Ordered almost ideals and fuzzy ordered almost ideals in ordered semigroups

Sudaporn Suebsung
Winita Yonthanthum
Ronnason Chinram*
Algebra and Applications Research Unit
Division of Computational Science
Faculty of Science
Prince of Songkla University
Hat Yai, Songkhla 90110
Thailand
Sudapornhmoo@gmail.com
winita.m@psu.ac.th
ronnason.c@psu.ac.th

Abstract. In this paper, we define ordered almost ideals and fuzzy ordered almost ideals in ordered semigroups and we study some of their properties. We give relationship between fuzzy ordered almost ideals and ordered almost ideals of ordered semigroups.
Keywords: ordered semigroups, ordered almost ideals, fuzzy ordered almost ideals.

1. Introduction

The notion of almost ideals (or A-ideals) of semigroups was introduced by Grosek and Satko [1] in 1980. Afterwards, they studied minimal almost ideals and smallest almost ideals of semigroups in [6] and [2], respectively.

In 1965, Zadeh [10] introduced the concept of fuzzy subsets. Rosenfeld [4] applied the concept of Zadeh to define fuzzy subgroups and fuzzy ideals in groups. In [3], Kuroki studied various kinds of fuzzy ideals in semigroups and characterized them. In 2018, Wattanatripop, Chinram and Changphas [9] introduced the notion of fuzzy almost ideals in semigroups by using the concept of almost ideals and fuzzy ideals of semigroups. In [7], Suebsung, Wattanatripop and Chinram defined and studied some properties of almost ideals and fuzzy almost ideals of ternary semigroups. Moreover, they introduced the notion of minimal fuzzy almost ideals of ternary semigroups and studied properties of them. Recently, Solano, Suebsung and Chinram [8] extended the ideal of almost ideals to n-ary semigroups.

In this paper, we define ordered almost ideals and fuzzy ordered almost ideals in ordered semigroups and we study some of their properties. We give

∗. Corresponding author

relationship between fuzzy ordered almost ideals and ordered almost ideals of ordered semigroups.

2. Preliminaries

In this section, we give some basic definitions and results, which are necessary for the subsequent sections.

Definition 2.1. Let S be a set with a binary operation $\cdot$ and a binary relation $\leq$. Then $(S, \cdot, \leq)$ is called an *ordered semigroup* if:

1. $(S, \cdot)$ is a semigroup;
2. $(S, \leq)$ is a partially ordered set;
3. for all $x, y, z \in S$, if $x \leq y$, then $xz \leq yz$ and $zx \leq zy$.

Let S be an ordered semigroup. For a nonempty subset A of S, we denote

$$(A] := \{x \in S \mid x \leq a \text{ for some a } \in A\}.$$

Proposition 2.2. *Let A and B be any two nonempty subsets of an ordered semigroup S.*

(1) $A \subseteq (A]$.

(2) *If $A \subseteq B$, then $(A] \subseteq (B]$.*

For nonempty subsets A and B of S, denote

$$AB := \{ab \mid a \in A \text{ and } b \in B\}.$$

If $s \in S$, we let $As := A\{s\}$, $sA := \{s\}A$.

Definition 2.3. ([1]) Let S be a semigroup.

1. A nonempty subset L of S is called a *left almost ideal* of S if $sL \cap L \neq \emptyset$ for any $s \in S$.
2. A nonempty subset R of S is called a *right almost ideal* of S if $Rs \cap R \neq \emptyset$ for any $s \in S$.
3. A nonempty subset I of S is called an *almost ideal* of S if I is both a left almost ideal and a right almost ideal of S.

Definition 2.4. Let S be an ordered semigroup. A nonempty subset A is called a *left (resp. right) ideal* of S if:

1. $SA \subseteq A$ (resp. $AS \subseteq A$);
2. if $a \in A$ and $s \in S$ such that $s \leq a$, then $s \in A$, that is $(A] \subseteq A$.

If A is both a left ideal and a right ideal of S, then it is called an *ideal* of S.

Definition 2.5. Let S be an ordered semigroup. The element $a \in S$ is called an *idempotent* if $a \leq a^2$.

A function f from a set S to the unit interval $[0, 1]$ is called a *fuzzy subset* of S. For any two fuzzy subsets f and g of a nonempty set S, define the *union* and *intersection* of f and g, denoted by $f \cup g$ and $f \cap g$, by for all $x \in S$,

$$(f \cup g)(x) = \max\{f(x), g(x)\},$$
$$(f \cap g)(x) = \min\{f(x), g(x)\}.$$

Let f and g be two fuzzy subsets of an ordered semigroup S. Define $f \subseteq g$ by

$$f \subseteq g \Longleftrightarrow f(x) \leq g(x), \quad \text{for all} \quad x \in S.$$

For a fuzzy subset f of S, the *support* of f is defined by

$$supp(f) = \{x \in S \mid f(x) \neq 0\}.$$

The *characteristic mapping* of a subset A of S is a fuzzy subset of S defined by

$$C_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

The definition of fuzzy points was given by Pu and Liu [5]. Let $s \in S$ and $\alpha \in (0, 1]$. A *fuzzy point* s_α of a set S is a fuzzy subset of S defined by

$$s_\alpha(x) = \begin{cases} \alpha, & x = s, \\ 0, & x \neq s. \end{cases}$$

Let $F(S)$ be the set of all fuzzy subsets of an ordered semigroup S. For any $f, g \in F(S)$, we define the *product* of f and g by

$$(f \circ g)(x) := \begin{cases} \sup_{x \leq uv} \min\{f(u), g(v)\}, & \text{if } x \leq uv \text{ where } u, v \in S, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 2.6. *Let f, g and h be fuzzy subsets of an ordered semigroup S.*

(1) *If $f \subseteq g$, then $f \circ h \subseteq g \circ h$ and $h \circ f \subseteq h \circ g$.*

(2) *If $f \subseteq g$, then $f \cap h \subseteq g \cap h$.*

(3) *If $f \subseteq g$, then $f \cup h \subseteq g \cup h$.*

(4) *If $f \subseteq g$, then $supp(f) \subseteq supp(g)$.*

Definition 2.7. Let S be an ordered semigroup. A fuzzy subset f of S is called a *fuzzy left (resp. right) ideal* of S if for all $x, y \in S$,

1. if $x \leq y$, then $f(x) \geq f(y)$;
2. $f(xy) \geq f(y)$ (resp. $f(xy) \geq f(x)$).

A fuzzy subset f of S is called a *fuzzy ideal* of S if it is both a fuzzy left ideal and a fuzzy right ideal of S, that is, for all $x, y \in S$,

1. if $x \leq y$, then $f(x) \geq f(y)$;
2. $f(xy) \geq \max\{f(x), f(y)\}$.

3. Ordered almost ideals

In this section, we introduce the definition of ordered almost ideals in ordered semigroups and we study some of their properties.

Definition 3.1. Let S be an ordered semigroup.

1. A nonempty subset L of S is called a *left ordered almost ideal* of S if $(sL] \cap L \neq \emptyset$, for all $s \in S$.
2. A nonempty subset R of S is called a *right ordered almost ideal* of S if $(Rs] \cap R \neq \emptyset$, for all $s \in S$.
3. A nonempty subset I of S is called an *ordered almost ideal* of S if it is both a left ordered almost ideal and a right ordered almost ideal of S.

Remark 3.2. Let S be an ordered semigroup.

(i) Every left ideal of S is a left ordered almost ideal of S.

(ii) Every right ideal of S is a right ordered almost ideal of S.

(iii) Every ideal of S is an ordered almost ideal of S.

Proof. We prove that (i) holds. Let L be a left ideal of S and $s \in S$. Then, $sL \neq \emptyset$ and $sL \subseteq L$. So, $\emptyset \neq (sL] \subseteq (L] \subseteq L$. Thus $(sL] \cap L \neq \emptyset$. Hence, L is a left ordered almost ideal of S. (ii) and (iii) can be seen in the same manner. □

The converse of Remark 3.2(i), does not hold true in general, as Example 3.3 shows.

Example 3.3. Consider the ordered semigroup $\mathbb{Z}_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ under the addition on $\mathbb{Z}_6$ and the order $\leq := \{(\bar{0}, \bar{0}), (\bar{1}, \bar{1}), (\bar{2}, \bar{2}), (\bar{3}, \bar{3}), (\bar{4}, \bar{4}), (\bar{5}, \bar{5})\}$. Let $L = \{\bar{1}, \bar{3}, \bar{4}\}$. We have
$(\bar{0} + L] \cap L = L \cap L = L$,
$(\bar{1} + L] \cap L = \{\bar{2}, \bar{4}, \bar{5}\} \cap L = \{\bar{4}\}$,
$(\bar{2} + L] \cap L = \{\bar{0}, \bar{3}, \bar{5}\} \cap L = \{\bar{3}\}$,
$(\bar{3} + L] \cap L = \{\bar{0}, \bar{1}, \bar{4}\} \cap L = \{\bar{1}, \bar{4}\}$,
$(\bar{4} + L] \cap L = \{\bar{1}, \bar{2}, \bar{5}\} \cap L = \{\bar{1}\}$,

$(\bar{5}+L]\cap L=\{\bar{0},\bar{2},\bar{3}\}\cap L=\{\bar{3}\}$.
Then $(s+L]\cap L\neq\emptyset$ for all $s\in\mathbb{Z}_6$. Thus L is a left ordered almost ideal of $\mathbb{Z}_6$. However, L is not a left ideal of $\mathbb{Z}_6$ because $\bar{0}=\bar{6}=\bar{5}+\bar{1}\in\bar{5}+L$ but $\bar{0}\notin L$.

Theorem 3.4. *Let S be an ordered semigroup. Then the following conditions hold.*

(1) *If L is a left ordered almost ideal of S, then every subset L' of S such that $L\subseteq L'$ is a left ordered almost ideal of S.*

(2) *If R is a right ordered almost ideal of S, then every subset R' of S such that $R\subseteq R'$ is a right ordered almost ideal of S.*

(3) *If I is an ordered almost ideal of S, then every subset I' of S such that $I\subseteq I'$ is an ordered almost ideal of S.*

Proof. Assume that L is a left ordered almost ideal of S. Let L' be a subset of S such that $L\subseteq L'$ and $s\in S$. Then $sL\subseteq sL'$, so $(sL]\subseteq(sL']$. This implies that $\emptyset\neq(sL]\cap L\subseteq(sL']\cap L'$. Thus L' is a left ordered almost ideal of S. This proves (1). In the case R is a right ordered almost ideal of S, the proof follows similarly. By conditions (1) and (2), it follows that the condition (3) holds. □

Corollary 3.5. *Let S be an ordered semigroup.*

(1) *If L_1 and L_2 are left ordered almost ideals of S, then $L_1\cup L_2$ is a left ordered almost ideal of S.*

(2) *If R_1 and R_2 are right ordered almost ideals of S, then $R_1\cup R_2$ is a right ordered almost ideal of S.*

(3) *If I_1 and I_2 are ordered almost ideals of S, then $I_1\cup I_2$ is an ordered almost ideal of S.*

Proof. Each of (1), (2), and (3) follows from the corresponding item in Theorem 3.4. □

For in case the intersection of two left ordered almost ideals is not true as in the following example:

Example 3.6. Consider the ordered semigroup $\mathbb{Z}_6=\{\bar{0},\bar{1},\bar{2},\bar{3},\bar{4},\bar{5}\}$ under the addition on $\mathbb{Z}_6$ and the order $\leq:=\{(\bar{0},\bar{0}),(\bar{1},\bar{1}),(\bar{2},\bar{2}),(\bar{3},\bar{3}),(\bar{4},\bar{4}),(\bar{5},\bar{5})\}$. Let $L_1=\{\bar{1},\bar{3},\bar{4}\}$ and $L_2=\{\bar{2},\bar{3},\bar{5}\}$. We have
$(\bar{0}+L_1]\cap L_1=L_1$ and $(\bar{0}+L_2]\cap L_2=L_2$,
$(\bar{1}+L_1]\cap L_1=\{\bar{4}\}$ and $(\bar{1}+L_2]\cap L_2=\{\bar{3}\}$,
$(\bar{2}+L_1]\cap L_1=\{\bar{3}\}$ and $(\bar{2}+L_2]\cap L_2=\{\bar{5}\}$,
$(\bar{3}+L_1]\cap L_1=\{\bar{1},\bar{4}\}$ and $(\bar{3}+L_2]\cap L_2=\{\bar{2},\bar{5}\}$,
$(\bar{4}+L_1]\cap L_1=\{\bar{1}\}$ and $(\bar{4}+L_2]\cap L_2=\{\bar{3}\}$,

$$(\bar{5}+L_1]\cap L_1=\{\bar{3}\} \quad \text{and} \quad (\bar{5}+L_2]\cap L_2=\{\bar{2}\}.$$

Thus $(s+L_1]\cap L_1\neq\emptyset$ and $(s+L_2]\cap L_2\neq\emptyset$ for all $s\in\mathbb{Z}_6$. Hence L_1 and L_2 are left ordered almost ideals of $\mathbb{Z}_6$. But $L_1\cap L_2=\{\bar{3}\}$ is not a left ordered almost ideal of $\mathbb{Z}_6$ because $(\bar{3}+\{\bar{3}\}]\cap\{\bar{3}\}=\{\bar{0}\}\cap\{\bar{3}\}=\emptyset$.

Lemma 3.7. *Let S be an ordered semigroup and $|S|>1$. Then the following properties hold.*

(1) *S has no proper left ordered almost ideals if and only if for all $a\in S$, there exists an element $s_a\in S$ such that $\big(s_a(S-\{a\})\big]=\{a\}$.*

(2) *S has no proper right ordered almost ideals if and only if for all $a\in S$, there exists an element $k_a\in S$ such that $\big((S-\{a\})k_a\big]=\{a\}$.*

(3) *S has no proper ordered almost ideals if and only if for all $a\in S$, there exist elements $s_a,k_a\in S$ such that $(s_a(S-\{a\})]=\{a\}$ and $((S-\{a\})k_a]=\{a\}$.*

Proof. We prove that property (1) holds. Assume S has no proper left ordered almost ideals and let $a\in S$. Then $S-\{a\}$ is not a left ordered almost ideal of S. That is there exists $s_a\in S$ such that $\big(s_a(S-\{a\})\big]\cap(S-\{a\})=\emptyset$. Thus $\big(s_a(S-\{a\})\big]=\{a\}$.

Conversely, let L be a proper nonempty subset of S. Then $L\subseteq S-\{a\}$ for some $a\in S$. By assumption, there exists $s_a\in S$ such that $\big(s_a(S-\{a\})\big]=\{a\}$. Since $s_aL\subseteq s_a(S-\{a\})$, it follows that $(s_aL]\subseteq\big(s_a(S-\{a\})\big]$. We have

$$(s_aL]\cap L\subseteq\big(s_a(S-\{a\})\big]\cap(S-\{a\})=\{a\}\cap(S-\{a\})=\emptyset.$$

Thus $(s_aL]\cap L=\emptyset$ which implies that L is not a left ordered almost ideal of S. Therefore S has no proper left ordered almost ideals. Property (2) and (3) can be proved in a similar manner. □

Theorem 3.8. *Let S be an ordered semigroup such that $|S|>1$ and $a\in S$. Then the following statements hold.*

(1) *If S has no proper left ordered almost ideals, then either a or a^2 is an idempotent.*

(2) *If S has no proper right ordered almost ideals, then either a or a^2 is an idempotent.*

(3) *If S has no proper ordered almost ideals, then either a or a^2 is an idempotent.*

Proof. (1) Assume S has no proper left ordered almost ideals. By Lemma 3.7(1), there exists an element $s_a\in S$ such that $\big(s_a(S-\{a\})\big]=\{a\}$.
Case 1. $a=a^2$. Then, $a=a^2\leq a^2$, so a is an idempotent.
Case 2. $a\neq a^2$. That is $a^2\in S-\{a\}$. So, $s_aa^2=a$.
Case 2.1. $s_a\leq a$. Then, $a=s_aa^2\leq a^3$, so $a^2\leq a^4$. Thus, a^2 is an idempotent.

Case 2.2. $s_a \nleq a$. That is $s_a \in S - \{a\}$. Thus, $s_a s_a = a$. If $s_a a \nleq a$, then $s_a a \in S - \{a\}$, so $s_a s_a a = a$. Then, we have $a^2 = aa = (s_a s_a)a = a$, this is a contradiction. Thus, $s_a a \leq a$. Hence, $a = s_a a^2 \leq a^2$. Therefore, a is an idempotent. For statements (2) and (3), we can prove similarly. □

Example 3.9. Consider the ordered semigroup $S = \{-1, 1\}$ under the usual multiplication on $\mathbb{Z}$ and the order $\leq := \{(-1,-1),(1,1)\}$. We have $\{-1\}, \{1\}$ and S are nonempty subsets of S. We see that $(-1 \cdot \{-1\}] \cap \{-1\} = \{1\} \cap \{-1\} = \emptyset$, $(-1 \cdot \{1\}] \cap \{1\} = \{-1\} \cap \{1\} = \emptyset$. Thus, $\{-1\}$ and $\{1\}$ are not left ordered almost ideals of S. Hence S has no proper left ordered almost ideals. We can see that $\big(-1\cdot(S-\{-1\})\big] = \big(-1\cdot\{1\}\big] = \{-1\}$, $\big(-1\cdot(S-\{1\})\big] = \big(-1\cdot\{-1\}\big] = \{1\}$. Thus, for any $a \in S$, there exists $s_a \in S$ such that $\big(s_a(S - \{a\})\big] = \{a\}$.

Example 3.9 shows that there is an ordered semigroup which correspond to Theorem 3.8

4. Fuzzy ordered almost ideals

In this section, we define fuzzy ordered almost ideals of ordered semigroups and we study some of their properties. Moreover, we study the relationship between fuzzy ordered almost ideals and ordered almost ideals of ordered semigroups.

Let S be an ordered semigroup. For a fuzzy subset f of S, we defined $(f] : S \longrightarrow [0,1]$ by

$$(f](x) = \sup_{x \leq y} f(y), \text{ for all } x \in S.$$

Proposition 4.1. *Let f, g and h be fuzzy subsets of an ordered semigroup S.*

(1) $f \subseteq (f]$.

(2) *If $f \subseteq g$, then $(f] \subseteq (g]$.*

(3) *If $f \subseteq g$, then $(f \circ h] \subseteq (g \circ h]$ and $(h \circ f] \subseteq (h \circ g]$.*

Proof. (1) Let $x \in S$. We have $x \leq x$. Then, $(f](x) = \sup_{x \leq y} f(y) \geq f(x)$. Thus $f \subseteq (f]$.

(2) Assume that $f \subseteq g$. Then $f(x) \leq g(x)$ for all $x \in S$. Let $x \in S$. Thus $(f](x) = \sup_{x\leq y} f(y) \leq \sup_{x \leq y} g(y) = (g](x)$. Hence $(f] \subseteq (g]$.

(3) Assume that $f \subseteq g$. By Proposition 2.6(1), $f \circ h \subseteq g \circ h$ and $h \circ f \subseteq h \circ g$. It follows from (1) that, $(f \circ h] \subseteq (g \circ h]$ and $(h \circ f] \subseteq (h \circ g]$. □

Proposition 4.2. *Let f be a fuzzy subset of an ordered semigroup S. The following statements are equivalent.*

(1) *If $x \leq y$, then $f(x) \geq f(y)$.*

(2) $(f] = f$.

Proof. Let $x \in S$. By assumption, $f(x) \geq f(y)$ for all $y \in S$ where $x \leq y$. Then $(f](x) = \sup_{x \leq y} f(y) = f(x)$. Hence, $(f] = f$. Conversely, assume that $x, y \in S$ and $x \leq y$. Then, we have $f(x) = (f](x) = \sup_{x \leq y} f(y) \geq f(y)$. Thus $f(x) \geq f(y)$. □

Definition 4.3. Let S be an ordered semigroup.

1. A fuzzy subset f of S is called a *fuzzy left ordered almost ideal* of S if $(s_\alpha \circ f] \cap f \neq 0$ for all $s \in S$ and $\alpha \in (0, 1]$.

2. A fuzzy subset f of S is called a *fuzzy right ordered almost ideal* of S if $(f \circ s_\alpha] \cap f \neq 0$ for all $s \in S$ and $\alpha \in (0, 1]$.

3. A fuzzy subset f of S is called a *fuzzy ordered almost ideal* of S if f is both a fuzzy left ordered almost ideal and a fuzzy right ordered almost ideal of S.

Remark 4.4. Let f be a nonzero fuzzy subset of an ordered semigroup S.

(i) Every fuzzy left ideal of S is a fuzzy left ordered almost ideal of S.

(ii) Every fuzzy right ideal of S is a fuzzy right ordered almost ideal of S.

(iii) Every fuzzy ideal of S is a fuzzy ordered almost ideal of S.

Proof. Assume that f is a fuzzy left ideal of S. Let $s \in S$ and $\alpha \in (0, 1]$. Since f is a nonzero function, there exists an element $a \in S$ such that $f(a) \neq 0$. Let $x = sa$. Since f is a fuzzy left ideal, it follows that $f(x) = f(sa) \geq f(a) \neq 0$, so $f(x) \neq 0$. Then we see that

$$\begin{aligned}
(s_\alpha \circ f](x) &\geq \sup_{x \leq u} (s_\alpha \circ f)(u) \\
&\geq (s_\alpha \circ f)(x) \\
&= \sup_{x \leq uv} \min\{s_\alpha(u), f(v)\} \\
&\geq \min\{s_\alpha(s), f(a)\} \\
&= \min\{\alpha, f(a)\} \neq 0.
\end{aligned}$$

Thus, $\big((s_\alpha \circ f] \cap f\big)(x) = \min\{(s_\alpha \circ f](x), f(x)\} \neq 0$. Hence, f is a fuzzy left ordered almost ideal of S. This prove (i). (ii) and (iii) can be seen in the same manner. □

Example 4.5. Consider the ordered semigroup $\mathbb{Z}_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ under the addition on $\mathbb{Z}_6$ and the order $\leq := \{(a, a) \mid a \in \mathbb{Z}_6\}$. Define a function $f : \mathbb{Z}_6 \longrightarrow [0, 1]$ by

$$f(\bar{0}) = 0, f(\bar{1}) = 0.3, f(\bar{2}) = 0, f(\bar{3}) = 0, f(\bar{4}) = 0.2, f(\bar{5}) = 0.1.$$

Then, for each $\alpha \in (0,1]$, we have

$$\begin{aligned} \big((\bar{0}_\alpha \circ f] \cap f\big)(\bar{1}) &\neq 0,\\ \big((\bar{1}_\alpha \circ f] \cap f\big)(\bar{5}) &\neq 0,\\ \big((\bar{2}_\alpha \circ f] \cap f\big)(\bar{1}) &\neq 0,\\ \big((\bar{3}_\alpha \circ f] \cap f\big)(\bar{4}) &\neq 0,\\ \big((\bar{4}_\alpha \circ f] \cap f\big)(\bar{5}) &\neq 0,\\ \big((\bar{5}_\alpha \circ f] \cap f\big)(\bar{4}) &\neq 0. \end{aligned}$$

Thus, for each $s \in \mathbb{Z}_6$ and $\alpha \in (0,1]$, there exists an element $x \in \mathbb{Z}_6$ such that $\big((s_\alpha \circ f] \cap f\big)(x) \neq 0$. Hence f is a fuzzy left ordered almost ideal of $\mathbb{Z}_6$ but not a fuzzy left ideal of $\mathbb{Z}_6$ because $f(\bar{2}+\bar{1}) = f(\bar{3}) = 0 \ngeq 0.3 = f(\bar{1})$.

From the example above, shows that the converse of the Remark 4.4(i) does not holds true.

Theorem 4.6. *Let S be an ordered semigroup.*

(1) *If f is a fuzzy left ordered almost ideal of S, then every fuzzy subset g of S such that $f \subseteq g$ is a fuzzy left ordered almost ideal of S.*

(2) *If f is a fuzzy right ordered almost ideal of S, then every fuzzy subset g of S such that $f \subseteq g$ is a fuzzy right ordered almost ideal of S.*

(3) *If f is a fuzzy ordered almost ideal of S, then every fuzzy subset g of S such that $f \subseteq g$ is a fuzzy ordered almost ideal of S.*

Proof. Assume f is a fuzzy left ordered almost ideal of S. Let g be a fuzzy subset of S such that $f \subseteq g$ and let $s \in S$ and $\alpha \in (0,1]$. Then we have $s_\alpha \circ f \subseteq s_\alpha \circ g$ and so $(s_\alpha \circ f] \subseteq (s_\alpha \circ g]$. Thus $(s_\alpha \circ f] \cap f \subseteq (s_\alpha \circ g] \cap g$. Since f is a fuzzy left ordered almost ideal, $(s_\alpha \circ f] \cap f \neq \emptyset$. Hence $(s_\alpha \circ g] \cap g \neq \emptyset$. Therefore g is a fuzzy left ordered almost ideal of S. This proves (1), and (2) and (3) follows in the same way. □

Corollary 4.7. *Let S be an ordered semigroup.*

(1) *If f and g are fuzzy left ordered almost ideals of S, then $f \cup g$ is a fuzzy left ordered almost ideal of S.*

(2) *If f and g are fuzzy right ordered almost ideals of S, then $f \cup g$ is a fuzzy right ordered almost ideal of S.*

(3) *If f and g are fuzzy ordered almost ideals of S, then $f \cup g$ is a fuzzy ordered almost ideal of S.*

Proof. The proof follows from Theorem 4.6. □

Example 4.8. Consider the ordered semigroup $\mathbb{Z}_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ under the addition on $\mathbb{Z}_6$ and the order $\leq := \{(a, a) \mid a \in \mathbb{Z}_6\}$. Define functions $f : \mathbb{Z}_6 \longrightarrow [0, 1]$ by

$$f(\bar{0}) = 0, f(\bar{1}) = 0.3, f(\bar{2}) = 0, f(\bar{3}) = 0, f(\bar{4}) = 0.2, f(\bar{5}) = 0.1,$$

and $g : \mathbb{Z}_6 \longrightarrow [0, 1]$ by

$$g(\bar{0}) = 0, g(\bar{1}) = 0.3, g(\bar{2}) = 0.1, g(\bar{3}) = 0, g(\bar{4}) = 0, g(\bar{5}) = 0.3.$$

Then, for each $\alpha \in (0, 1]$, we have

$$\begin{array}{lll}
\big((\bar{0}_\alpha \circ f] \cap f\big)(\bar{1}) \neq 0 & \text{and} & \big((\bar{0}_\alpha \circ g] \cap g\big)(\bar{1}) \neq 0, \\
\big((\bar{1}_\alpha \circ f] \cap f\big)(\bar{5}) \neq 0 & \text{and} & \big((\bar{1}_\alpha \circ g] \cap g\big)(\bar{2}) \neq 0, \\
\big((\bar{2}_\alpha \circ f] \cap f\big)(\bar{1}) \neq 0 & \text{and} & \big((\bar{2}_\alpha \circ g] \cap g\big)(\bar{1}) \neq 0, \\
\big((\bar{3}_\alpha \circ f] \cap f\big)(\bar{4}) \neq 0 & \text{and} & \big((\bar{3}_\alpha \circ g] \cap g\big)(\bar{5}) \neq 0, \\
\big((\bar{4}_\alpha \circ f] \cap f\big)(\bar{5}) \neq 0 & \text{and} & \big((\bar{4}_\alpha \circ g] \cap g\big)(\bar{5}) \neq 0, \\
\big((\bar{5}_\alpha \circ f] \cap f\big)(\bar{4}) \neq 0 & \text{and} & \big((\bar{5}_\alpha \circ g] \cap g\big)(\bar{1}) \neq 0.
\end{array}$$

Thus, for each $s \in \mathbb{Z}_6$ and $\alpha \in (0, 1]$, there are elements $x, y \in \mathbb{Z}_6$ such that $\big((s_\alpha \circ f] \cap f\big)(x) \neq 0$ and $\big((s_\alpha \circ g] \cap g\big)(y) \neq 0$. Hence f and g are fuzzy left ordered almost ideals of $\mathbb{Z}_6$.

Next, consider $f \cap g : \mathbb{Z}_6 \longrightarrow [0, 1]$ where

$$\begin{array}{lll}
(f \cap g)(\bar{0}) = 0, & (f \cap g)(\bar{1}) = 0.3, & (f \cap g)(\bar{2}) = 0, \\
(f \cap g)(\bar{3}) = 0, & (f \cap g)(\bar{4}) = 0, & (f \cap g)(\bar{5}) = 0.1.
\end{array}$$

Then, $f \cap g$ is not a fuzzy left ordered almost ideal of $\mathbb{Z}_6$ because $(\bar{1}_\alpha \circ (f \cap g)] \cap (f \cap g) = 0$ for all $\alpha \in (0, 1]$.

Example 4.8 implies that the intersection of two fuzzy left ordered almost ideals need not be a fuzzy left ordered almost ideal.

Theorem 4.9. *Let A be a nonempty subset of an ordered semigroup S. Then, the following properties hold:*

(1) *A is a left ordered almost ideal of S if and only if C_A is a fuzzy left ordered almost ideal of S;*

(2) *A is a right ordered almost ideal of S if and only if C_A is a fuzzy right ordered almost ideal of S;*

(3) *A is an ordered almost ideal of S if and only if C_A is a fuzzy ordered almost ideal of S.*

Proof. Assume A is a left ordered almost ideal of S. Let $s \in S$ and $\alpha \in (0,1]$. Then, $(sA] \cap A \neq \emptyset$. That is there exists $x \in A$ and $x \in (sA]$. So, $C_A(x) = 1 \neq 0$ and $x \leq sa$, for some $a \in A$. We have

$$\begin{aligned}(s_\alpha \circ C_A](x) &= \sup_{x \leq y}(s_\alpha \circ C_A)(y) \geq (s_\alpha \circ C_A)(x)\\ &= \sup_{x \leq uv} \min\{s_\alpha(u), C_A(v)\}\\ &\geq \min\{s_\alpha(s), C_A(a)\} = \min\{\alpha, 1\} \neq 0.\end{aligned}$$

Hence $\big((s_\alpha \circ C_A] \cap C_A\big)(x) \neq 0$. Therefore, C_A is a fuzzy left ordered almost ideal of S.

Conversely, assume that C_A is a fuzzy left ordered almost ideal of S. Let $s \in S$. There exists $x \in S$ such that $\big((s_\alpha \circ C_A] \cap C_A\big)(x) \neq 0$ for all $\alpha \in (0,1]$. That is $C_A(x) \neq 0$ and $(s_\alpha \circ C_A](x) \neq 0$ for all $\alpha \in (0,1]$. So, $x \in A$. Since $(s_\alpha \circ C_A](x) = \sup_{x \leq y}(s_\alpha \circ C_A)(y) \neq 0$, there is $y \in S$ such that $x \leq y$ and

$$(s_\alpha \circ C_A)(y) = \sup_{y \leq uv} \min\{(s_\alpha(u), C_A(v)\} \neq 0.$$

Thus, $y \leq sa$ for some $a \in A$. Since $x \leq y$, we get $x \leq sa$, so $x \in (sA]$. Hence $x \in (sA] \cap A$. Therefore, A is a left ordered almost ideal of S. In the case property (2), the proof follows similarly. By property (1) and (2), it can see that property (3) holds. □

Theorem 4.10. *Let S be an ordered semigroup. Then the following properties hold.*

(1) *f is a fuzzy left ordered almost ideal of S if and only if $supp(f)$ is a left ordered almost ideal of S.*

(2) *f is a fuzzy right ordered almost ideal of S if and only if $supp(f)$ is a right ordered almost ideal of S.*

(3) *f is a fuzzy ordered almost ideal of S if and only if $supp(f)$ is an ordered almost ideal of S.*

Proof. Assume that f is a fuzzy left ordered almost ideal of S. Let $s \in S$. There exists $x \in S$ such that $\big((s_\alpha \circ f] \cap f\big)(x) \neq 0$ for all $\alpha \in (0,1]$. That is $f(x) \neq 0$ and $(s_\alpha \circ f](x) \neq 0$ for all $\alpha \in (0,1]$. So $x \in supp(f)$. For all $\alpha \in (0,1]$, there is an element $y \in S$ such that $x \leq y$ and $(s_\alpha \circ f)(y) \neq 0$. Since $(s_\alpha \circ f)(y) \neq 0$, we obtain $y \leq sa$ for some $a \in S$ such that $f(a) \neq 0$. Thus $x \leq sa$ for some $a \in supp(f)$. This implies that $x \in \big(s(supp(f))\big]$. Hence $x \in \big(s(supp(f))\big] \cap supp(f)$. Therefore, $supp(f)$ is a left ordered almost ideal of S.

Conversely, assume that $supp(f)$ is a left ordered almost ideal of S. By Theorem 4.9 (1), $C_{supp(f)}$ is a fuzzy left ordered almost ideal of S. Let $s \in S$ and

$\alpha \in (0,1]$. Then, there exists $x \in S$ such that $\big((s_\alpha \circ C_{supp(f)}] \cap C_{supp(f)}\big)(x) \neq 0$ which implies that $C_{supp(f)}(x) \neq 0$ and $(s_\alpha \circ C_{supp(f)}](x) \neq 0$. So, $x \in supp(f)$ and there exists $y \in S$ such that $x \leq y$ and $(s_\alpha \circ C_{supp(f)})(y) \neq 0$. Thus, $y \leq sa$ for some $a \in supp(f)$. This implies that $x \leq sa$ where $f(x) \neq 0$ and $f(a) \neq 0$. Hence $\big((s_\alpha \circ f] \cap f\big)(x) \neq 0$. Therefore, f is a fuzzy left ordered almost ideal of S. Property (2), we can prove similarly. The proof of property (3) follows from properties (1) and (2). □

Acknowledgments

This paper was supported by Algebra and Applications Research Unit, Faculty of Science, Prince of Songkla University.

References

[1] O. Grosek, L. Satko, *A new notion in the theory of semigroups*, Semigroup Forum, 20 (1980), 233-240.

[2] O. Grosek, L. Satko, *Smallest A-ideals in semigroups*, Semigroup Forum, 23 (1981), 297-309.

[3] N. Kuroki, *Fuzzy bi-ideals in semigroups*, Commentarii Mathematici Universitatis Sancti Pauli, 28 (1979), 17-21.

[4] A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl., 35 (1971), 512-517.

[5] P. Pao-Ming, L. Ying-Ming, *Fuzzy topology I. Neighborhood structure of a fuzzy point and Moore-Smith convergence*, J. Math. Anal. Appl., 76 (1980), 571-599.

[6] L. Satko, O. Grosek, *On minimal A-ideals of semigroups,* Semigroup Forum, 23 (1981), 283-295.

[7] S. Suebsung, K. Wattanatripop, R. Chinram, *A-ideals and fuzzy A-ideals of ternary semigroups*, Songklanakarin J. Sci. Techno., 41 (2019), 299-304.

[8] J. P. F. Solano, S. Suebsung, R. Chinram, *On almost i-ideals and fuzzy almost i-ideals in n-ary semigroups*, JP J. Algebra, Number Theory Appl., 40 (2018), 833-842.

[9] K. Wattanatripop, R. Chinram, T. Changphas, *Quasi-A-ideals and fuzzy A-ideals in semigroups*, J. Discrete Math. Sci. Crypt., 21 (2018), 1131-1138.

[10] L. A. Zadeh, *Fuzzy sets,* Inf. Control, 8 (1965), 338-353.

Accepted: May 10, 2021