, secure communications, artificial neural networks and so on. Unfortunately, chaotic systems are difficult to synchronize due to sensitive dependence on initial conditions. Nevertheless, many fundamental and meaningful results have been obtained by scholars from mathematics, physics, engineering, etc. There are many methods to synchronize of chaotic systems. For example, fuzzy control [2], switch control [3], impulsive control [4-6], sliding mode control [7], adaptive control [8], intermittent control [9], feedback control [10, 11]. For more information on stabilization and synchronization of chaotic systems have been investigated in the literature (see, [12-19]).

This paper is organized as follows. In 2, some preliminaries are introduced. The problem synchronization of chaotic systems via periodically piecewise feedback control is described. In Section 3, the criterion of synchronization is strictly derived. In Section 4, an example is given to illustrate the effectiveness of the theoretical result.

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2. Problem formulation and preliminaries

Notations. Let R be the set of real numbers, R^n the $n-$ dimensional real Euclidean space with norm $\|\cdot\|$, $R^{m\times n}$ the set of all $m\times n-$ dimensional real matrices. The notation $\lambda_{\max}(A)$, $\lambda_{\min}(A)$ and A^T denote the maximum, the minimum eigenvalue and the transpose of real symmetric matrix A, respectively. $A>0$ means that A is a positive definite matrix. I stands for the identity matrix with appropriate dimensions. Set $f\left(x\left(t_0^-\right)\right)=\lim\limits_{t\to t_0^-} f\left(x\left(t\right)\right)$.

The chaotic master (driver) system considered in [20] can be shown by:

$$\begin{cases} \dot{x}(t)=Ax(t)+Qf(x(t)), & t>0,\\ x(t_0)=x_0. \end{cases} \tag{2.1}$$

In order to synchronize system (2.1) via periodically piecewise feedback control, the slave (response) system is described as

$$\begin{cases} \dot{y}(t)=Ay(t)+Qf(y(t))+g(t)(x(t)-y(t)), & t>0,\\ y(t_0)=y_0, \end{cases} \tag{2.2}$$

where $x(t),y(t)\in R^n$ denote the state vectors of systems (2.1) and (2.2), respectively, $A,Q\in R^{n\times n}$ are known matrices, $f:R^n\to R^n$ is said to be a continuous nonlinear function satisfying $f(0)=0$ and $||f(x)-f(y)||\le l||x-y||$, $l\ge 0$ is a constant, $g(t)$ is the periodically piecewise feedback gain defined as follows:

$$g(t)=\begin{cases} G_1, & kT+t_0\le t<kT+\tau+t_0,\\ G_2, & kT+\tau+t_0\le t<(k+1)T+t_0, \end{cases}$$

where $G_1,G_2\in R^{n\times n}$ are two constant control gains, $T>0$ represents control period, $\tau>0$ is the control width. Our target is to design suitable T,τ, G_1 and G_2 such that system (2.2) synchronizes system (2.1).

Define $e(t)=y(t)-x(t)$. $e(t)$ is the periodically synchronization error between the states of drive system (2.1) and response system (2.2). Then, the error system can be written as follows:

$$\dot{e}(t)=\begin{cases} Ae(t)+Q(f(y(t))-f(x(t)))-G_1e(t),\\ \qquad\qquad kT+t_0\le t<kT+\tau+t_0,\\ Ae(t)+Q(f(y(t))-f(x(t)))-G_2e(t),\\ \qquad\qquad kT+\tau+t_0\le t<(k+1)T+t_0. \end{cases} \tag{2.3}$$

Obviously (2.3) is a classical switched system where the switching rule depends on time.

Remark 2.1. When $G_2e(t)=0$, the error system (2.3) becomes the classical intermittent error system [20].

In this paper, we show a new sufficient exponential synchronization of system (2.1) and system (2.2) criterion condition. The computational cost of solving conditions of our result is lower than ones shown in [20].

We need a lemma which plays a major role in the proof of theorem.

Lemma 2.1 ([21]). *Suppose that $A \in R^{n\times n}$ is a symmetric matrix. Then, for all $x \in R^n$, $\lambda_{\min}(A)x^Tx \le x^TAx \le \lambda_{\max}(A)x^Tx$.*

3. Main result

In what follows, we will obtain the main result.

Theorem 3.1. *Let $0 < P \in R^{n\times n}$ such that the following two conditions are satisfied:*

(1) $r_1 < 0$,

(2) $r_1\tau + r_2(T-\tau) < 0$,

where $\beta_1 = \lambda_{\max}\left(P^{-1}\left(PA + A^TP - PG_1 - G_1^TP\right)\right)$, $\beta_2 = \lambda_{\max}\left(Q^TPQ\right)$, $\beta_3 = \lambda_{\min}(P)$, $\beta_4 = \lambda_{\max}\left(P^{-1}\left(PA + A^TP - PG_2 - G_2^TP\right)\right)$, $r_1 = \beta_1 + 2l\sqrt{\frac{\beta_2}{\beta_3}}$, $r_2 = \beta_4 + 2l\sqrt{\frac{\beta_2}{\beta_3}}$. *Then, master system* (2.1) *and slave system* (2.2) *are exponential synchronization via periodically piecewise feedback control.*

Proof. Define $V(e(t)) = e^T(t)Pe(t)$. For $t \in [kT + t_0, kT + \tau + t_0)$, by Cauchy-Schwarz inequality and Lemma 2.1, we obtain

$$\begin{aligned}
D^+\left(V\left(e\left(t\right)\right)\right) &= 2e^T(t)P[Ae(t) + Q(f(y(t)) - f(x(t))) - G_1e(t)] \\
&= 2e^T(t)PQ(f(y(t)) - f(x(t))) \\
&\quad + e^T\left(t\right)\left(PA + A^TP - PG_1 - G_1^TP\right)e\left(t\right) \\
&\le 2\sqrt{e^T(t)Pe(t)(f(y(t)) - f(x(t)))^TQ^TPQ(f(y(t)) - f(x(t)))} \\
&\quad + \beta_1e^T\left(t\right)Pe\left(t\right) \\
&\le 2\sqrt{e^T(t)Pe(t)\beta_2(f(y(t)) - f(x(t)))^T(f(y(t)) - f(x(t)))} \\
&\quad + \beta_1e^T\left(t\right)Pe\left(t\right) \\
&\le \beta_1e^T\left(t\right)Pe\left(t\right) + 2l\sqrt{e^T(t)Pe(t)\beta_2e^T(t)e(t)} \\
&\le \beta_1e^T\left(t\right)Pe\left(t\right) + 2l\sqrt{e^T\left(t\right)Pe\left(t\right)\frac{\beta_2}{\beta_3}e^T\left(t\right)Pe\left(t\right)} \\
&= r_1V\left(e\left(t\right)\right),
\end{aligned}$$

which deduces

$$V\left(e\left(t\right)\right) \le V\left(e\left(\left(kT + t_0\right)^-\right)\right)e^{r_1(t-kT-t_0)}. \tag{3.1}$$

In the same way, for $t \in [kT+\tau+t_0, (k+1)T+t_0)$, we also obtain

$$\begin{aligned} D^+\left(V\left(e\left(t\right)\right)\right) &= 2e^T(t)P[Ae(t)+Q(f(y(t))-f(x(t)))-G_2e(t)] \\ &= e^T\left(t\right)\left(PA+A^TP-PG_2-G_2^TP\right)e\left(t\right) \\ &+ 2e^T(t)PQ(f(y(t))-f(x(t))) \\ &\leq \beta_4 e^T\left(t\right)Pe\left(t\right)+2l\sqrt{e^T\left(t\right)Pe\left(t\right)\frac{\beta_2}{\beta_3}e^T\left(t\right)Pe\left(t\right)} \\ &= r_2V\left(e\left(t\right)\right), \end{aligned}$$

which means

$$V\left(e\left(t\right)\right) \leq V\left(e\left(\left(kT+\tau+t_0\right)^-\right)\right)e^{r_2(t-kT-\tau-t_0)}. \tag{3.2}$$

When $k=0$, for $t \in [t_0, \tau+t_0)$, from (3.1), we have

$$V\left(e\left(t\right)\right) \leq V\left(e\left(t_0\right)\right)e^{r_1(t-t_0)},$$

hence

$$V\left(e\left(\left(\tau+t_0\right)^-\right)\right) \leq V\left(e\left(t_0\right)\right)e^{r_1\tau}. \tag{3.3}$$

For $t \in [\tau+t_0, T+t_0)$, by (3.2) and (3.3), we have

$$V\left(e\left(t\right)\right) \leq V\left(e\left(\left(\tau+t_0\right)^-\right)\right)e^{r_2(t-\tau-t_0)} \leq V\left(e\left(t_0\right)\right)e^{r_1\tau+r_2(t-\tau-t_0)},$$

so

$$V\left(e\left(\left(T+t_0\right)^-\right)\right) \leq V\left(e\left(t_0\right)\right)e^{r_1\tau+r_2(T-\tau)}. \tag{3.4}$$

When $k=1$, for $t \in [T+t_0, T+\tau+t_0)$, by (3.1) and (3.4), we have

$$\begin{aligned} V\left(e\left(t\right)\right) &\leq V\left(e\left(\left(T+t_0\right)^-\right)\right)e^{r_1(t-T-t_0)} \\ &\leq V\left(e\left(t_0\right)\right)e^{r_1\tau+r_2(T-\tau)+r_1(t-T-t_0)}, \end{aligned}$$

so

$$V\left(e\left(\left(T+\tau+t_0\right)^-\right)\right) \leq V\left(e\left(t_0\right)\right)e^{2r_1\tau+r_2(T-\tau)}. \tag{3.5}$$

For $t \in [T+\tau+t_0, 2T+t_0)$, by (3.2) and (3.5), we have

$$\begin{aligned} V\left(e\left(t\right)\right) &\leq V\left(e\left(\left(T+\tau+t_0\right)^-\right)\right)e^{r_2(t-T-\tau-t_0)} \\ &\leq V\left(e\left(t_0\right)\right)e^{2r_1\tau+r_2(T-\tau)+r_2(t-T-\tau-t_0)}. \end{aligned}$$

By induction, when k=m, m=$0, 1, \ldots$, for $t \in [mT+t_0, mT+\tau+t_0)$, we have

$$V\left(e\left(t\right)\right) \leq V\left(e\left(t_0\right)\right)e^{mr_1\tau+mr_2(T-\tau)+r_1(t-mT-t_0)}, \tag{3.6}$$

so

$$V\left(e\left(\left(mT+\tau+t_0\right)^-\right)\right) \leq V\left(e\left(t_0\right)\right)e^{(m+1)r_1\tau+mr_2(T-\tau)}. \tag{3.7}$$

For $t \in [mT+\tau+t_0, (m+1)T+t_0)$, by (3.2) and (3.7), we have

$$V(e(t)) \leq V\left(e\left((mT+\tau+t_0)^-\right)\right)e^{r_2(t-mT-\tau-t_0)} \leq V(e(t_0))\,e^{(m+1)r_1\tau+mr_2(T-\tau)+r_2(t-mT-\tau-t_0)}. \quad (3.8)$$

By (3.6), we have

$$\begin{aligned} V(e(t)) &\leq V(e(t_0))\,e^{mr_1\tau+mr_2(T-\tau)} \\ &< V(e(t_0))\,e^{\frac{t-\tau-t_0}{T}(r_1\tau+r_2(T-\tau))} \\ &< V(e(t_0))\,e^{\frac{t-T-t_0}{T}(r_1\tau+r_2(T-\tau))}, \end{aligned} \quad (3.9)$$

where $t \in [mT+t_0, mT+\tau+t_0)$. By (3.8), we know
Case 1. When $r_2 > 0$, we have

$$\begin{aligned} V(e(t)) &< V(e(t_0))\,e^{(m+1)r_1\tau+(m+1)r_2(T-\tau)} \\ &< V(e(t_0))\,e^{\frac{t-t_0}{T}(r_1\tau+r_2(T-\tau))} \\ &< V(e(t_0))\,e^{\frac{t-T-t_0}{T}(r_1\tau+r_2(T-\tau))}. \end{aligned} \quad (3.10)$$

Case 2. When $r_2 \leq 0$, we have

$$\begin{aligned} V(e(t)) &\leq V(e(t_0))\,e^{(m+1)r_1\tau+mr_2(T-\tau)} \\ &< V(e(t_0))\,e^{mr_1\tau+mr_2(T-\tau)} \\ &< V(e(t_0))\,e^{\frac{t-T-t_0}{T}(r_1\tau+r_2(T-\tau))}. \end{aligned} \quad (3.11)$$

So, for any $r_2 \in R$, by (3.10) and (3.11), we have

$$V(e(t)) < V(e(t_0))\,e^{\frac{t-T-t_0}{T}(r_1\tau+r_2(T-\tau))}, \quad (3.12)$$

where $t \in [mT+\tau+t_0, (m+1)T+t_0)$. For all $t > 0$, we can conclude from (3.9) and (3.12) that

$$V(e(t)) < V(e(t_0))\,e^{\frac{t-T-t_0}{T}(r_1\tau+r_2(T-\tau))}. \quad (3.13)$$

Let $\eta = V(e(t_0))e^{-[r_1\tau+r_2(T-\tau)]}e^{\frac{-[r_1\tau+r_2(T-\tau)]t_0}{T}}$. By Lemma 2.1 and (3.13), we get

$$\lambda_{\min}(P)\,\|e(t)\|^2 < \eta e^{\frac{[r_1\tau+r_2(T-\tau)]t}{T}}.$$

Namely,

$$\|e(t)\| < \sqrt{\frac{\eta}{\lambda_{\min}(P)}}\,e^{\frac{[r_1\tau+r_2(T-\tau)]t}{2T}},$$

which concludes that system (2.1) and system (2.2) synchronize exponentially. This completes the proof. □

Remark 3.1. Theorem 1 of [20] needs to solve the linear matrix inequalities, however, Theorem 3.1 does not need to solve them.

4. A numerical example

In this section, we study the synchronization of Chua's oscillator via employing above theoretical result with the initial condition $x(0) = (0.2, -0.1, 0.2)^T$ for the master system and $y(0) = (-0.2, 0.1, -0.2)^T$ for the slave system.

Example 4.1. The Chua's system [22] is given as follows:

$$\begin{cases} \dot{x}_1 = \alpha(x_2 - x_1 - g(x_1)), \\ \dot{x}_2 = x_1 - x_2 + x_3, \\ \dot{x}_3 = -\eta x_2, \end{cases} \tag{4.1}$$

where α and η are two parameters $g(x_1) = bx_1 + 0.5(a-b)(|x_1+1| - |x_1-1|)$, where a and b are two given constants satisfying $a < b < 0$.

In order to use the above result, we rewrite system (4.1) as follows $\dot{x}(t) = Ax + f(x)$, where

$$A = \begin{bmatrix} -\alpha - \alpha b & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\eta & 0 \end{bmatrix},$$

$$f(x) = \begin{bmatrix} -0.5\alpha(a-b)(|x_1+1| - |x_1-1|) \\ 0 \\ 0 \end{bmatrix}.$$

By simple calculation, we can choose $l^2 = \alpha^2(a-b)^2$.

The Chua's system exhibits chaotic phenomenon when $\alpha = 9.2156,\ \eta = 15.9946,\ a = -1.24905,\ b = -0.75735$ (see, [20]).

The two Chua systems are not synchronization without periodically piecewise feedback control as shown in Figure 1.

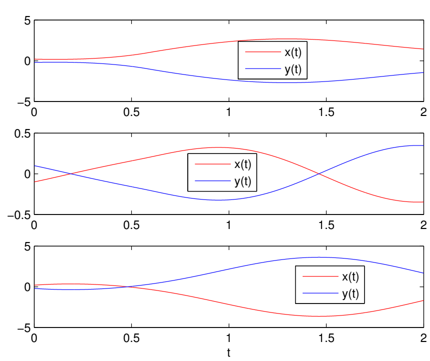

Figure 1: The two Chua systems are not synchronization without periodically piecewise feedback control .

At the same time, for convenience of calculations, we choose $Q = P = I$, $G_1 = diag(18, 18, 18)$, $G_2 = diag(8, 8, 8)$, $T = 1$ and $\tau = 0.5$. A small calculations show that $\beta_1 = -19.4502$, $\beta_2 = \beta_3 = 1$, $\beta_4 = 0.5498$, $l = 4.5313$, $r_1 = -10.3876$, $r_2 = 9.6124$ and $r_1\tau + r_2(T - \tau) = -0.3876 < 0$. Thus, system (2.1) and system (2.2) are exponential synchronization by Theorem 3.1, as shown in Figure 2.

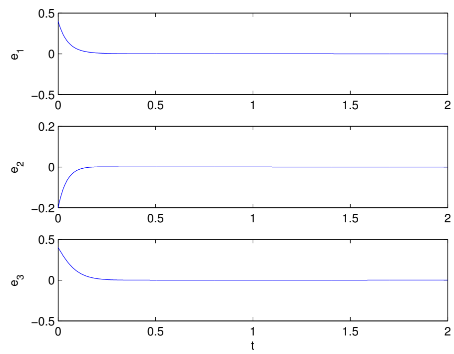

Figure 2: The two Chua systems synchronize via periodically piecewise feedback control.

5. Conclusions

This paper first introduces a generalized model of chaotic systems. We obtain a new exponential synchronization criterion of chaotic systems via periodically piecewise feedback control. Finally, an example based on Chua's circuit is provided to show the validity and superiority of the result.

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