Inverse int-fuzzy soft (left, right) ideals over semigroups

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Abstract. In this paper, we define inverse int-soft sets, inverse int-fuzzy soft sets, inverse int-soft subsemigroups, inverse int-fuzzy soft subsemigroups and inverse int-soft (left, right) ideals, inverse int-fuzzy soft (left, right) ideals over a semigroup. We also give some of their properties such as AND operation, OR operation, union and intersection of them over semigroups with supported examples. Moreover, we show that the images of inverse int-fuzzy soft (left, right) ideals over semigroups are the inverse int-fuzzy soft (left, right) ideals over semigroups under some conditions.

Keywords: inverse int-fuzzy soft set, inverse int-fuzzy soft subsemigroups, inverse int-fuzzy soft (left, right) ideals.

1. Introduction

The solution to real-world problems in many fields such as engineering, economics, computer science, environment, medical science involve data that contain uncertainties. There is a wide range of theories that can be used when dealing with uncertainties in data, such as the theory of fuzzy sets [22], rough sets [20], intuitionistic fuzzy sets [1], interval mathematics [4], as well as other mathematic tools. In 1999, Molodtsov [14] defined soft set theory as a new mathematical tool for dealing with uncertainties that are are free from the difficulties. He pointed out directions of soft sets for the applications like soft analysis, operation research, probability and game theory. Later, Maji et al. [13] defined soft subset and soft super set, equality of two soft sets. They presented soft binary operation, such as AND, OR, intersection, union and studied De morgan's Laws of soft set.

Jun [6] applied the notion of soft sets to BCK/BCI-algebras. Next, Muhiuddin and Al-Roqi [16] introduced the notions of (internal, external) cubic soft sets, P-cubic (resp. R-cubic) soft subsets, R-union (resp. R-intersection, P-union, P-intersection) of cubic soft sets, and the complement of a cubic soft set and investigated some of their properties. They applied the notion of cubic soft sets to BCK/BCI-algebras. Later, Muhiuddin et al.[17] considered some operations of cubic soft sets, such as AND operation and OR operation based on the P-order and the R-order. They discussed conditions for the R-union of two internal cubic soft sets to be an internal cubic soft set and investigated some properties of cubic soft subalgebras in BCK/BCI-algebras. Jun et al. [7] introduced the notions of cubic soft ∘-subalgebras and (closed) cubic soft ideals in BCK/BCI-algebras and investigated some of their properties. They considered relations between cubic soft subalgebras, cubic soft ∘-subalgebras and (closed) cubic soft ideals and discussed characterizations of cubic soft ideals.

In 2001, Maji et al. [12] extended the soft sets to fuzzy soft sets. They introduced the concept of fuzzy soft sets and defined fuzzy soft subsets, the intersection and union of fuzzy soft sets over common universe. The fuzzy soft sets are developed to fuzzy soft semigroups by Yang [21] (2011). He defined fuzzy soft [left, right] ideals over semigroups and fuzzy soft semigroups, and studied sufficient and necessary conditions for α-level set, intersection and union of fuzzy soft [left, right] ideals. In 2014, Song et al. [19] introduced int-soft semigroups and int-soft left (right) ideals of semigroups. They proved that the soft intersection of int-soft left (right) ideals (int-soft semigroups) is also int-soft left (right) ideals (int-soft semigroups).

The soft sets and fuzzy soft sets are developed to inverse soft sets and inverse fuzzy soft sets over parameters by Çetkin et al. [3] (2016). They defined inverse soft sets and inverse fuzzy soft sets over parameters, and discussed the application in decision making problems of them. In 2019, Khalil and Hassan [8] studied inverse soft sets and inverse fuzzy soft sets and defined operations, such as AND operation, OR operation and investigated some of their properties. They constructed an algorithm using max-min and min-max decision of inverse fuzzy soft set for a fuzzy decision making problem. Motivated and inspired by the works above, we applied the notion of inverse soft sets and inverse fuzzy soft sets to int-soft sets and int-fuzzy soft sets over semigroups.

In this paper, we define inverse int-soft sets, inverse int-fuzzy soft sets, inverse int-soft subsemigroups, inverse int-fuzzy soft subsemigroups and inverse int-soft (left, right) ideals, inverse int-fuzzy soft (left, right) over a semigroup. We investigate some of their properties such as AND operation, OR operation, union and intersection of them over semigroups with supported examples. Finally, we show that the images of inverse int-fuzzy soft (left, right) ideals over semigroups are the inverse int-fuzzy soft (left, right) ideals over semigroups under some conditions.

2. Preliminaries

In this section, we shall give some of basic definitions and results that will be used.

Let S be a semigroup. A non-empty subset T of a semigroup S is called a subsemigroup if $T^2 \subseteq T$. A subsemigroup T of a semigroup S is called a left (right) ideal of S if $ST \subseteq T(TS \subseteq T)$. A subsemigroup T on a semigroup S is called an ideal on S if it is both a left and a right ideal on S [5].

The concept of fuzzy semigroups are introduced by Kuroki [10, 11].

For any $a, b \in [0,1]$, define $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. Then $a \wedge b$ and $a \vee b$ are element in $[0,1]$. A function f from a semigroup S to the unit interval $[0,1]$ is called a fuzzy set on S. A fuzzy set f on S is called a fuzzy subsemigroup on S if $f(xy) \geq f(x) \wedge f(y)$ for all $x, y \in S$.

A fuzzy set f on a semigroup S is called a fuzzy left [right] ideal on S if $f(xy) \geq f(y)[f(xy) \geq f(x)]$ for all $x, y \in S$. A fuzzy set f on a semigroup S is called a fuzzy ideal on S if it is both a fuzzy left and a fuzzy right ideal on S. We note that any fuzzy left [right] ideal on S is a fuzzy subsemigroup on S. Let T be a nonempty subset of a semigroup S, the characteristic function on T is the function $\chi_T : S \to [0,1]$ defined by $\chi_T(x) = 1$ if $x \in T$ and $\chi_T(x) = 0$ if $x \notin T$.

If f and g are fuzzy sets on a semigroup S then $f \leq g$, $f \vee g$ and $f \wedge g$ (some authors [15, 11] use notations $f \subseteq g$, $f \cup g$ and $f \cap g$, respectively), respectively, are defined as follows:

$$f \leq g \text{ if } f(x) \leq g(x), \quad \text{for all } \; x \in S,$$
$$(f \vee g)(x) = f(x) \vee g(x), \text{ for all } x \in S,$$
$$(f \wedge g)(x) = f(x) \wedge g(x), \text{ for all } x \in S.$$

Lemma 2.1. [15] *Let T be a nonempty subset of a semigroup S. Then, the following properties hold.:*

(*i*) *T is a subsemigroup of S if and only if χ_T is a fuzzy subsemigroup on S.*

(*ii*) *T is a (left, right) ideal of S if and only if χ_T is a fuzzy (left, right) ideal on S.*

Lemma 2.2. [15] *Let S be a semigroup. Then, the following properties hold:*

(*i*) *Let f and g be two fuzzy subsemigroups on S. Then, $f \cap g$ is also a fuzzy subsemigroup on S.*

(*ii*) *Let f and g be two fuzzy (left, right) ideals of S. Then, $f \cap g$ is also a fuzzy (left, right) ideal on S.*

Now, we give some definition of soft sets and fuzzy soft sets over an initial universe set.

Definition 2.1 ([14])**.** *Let U be an initial universe set and let E be a set of parameters. Let $P(U)$ denotes the power set of U and let $\emptyset \neq A \subseteq E$. A pair $(\widehat{F}, A)$ is called a soft set over U, where $\widehat{F}$ is a mapping given by $\widehat{F} : A \to P(U)$.*

Definition 2.2 ([12])**.** *Let E be a set of parameters and let $\emptyset \neq A \subseteq E$. A pair (F, A) is called a fuzzy soft set over U, where F is a mapping given by $F : A \to Fuz(U)$ and $Fuz(U)$ is the set of all fuzzy sets on U.*

The concept of the inverse fuzzy soft set over an initial universe set are introduced by Çetkin et al. [3].

Definition 2.3 ([3]). *Let U be an initial universe set and let E be a set of parameters. A pair $(\widehat{F}, U)$ is called an inverse soft set over E, where $\widehat{F}$ is a mapping given by $\widehat{F} : U \to P(E)$ and $P(E)$ is the power set of E.*

Definition 2.4 ([3]). *Let E be a set of parameters. A pair $(\mathcal{F}, U)$ is called an inverse fuzzy soft set over E, where $\mathcal{F}$ is a mapping given by $\mathcal{F} : U \to Fuz(E)$ is a mapping and $Fuz(E)$ is the set of all fuzzy parameter set on E.*

Song et al. [19] provided new definitions and various results on int-soft set theory. In what follows, we take $E = S$, as a set of parameters, which is a semigroup unless otherwise specified.

Definition 2.5 ([19]). *A soft set $(\widehat{f}, S)$ over U is called an int-soft semigroup over U if it satisfies:*

$$\widehat{f}(xy) \supseteq \widehat{f}(x) \cap \widehat{f}(y), \text{ for all } x, y \in S.$$

Definition 2.6 ([19]). *A soft set $(\widehat{f}, S)$ over U is called an int-soft left [right] ideal over U if it satisfies:*

$$\widehat{f}(xy) \supseteq \widehat{f}(y)[\widehat{f}(xy) \supseteq \widehat{f}(x)], \text{ for all } x, y \in S.$$

If a soft set $(\widehat{f}, S)$ over U is both an int-soft left ideal and an int-soft right ideal over U, we say that $(\widehat{f}, S)$ is an int-soft [two-sided] ideal over U.

Next, we give some definition of fuzzy soft sets over semigroups.

Definition 2.7 ([21]). *Let E be a set of parameters and let $\emptyset \neq A \subseteq E$. A pair (F, A) is called a fuzzy soft set over a semigroup S, where $F : A \to Fuz(S)$ is a mapping and $Fuz(S)$ is the set of all fuzzy sets on S.*

Let (F, A) be a fuzzy soft set over a semigroup S. For $p \in A, F(p) \in Fuz(S)$. Set $F_p := F(p)$. Then $F_p \in Fuz(S)$.

Definition 2.8 ([18]). *Let T be a subset of a semigroup S and A be a set of parameter. A fuzzy soft characteristic function $(\mathcal{C}^T, A)$ over S is defined by $(\mathcal{C}^T)_p = \chi_T$ for all $p \in A$.*

Definition 2.9 ([2]). *Let (F, A) be a fuzzy soft set over a semigroup S. For each $\alpha \in [0, 1]$, the set $(F, A)^\alpha$ is called an α*-level set *of (F, A), where*

$$(F_p)^\alpha = \{x \in S \mid F_p(x) \geq \alpha\}, \text{ for all } p \in A.$$

Definition 2.10 ([12]). *Let (F, A) and (G, B) be two fuzzy soft sets over a semigroup S. Define $(F, A) \leq (G, B)$ if*

(*i*) *$A \subseteq B$ and*

(ii) $F_p \leq G_p$ *for all* $p \in A$.

Definition 2.11 ([12]). *Let* (F, A) *and* (G, B) *be two fuzzy soft sets over a semigroup* S *with* $A \cap B \neq \emptyset$. *Define* $(F, A)\tilde{\wedge}(G, B) := (F \wedge G, A \cap B)$. *For each* $p \in A \cap B$, $(F \wedge G)_p = F_p \wedge G_p$. *Then,* $(F, A)\tilde{\wedge}(G, B)$ *is a fuzzy soft set over* S.

Some authors call $(F, A)\tilde{\wedge}(G, B)$, *the intersection of* (F, A) *and* (G, B).

Definition 2.12 ([12]). *Let* (F, A) *and* (G, B) *be two fuzzy soft sets over a semigroup* S. *Define* $(F, A)\tilde{\vee}(G, B) := (F \vee G, A \cup B)$, *where for all* $p \in A \cup B$,

$$(F \vee G)_p = \begin{cases} F_p, & \text{if } p \in A - B, \\ G_p, & \text{if } p \in B - A, \\ F_p \vee G_p, & \text{if } p \in A \cap B. \end{cases}$$

Then $(F, A)\tilde{\vee}(G, B)$ *is a fuzzy soft set over* S.

Some authors call $(F, A)\tilde{\vee}(G, B)$, *the union of* (F, A) *and* (G, B).

Definition 2.13 ([9]). *Let* U *be an initial universe set and let* E *be a set of parameters. A collection of all fuzzy soft sets over* U *with parameters from* E *is called a fuzzy soft class and is denoted by* $FS(U, E)$.

That is, $FS(U, E) = \{(F, A) | A \subseteq E \text{ and } F : A \to Fuz(U)\}$.

3. Inverse int-fuzzy soft (left, right) ideals over semigroups

In this section, we define inverse int-soft subsemigroup, inverse int-soft (left, right) ideals, inverse int-fuzzy soft subsemigroup and inverse int-fuzzy soft (left, right) ideals over semigroups and investigate some of their properties and support examples.

Definition 3.1. *Let* U *be an initial universe and let* S *be a semigroup. A pair* $(\widehat{F}, U)$ *is called an inverse int-soft set over* S, *where* $\widehat{\mathcal{F}}$ *is a mapping given by* $\widehat{\mathcal{F}} : U \to P(S)$ *and* $P(S)$ *is the power set of* S.

Definition 3.2. *Let* $(\widehat{\mathcal{F}}, U)$ *be an inverse int-soft set over a semigroup* S. $(\widehat{\mathcal{F}}, U)$ *is called an inverse int-soft subsemigroup if and only if* $\widehat{\mathcal{F}}(u)$ *is a subsemigroup of* S *for all* $u \in U$.

Definition 3.3. *Let* $(\widehat{\mathcal{F}}, U)$ *be an inverse int-soft set over a semigroup* S. $(\widehat{\mathcal{F}}, U)$ *is called an inverse int-soft left (right) ideal if and only if* $\widehat{\mathcal{F}}(u)$ *is a left (right) ideal of* S *for all* $u \in U$.

Definition 3.4. *Let* $(\widehat{\mathcal{F}}, U)$ *be an inverse int-soft set over* S. $(\widehat{\mathcal{F}}, U)$ *is called an inverse int-soft ideal if and only if* $(\widehat{\mathcal{F}}, U)$ *is both an inverse int-soft left and an inverse int-soft right ideal over* S.

Definition 3.5. *Let* U *be an initial universe and let* S *be a semigroup. A pair* $(\widehat{\mathbf{F}}, S)$ *is called an int-fuzzy soft set over* S, *where* $\widehat{\mathbf{F}}$ *is a mapping given by* $\widehat{\mathbf{F}} : S \to Fuz(U)$ *is a mapping and* $Fuz(U)$ *is the set of all fuzzy sets on* U.

Table 1: Multiplication table of a semigroup S

$\cdot$	x	y	z
x	x	z	z
y	z	y	z
z	z	z	z

Example 3.1. Let $U = \{u_1, u_2, u_3\}$ and $S = \{x, y, z\}$ be a semigroup having the following multiplication table (Table 1):

We choose the int-fuzzy soft set over U by Definition 3.5 as follows:

$$\begin{aligned}\widetilde{\mathbf{F}}_x &= \{u_1/0.1, u_2/0.4, u_3/0.8\},\\ \widetilde{\mathbf{F}}_y &= \{u_1/0.2, u_2/0.5, u_3/0.9\},\\ \widetilde{\mathbf{F}}_z &= \{u_1/0.3, u_2/0.6, u_3/0.9\}.\end{aligned}$$

Thus, we can view the int-fuzzy soft set $(\widetilde{\mathbf{F}}, S)$ as a collection of approximations as follow:

$$\begin{aligned}(\widetilde{\mathbf{F}}, S) = \{\widetilde{\mathbf{F}}_x &= \{u_1/0.1, u_2/0.4, u_3/0.8\},\\ \widetilde{\mathbf{F}}_y &= \{u_1/0.2, u_2/0.5, u_3/0.9\},\\ \widetilde{\mathbf{F}}_z &= \{u_1/0.3, u_2/0.6, u_3/0.9\}\}.\end{aligned}$$

Now, we define an inverse int-fuzzy soft set over a semigroup and give some their operations.

Definition 3.6. *Let U be an initial universe and let S be a semigroup. A pair $(\widetilde{\mathcal{F}}, U)$ is called an inverse int-fuzzy soft set over S, where $\widetilde{\mathcal{F}}$ is a mapping given by $\widetilde{\mathcal{F}} : U \to Fuz(S)$ is a mapping and $Fuz(S)$ is the set of all fuzzy set on S.*

Example 3.2. Let $U = \{u_1, u_2, u_3\}$ and $S = \{x, y, z\}$ be a semigroup having the same multiplication table as Example 3 (Table 1): We choose the inverse int-fuzzy soft set over S by Definition 3.6 as follows:

$$\begin{aligned}\widetilde{\mathcal{F}}_{u_1} &= \{x/0.1, y/0.4, z/0.8\},\\ \widetilde{\mathcal{F}}_{u_2} &= \{x/0.2, y/0.5, z/0.9\},\\ \widetilde{\mathcal{F}}_{u_3} &= \{x/0.3, y/0.6, z/0.9\}.\end{aligned}$$

Thus, we can view the inverse int-fuzzy soft set $(\widetilde{\mathcal{F}}, U)$ as a collection of approximations as follow:

$$\begin{aligned}(\widetilde{\mathcal{F}}, U) = \{\widetilde{\mathcal{F}}_{u_1} &= \{x/0.1, y/0.4, z/0.8\},\\ \widetilde{\mathcal{F}}_{u_2} &= \{x/0.2, y/0.5, z/0.9\},\\ \widetilde{\mathcal{F}}_{u_3} &= \{x/0.3, y/0.6, z/0.9\}\}.\end{aligned}$$

Definition 3.7. *Let $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ be two inverse int-fuzzy soft sets over S. The "$(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$", denoted by $(\widetilde{\mathcal{F}}, U)\widetilde{\sqcap}(\widetilde{\mathcal{G}}, V)$ is defined as*

$$(\widetilde{\mathcal{F}}, U)\widetilde{\sqcap}(\widetilde{\mathcal{G}}, V) = (\widetilde{\mathcal{H}}, U \times V),$$

where $\widetilde{\mathcal{H}}(u, v) = \widetilde{\mathcal{F}}(u) \wedge \widetilde{\mathcal{G}}(v)$ for all $(u, v) \in U \times V$ and $\wedge$ is the operation "fuzzy intersection" of two of fuzzy parameter sets.

Definition 3.8. *Let $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ be two inverse int-fuzzy soft sets over S. The "$(\widetilde{\mathcal{F}}, U)$ OR $(\widetilde{\mathcal{G}}, V)$", denoted by $(\widetilde{\mathcal{F}}, U)\widetilde{\sqcup}(\widetilde{\mathcal{G}}, V)$ is defined as*

$$(\widetilde{\mathcal{F}}, U)\widetilde{\sqcup}(\widetilde{\mathcal{G}}, V) = (\widetilde{\mathcal{K}}, U \times V),$$

where $\widetilde{\mathcal{K}}(u, v) = \widetilde{\mathcal{F}}(u) \vee \widetilde{\mathcal{G}}(v)$ for all $(u, v) \in U \times V$ and $\vee$ is the operation "fuzzy union" of two of fuzzy parameter sets.

Theorem 3.1. *Let $(\widetilde{\mathcal{F}}, U), (\widetilde{\mathcal{G}}, V)$ and $(\widetilde{\mathcal{H}}, W)$ be three inverse int-fuzzy soft sets over a semigroup S. Then:*
(i) $(\widetilde{\mathcal{F}}, U)\widetilde{\sqcap}((\widetilde{\mathcal{G}}, V)\widetilde{\sqcap}(\widetilde{\mathcal{H}}, W)) = ((\widetilde{\mathcal{F}}, U)\widetilde{\sqcap}(\widetilde{\mathcal{G}}, V))\widetilde{\sqcap}(\widetilde{\mathcal{H}}, W)$.
(ii) $(\widetilde{\mathcal{F}}, U)\widetilde{\sqcup}((\widetilde{\mathcal{G}}, V)\widetilde{\sqcup}(\widetilde{\mathcal{H}}, W)) = ((\widetilde{\mathcal{F}}, U)\widetilde{\sqcup}(\widetilde{\mathcal{G}}, V))\widetilde{\sqcup}(\widetilde{\mathcal{H}}, W)$.
(iii) $(\widetilde{\mathcal{F}}, U)\widetilde{\sqcap}((\widetilde{\mathcal{G}}, V)\widetilde{\sqcup}(\widetilde{\mathcal{H}}, W)) = ((\widetilde{\mathcal{F}}, U)\widetilde{\sqcap}(\widetilde{\mathcal{G}}, V))\widetilde{\sqcup}((\widetilde{\mathcal{F}}, U)\widetilde{\sqcap}(\widetilde{\mathcal{H}}, W))$.
(iv) $(\widetilde{\mathcal{F}}, U)\widetilde{\sqcup}((\widetilde{\mathcal{G}}, V)\widetilde{\sqcap}(\widetilde{\mathcal{H}}, W)) = ((\widetilde{\mathcal{F}}, U)\widetilde{\sqcup}(\widetilde{\mathcal{G}}, V))\widetilde{\sqcap}((\widetilde{\mathcal{F}}, U)\widetilde{\sqcup}(\widetilde{\mathcal{H}}, W))$.

Proof. For any $u \in U, v \in V$ and $w \in W$, we have

$$\widetilde{\mathcal{F}}_u \wedge (\widetilde{\mathcal{G}}_v \wedge \widetilde{\mathcal{H}}_w) = (\widetilde{\mathcal{F}}_u \wedge \widetilde{\mathcal{G}}_v) \wedge \widetilde{\mathcal{H}}_w.$$

Thus, $(\widetilde{\mathcal{F}}, U)\widetilde{\sqcap}((\widetilde{\mathcal{G}}, V)\widetilde{\sqcap}(\widetilde{\mathcal{H}}, W)) = ((\widetilde{\mathcal{F}}, U)\widetilde{\sqcap}(\widetilde{\mathcal{G}}, V))\widetilde{\sqcap}(\widetilde{\mathcal{H}}, W)$.

Similarly, we have $(\widetilde{\mathcal{F}}, U)\widetilde{\sqcup}((\widetilde{\mathcal{G}}, V)\widetilde{\sqcup}(\widetilde{\mathcal{H}}, W)) = ((\widetilde{\mathcal{F}}, U)\widetilde{\sqcup}(\widetilde{\mathcal{G}}, V))\widetilde{\sqcup}(\widetilde{\mathcal{H}}, W)$. For any $u \in U, v \in V$ and $w \in W$, we have

$$\widetilde{\mathcal{F}}_u \wedge (\widetilde{\mathcal{G}}_v \vee \widetilde{\mathcal{H}}_w) = (\widetilde{\mathcal{F}}_u \wedge \widetilde{\mathcal{G}}_v) \vee (\widetilde{\mathcal{F}}_u \wedge \widetilde{\mathcal{H}}_w).$$

Thus, $(\widetilde{\mathcal{F}}, U)\widetilde{\sqcap}((\widetilde{\mathcal{G}}, V)\widetilde{\sqcup}(\widetilde{\mathcal{H}}, W)) = ((\widetilde{\mathcal{F}}, U)\widetilde{\sqcap}(\widetilde{\mathcal{G}}, V))\widetilde{\sqcup}((\widetilde{\mathcal{F}}, U)\widetilde{\sqcap}(\widetilde{\mathcal{H}}, W))$.

Similarly, we have

$$(\widetilde{\mathcal{F}}, U)\widetilde{\sqcup}((\widetilde{\mathcal{G}}, V)\widetilde{\sqcap}(\widetilde{\mathcal{H}}, W)) = ((\widetilde{\mathcal{F}}, U)\widetilde{\sqcup}(\widetilde{\mathcal{G}}, V))\widetilde{\sqcap}((\widetilde{\mathcal{F}}, U)\widetilde{\sqcup}(\widetilde{\mathcal{H}}, W)). \qquad \square$$

Now, we define an inverse int-fuzzy soft subsemigroup and an inverse int-fuzzy soft (left, right) ideal over a semigroup and give some their properties with supported examples.

Definition 3.9. *Let $(\widetilde{\mathcal{F}}, U)$ be an inverse int-fuzzy soft set over a semigroup S. $(\widetilde{\mathcal{F}}, U)$ is called an inverse int-fuzzy soft subsemigroup if and only if $\widetilde{\mathcal{F}}_u$ is a fuzzy subsemigroup on S for all $u \in U$.*

Definition 3.10. *Let $(\widetilde{\mathcal{F}}, U)$ be an inverse int-fuzzy soft set over a semigroup S. $(\widetilde{\mathcal{F}}, U)$ is called an inverse int-fuzzy soft left (right) ideal if and only if $\widetilde{\mathcal{F}}_u$ is a fuzzy left (right) on S for all $u \in U$.*

Example 3.3. Let $U = \{u_1, u_2, u_3\}$ and $S = \{x, y, z\}$ be a semigroup having the same multiplication table as Example 3 (Table 1):

Choose $(\widetilde{\mathcal{F}}, U)$ is an inverse int-fuzzy soft set over S as:

$$\widetilde{\mathcal{F}}_{u_1} = \{x/0.1, y/0.4, z/0.8\},$$
$$\widetilde{\mathcal{F}}_{u_2} = \{x/0.2, y/0.5, z/0.9\},$$
$$\widetilde{\mathcal{F}}_{u_3} = \{x/0.3, y/0.6, z/0.9\}.$$

It is easy to verify that $\widetilde{\mathcal{F}}_{u_1}, \widetilde{\mathcal{F}}_{u_2}, \widetilde{\mathcal{F}}_{u_3}$ are both fuzzy left ideals on S.

Hence, $(\widetilde{\mathcal{F}}, U)$ is an inverse int-fuzzy soft left ideal over S.

Definition 3.11. *Let $(\widetilde{\mathcal{F}}, U)$ be an inverse int-fuzzy soft set over S. $(\widetilde{\mathcal{F}}, U)$ is called an inverse int-fuzzy soft ideal if and only if $(\widetilde{\mathcal{F}}, U)$ is both an inverse int-fuzzy soft left and an inverse int-fuzzy soft right ideal over S.*

Example 3.4. Let $U = \{u_1, u_2, u_3\}$ and $S = \{x, y, z\}$ be a semigroup having the same multiplication table as Example 3 (Table 1):

Choose $(\widetilde{\mathcal{F}}, U)$ is an inverse int-fuzzy soft set over S as:

$$\widetilde{\mathcal{F}}(u_1) = \{x/0.1, y/0.3, z/0.5\},$$
$$\widetilde{\mathcal{F}}(u_2) = \{x/0.2, y/0.5, z/0.9\},$$
$$\widetilde{\mathcal{F}}(u_3) = \{x/0.2, y/0.4, z/0.8\}.$$

It is easy to verify that $\widetilde{\mathcal{F}}_{u_1}, \widetilde{\mathcal{F}}_{u_2}, \widetilde{\mathcal{F}}_{u_3}$ are both fuzzy left ideals on S, at the same time, $\widetilde{\mathcal{F}}_{u_1}, \widetilde{\mathcal{F}}_{u_2}, \widetilde{\mathcal{F}}_{u_3}$ are both fuzzy right ideals on S. Hence $(\widetilde{\mathcal{F}}, U)$ is an inverse int-fuzzy soft ideal over S.

Definition 3.12. *Let T be a nonempty subset of a semigroup S and let U be an initial universe. An inverse int-fuzzy soft characteristic function $(\widetilde{\mathcal{C}}^T, U)$ over S if $(\widetilde{\mathcal{C}}^T)_u = \chi_T$ for all $u \in U$.*

Definition 3.13. *Let $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ be two inverse int-fuzzy soft sets over a semigroup S. $(\widetilde{\mathcal{F}}, U)$ is called an inverse int-fuzzy soft subset of $(\widetilde{\mathcal{G}}, V)$, denoted by $(\widetilde{\mathcal{F}}, U) \widetilde{\leq} (\widetilde{\mathcal{G}}, V)$, if*

(i) $U \subseteq V$;

(ii) *for all $u \in U$ there exists $v \in V$ such that $\widetilde{\mathcal{F}}_u \leq \widetilde{\mathcal{G}}_v$.*

Definition 3.14. *Two inverse int-fuzzy soft sets $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ over a semigroup S are called an inverse int-fuzzy soft equal if $(\widetilde{\mathcal{F}}, U) \widetilde{\leq} (\widetilde{\mathcal{G}}, V)$ and $(\widetilde{\mathcal{G}}, V) \widetilde{\leq} (\widetilde{\mathcal{F}}, U)$. We denote by $(\widetilde{\mathcal{F}}, U) = (\widetilde{\mathcal{G}}, V)$.*

Definition 3.15. *Let $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ be two inverse int-fuzzy soft sets over a semigroup S with $U \cap V \neq \emptyset$. The intersection of them denoted by*

$$(\widetilde{\mathcal{F}}, U) \tilde{\wedge} (\widetilde{\mathcal{G}}, V) := (\widetilde{\mathcal{F}} \wedge \widetilde{\mathcal{G}}, U \cap V).$$

Define for each $u \in U \cap V$, $(\widetilde{\mathcal{F}} \wedge \widetilde{\mathcal{G}})_u = \widetilde{\mathcal{F}}_u \wedge \widetilde{\mathcal{G}}_u$. Then, $(\widetilde{\mathcal{F}}, U) \tilde{\wedge} (\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft set over S.

Definition 3.16. *Let $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ be two inverse int-fuzzy soft sets over a semigroup S. The union of them denoted by $(\widetilde{\mathcal{F}}, U)\tilde{\vee}(\widetilde{\mathcal{G}}, V) := (\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}}, U \cup V)$, where for all $u \in U \cup V$,*

$$(\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_u = \begin{cases} \widetilde{\mathcal{F}}_u, & \text{if } u \in U - V, \\ \widetilde{\mathcal{G}}_u, & \text{if } u \in V - U, \\ \widetilde{\mathcal{F}}_u \vee \widetilde{\mathcal{G}}_u, & \text{if } u \in U \cap V. \end{cases}$$

Then, $(\widetilde{\mathcal{F}}, U)\tilde{\vee}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft set over S.

Lemma 3.1. *Let T be a nonempty subset of a semigroup S. Then, the following properties hold:*

(*i*) *T is a subsemigroup of S if and only if $(\widetilde{\mathcal{C}}^T, U)$ is an inverse int-fuzzy soft subsemigroup over S.*

(*ii*) *T is a (left, right) ideal of S if and only if $(\widetilde{\mathcal{C}}^T, U)$ is an inverse int-fuzzy soft (left, right) ideal on S.*

Proof. It follows from Lemma 2.1. □

In the following theorem, we show that an inverse int-fuzzy soft set over a semigroup is an inverse int-fuzzy soft subsemigroup over a semigroup if and only if an α-level set of inverse int-fuzzy soft set over the semigroup is an inverse int-soft subsemigroup over the semigroup for all $\alpha \in [0, 1]$.

Theorem 3.2. *Let $(\widetilde{\mathcal{F}}, U)$ is an inverse int-fuzzy soft set over a semigroup S. Then $(\widetilde{\mathcal{F}}, U)$ is an inverse int-fuzzy soft subsemigroup over S if and only if $(\widetilde{\mathcal{F}}, U)^\alpha$ is an inverse int-soft subsemigroup over S for all $\alpha \in [0, 1]$.*

Proof. Assume that $(\widetilde{\mathcal{F}}, U)$ is an inverse int-fuzzy soft subsemigroup over S. Let $\alpha \in [0, 1]$, $u \in U, y, z \in (\widetilde{\mathcal{F}}_u)^\alpha$. Then $\widetilde{\mathcal{F}}_u(y) \geq \alpha$ and $\widetilde{\mathcal{F}}_u(z) \geq \alpha$. By the assumption, $\widetilde{\mathcal{F}}_u$ is a fuzzy subsemigroup on S. We then have

$$\widetilde{\mathcal{F}}_u(y \cdot z) \geq \widetilde{\mathcal{F}}_u(y) \wedge \widetilde{\mathcal{F}}_u(z) \geq \alpha.$$

It follows that $y \cdot z \in (\widetilde{\mathcal{F}}_u)^\alpha$. Thus $(\widetilde{\mathcal{F}}_u)^\alpha$ is a subsemigroup of S. Therefore, $(\widetilde{\mathcal{F}}, U)^\alpha$ is an inverse int-soft subsemigroup over S.

Conversely, suppose that $(\widetilde{\mathcal{F}}, U)^\alpha$ is an inverse int-soft subsemigroup over S for all $\alpha \in [0, 1]$. Let $u \in U$ and $y, z \in S$. Choose $\alpha := \widetilde{\mathcal{F}}_u(y) \wedge \widetilde{\mathcal{F}}_u(z)$. Then $y, z \in (\widetilde{\mathcal{F}}_u)^\alpha$. Since $(\widetilde{\mathcal{F}}_u)^\alpha$ is a subsemigroup of S, we then have $y \cdot z \in (\widetilde{\mathcal{F}}_u)^\alpha$. This implies that $\widetilde{\mathcal{F}}_u(y \cdot z) \geq \alpha = \widetilde{\mathcal{F}}_u(y) \wedge \widetilde{\mathcal{F}}_u(z)$. Thus $\widetilde{\mathcal{F}}_u$ is a fuzzy subsemigroup on S. By the assumption, $y \cdot z \in (\widetilde{\mathcal{F}})^\alpha$. Thus

$$\widetilde{\mathcal{F}}_u(y \cdot z) \geq \alpha = \widetilde{\mathcal{F}}_u(y) \wedge \widetilde{\mathcal{F}}_u(z).$$

Hence $\widetilde{\mathcal{F}}_u$ is a fuzzy subsemigroup on S. Therefore $(\widetilde{\mathcal{F}}, U)$ is an inverse int-fuzzy soft subsemigroup over S. □

In the following theorem, we show that an inverse int-fuzzy soft set over a semigroup is an inverse int-fuzzy soft left (right) ideal over a semigroup if and only if an α-level set of inverse int-fuzzy soft set over the semigroup is an inverse int-soft left (right) ideal over the semigroup for all $\alpha \in [0, 1]$.

Theorem 3.3. *Let $(\widetilde{\mathcal{F}}, U)$ be an inverse int-fuzzy soft set over a semigroup S. Then $(\widetilde{\mathcal{F}}, U)$ is an inverse int-fuzzy soft left (right) ideal over S if and only if $(\widetilde{\mathcal{F}}, U)^{\alpha}$ is an inverse int-soft left (right) ideal over S for all $\alpha \in [0, 1]$.*

Proof. We prove only an inverse int-fuzzy soft left ideal, an inverse int-soft right ideal is similar. Assume that $(\widetilde{\mathcal{F}}, U)$ is an inverse int-fuzzy soft left ideal over S. Let $\alpha \in [0, 1]$, $u \in U, y \in (\widetilde{\mathcal{F}}_u)^{\alpha}$ and $x \in S$. Then $\widetilde{\mathcal{F}}_u(y) \geq \alpha$. By the assumption, $\widetilde{\mathcal{F}}_u$ is a fuzzy left ideal on S. We then have

$$\widetilde{\mathcal{F}}_u(x \cdot y) \geq \widetilde{\mathcal{F}}_u(y) \geq \alpha.$$

It follows that $x \cdot y \in (\widetilde{\mathcal{F}}_u)^{\alpha}$. Thus $(\widetilde{\mathcal{F}}_u)^{\alpha}$ is a left ideal of S. Therefore $(\widetilde{\mathcal{F}}, U)^{\alpha}$ is an inverse int-soft left ideal over S.

Conversely, suppose that $(\widetilde{\mathcal{F}}, U)^{\alpha}$ is an inverse int-soft left ideal over S for all $\alpha \in [0, 1]$. Let $u \in U$ and $y \in S$. Choose $\alpha := \widetilde{\mathcal{F}}_u(y)$. Then $y \in (\widetilde{\mathcal{F}}_u)^{\alpha}$. Since $(\widetilde{\mathcal{F}}_u)^{\alpha}$ is a left ideal of S, we then have $x \cdot y \in (\widetilde{\mathcal{F}}_u)^{\alpha}$ for all $x \in S$. This implies that

$$\widetilde{\mathcal{F}}_u(x \cdot y) \geq \alpha = \widetilde{\mathcal{F}}_u(y).$$

Thus $\widetilde{\mathcal{F}}_u$ is a fuzzy left ideal on S. Therefore $(\widetilde{\mathcal{F}}, U)$ is an inverse int-fuzzy soft left ideal over S. □

Theorem 3.4. *Let $(\widetilde{\mathcal{F}}, U)$ be an inverse int-fuzzy soft set over a semigroup S. Then $(\widetilde{\mathcal{F}}, U)$ is an inverse int-fuzzy soft ideal over S if and only if $(\widetilde{\mathcal{F}}, U)^{\alpha}$ is an inverse int-soft ideal over S for all $\alpha \in [0, 1]$.*

Proof. It follows from Theorem 3.3. □

The following theorem shows that $(\widetilde{\mathcal{F}}, U)\tilde{\wedge}(\widetilde{\mathcal{G}}, V)$ with $U \cap V \neq \emptyset$ is an inverse int-fuzzy soft subsemigroup over a semigroup.

Theorem 3.5. *Let $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ be two inverse int-fuzzy soft subsemigroups over a semigroup S. If $U \cap V \neq \emptyset$ then $(\widetilde{\mathcal{F}}, U)\tilde{\wedge}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft subsemigroup over S.*

Proof. Let $u \in U \cap V \neq \emptyset$. Thus $(\widetilde{\mathcal{F}} \wedge \widetilde{\mathcal{G}})_u = \widetilde{\mathcal{F}}_u \wedge \widetilde{\mathcal{G}}_u$. Since $\widetilde{\mathcal{F}}_u$ and $\widetilde{\mathcal{G}}_u$ are both fuzzy subsemigroup on S and the intersection of two fuzzy subsemigroups is a fuzzy subsemigroup (Lemma 2.2), we have $(\widetilde{\mathcal{F}} \wedge \widetilde{\mathcal{G}})_u$ is a fuzzy subsemigroup on S. Therefore $(\widetilde{\mathcal{F}}, U)\tilde{\wedge}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft subsemigroups over S. □

The following theorem shows that $(\widetilde{\mathcal{F}}, U)\tilde{\wedge}(\widetilde{\mathcal{G}}, V)$ with $U \cap V \neq \emptyset$ is an inverse int-fuzzy soft (left, right) ideal over a semigroup.

Theorem 3.6. *Let $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ be two inverse int-fuzzy (left, right) ideals over a semigroup S. If $U \cap V \neq \emptyset$ then $(\widetilde{\mathcal{F}}, U)\tilde{\wedge}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft (left, right) ideal over S.*

Proof. Let $u \in U \cap V \neq \emptyset$. Thus $(\widetilde{\mathcal{F}} \wedge \widetilde{\mathcal{G}})_u = \widetilde{\mathcal{F}}_u \wedge \widetilde{\mathcal{G}}_u$. Since $\widetilde{\mathcal{F}}_u$ and $\widetilde{\mathcal{G}}_u$ are both fuzzy (left, right) ideals on S and the intersection of two fuzzy (left, right) ideal is a fuzzy (left, right) ideal (Lemma 2.2), we have $(\widetilde{\mathcal{F}} \wedge \widetilde{\mathcal{G}})_u$ is a fuzzy (left, right) ideal on S. Therefore $(\widetilde{\mathcal{F}}, U)\tilde{\wedge}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft (left, right) ideal over S. □

In general, the union of two inverse int-fuzzy soft semigroups is not an inverse int-fuzzy soft semigroup, as is shown in the following:

Example 3.5. Let $U = \{u_1\}, V = \{u_1\}$ and $S = \{x, y, z\}$ be a semigroup having the same multiplication table as Example 3.1 (Table 1): Choose $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ are inverse int-fuzzy soft sets over S as:

$$\widetilde{\mathcal{F}}(u_1) = \{x/0.2, y/0.7, z/0.3\}$$

and

$$\widetilde{\mathcal{G}}(u_1) = \{x/0.8, y/0.3, z/0.4\}.$$

It is easy to verify that $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ are both inverse int-fuzzy soft subsemigroups over S. Then, $(\widetilde{\mathcal{F}}, U)\tilde{\vee}(\widetilde{\mathcal{G}}, V)$ as $\widetilde{\mathcal{F}}_u \vee \widetilde{\mathcal{G}}_{u_1} = \{x/0.8, y/0.7, z/0.4\}$. We see that

$$(\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_{u_1}(x \cdot y) = (\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_{u_1}(z) = 0.4 < 0.7 = (\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_{u_1}(x) \cap (\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_{u_1}(y).$$

Thus, $(\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_{u_1}$ is not a fuzzy subsemigroup on S. Hence, the union of two inverse int-fuzzy soft semigroups is not an inverse int-fuzzy soft semigroup.

The next theorem shows that $(\widetilde{\mathcal{F}}, U)\tilde{\vee}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft subsemigroup over a semigroup under some conditions.

Theorem 3.7. *Let $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ be two inverse int-fuzzy soft subsemigroups over a semigroup S. If $U \cap V = \emptyset$ then $(\widetilde{\mathcal{F}}, U)\tilde{\vee}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft subsemigroup over S.*

Proof. Suppose that $U \cap V = \emptyset$. Thus $(\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_u = \widetilde{\mathcal{F}}_u$ or $\widetilde{\mathcal{G}}_u$.

Since $\widetilde{\mathcal{F}}_u$ and $\widetilde{\mathcal{G}}_u$ are both fuzzy subsemigroup on S, we have $(\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_u$ is a fuzzy subsemigroup on S. Therefore, $(\widetilde{\mathcal{F}}, U)\tilde{\vee}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft subsemigroups over S. □

Theorem 3.8. *Let $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ be two inverse int-fuzzy soft subsemigroups over a semigroup S. If $U \subseteq V$ or $V \subseteq U$ then $(\widetilde{\mathcal{F}}, U)\tilde{\vee}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft subsemigroup over S.*

Proof. It suffices to show the theorem for the case $U \subseteq V$.

Let $u \in U \cup V$. Since $U \subseteq V$, we have $U \cup V = V$.

If $u \in V - U$ then $(\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_u = \widetilde{\mathcal{G}}_u$.

If $u \in U \cap V = U$ then $(\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_u = \widetilde{\mathcal{F}}_u$. Thus, $(\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_u$ is a fuzzy subsemigroups on S. Hence $(\widetilde{\mathcal{F}}, U)\tilde{\vee}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft subsemigroup over S. □

The next theorem shows that $(\widetilde{\mathcal{F}}, U)\tilde{\vee}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft (left, right) ideals over a semigroup.

Theorem 3.9. *Let $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ be two inverse int-fuzzy (left, right) ideals over a semigroup S. Then $(\widetilde{\mathcal{F}}, U)\tilde{\vee}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft (left, right) ideal over S.*

Proof. Let $u \in U \cup V$.

If $u \in U - V$ or $u \in V - U$ then $(\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_u = \widetilde{\mathcal{F}}_u$ or $\widetilde{\mathcal{G}}_u$.

Since $\widetilde{\mathcal{F}}_u$ and $\widetilde{\mathcal{G}}_u$ are both fuzzy (left, right) ideals on S, we have $(\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_u$ is a fuzzy (left, right) ideal on S.

If $u \in U \cap V$ then $(\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_u = \widetilde{\mathcal{F}}_u \vee \widetilde{\mathcal{G}}_u$. Since $\widetilde{\mathcal{F}}_u$ and $\widetilde{\mathcal{G}}_u$ are both fuzzy (left, right) ideals on S for all $x, y \in S$, we have

$$(\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_u(x \cdot y) = (\widetilde{\mathcal{F}})_u(x \cdot y) \vee (\widetilde{\mathcal{G}})_u(x \cdot y) \geq (\widetilde{\mathcal{F}})_u(y) \vee (\widetilde{\mathcal{G}})_u(y) = (\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_u(y).$$

Hence $(\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_u$ is a fuzzy (left, right) ideal on S. Therefore, $(\widetilde{\mathcal{F}}, U)\tilde{\vee}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft (left, right) ideal over S. □

4. Mapping on classes of inverse int-fuzzy soft (left, right) ideals over semigroups

In this section, we construct the image of inverse int-fuzzy soft subsemigroups and int-fuzzy soft (left, right) ideals over semigroups.

Now, we define notations and definitions of the images of functions as follows.

Definition 4.1. *Let U be an initial universe set and let S be a semigroup. A collection of all inverse int-fuzzy soft sets over S is called an inverse int-fuzzy soft class and is denoted by $\widetilde{FS}(S, U)$. That is, $\widetilde{FS}(S, U) = \{(\widetilde{\mathcal{F}}, U) | \widetilde{\mathcal{F}} : U \to Fuz(S)\}$.*

For any sets A and B, define $M(U, V) := \{f | f : U \to V\}$.

Let $\widetilde{FS}(S, U)$ and $\widetilde{FS}(T, V)$ be inverse int-fuzzy soft classes over a semigroup S and a semigroup T with the initial universes from U and V, respectively. That is,

$\widetilde{FS}(S, U) = \{(\widetilde{\mathcal{F}}, U) | \widetilde{\mathcal{F}} : U \to \mathrm{Fuz}(S)\}$,

$\widetilde{FS}(T, V) = \{(\widetilde{\mathcal{G}}, V) | \widetilde{\mathcal{G}} : V \to \mathrm{Fuz}(T)\}$.

Let $u : S \to T$ and $v : U \to V$ be two mappings.

We want to construct two functions:

(i) $\Psi : \widetilde{FS}(S, U) \to \widetilde{FS}(T, V)$;

(ii) $\Sigma : \widetilde{FS}(T,V) \to \widetilde{FS}(S,U)$.

(i) Let $(\widetilde{\mathcal{F}},U) \in \widetilde{FS}(S,U)$.

Let $\Omega : M(U, \mathrm{Fuz}(S)) \to M(v(U), \mathrm{Fuz}(T))$ be defined as follows:

Given $y \in T$ and $\beta \in v(U)$, define

$$[\Omega(\widetilde{\mathcal{F}})]_\beta(y) = \begin{cases} \bigvee_{x\in u^{-1}(y)} [\bigvee_{\delta\in v^{-1}(\beta)\cap U} \widetilde{\mathcal{F}}_\delta(x)], & \text{if } u^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\Psi(\widetilde{\mathcal{F}},U) := (\Omega(\widetilde{\mathcal{F}}), v(U))$. Then, $(\Omega(\widetilde{\mathcal{F}}), v(U))$ is an image of $(\widetilde{\mathcal{F}},U)$ under the function Ψ.

(ii) Let $(\widetilde{\mathcal{G}},V) \in \widetilde{FS}(V,T)$.

Let $\Gamma : M(V, \mathrm{Fuz}(T)) \to M(v^{-1}(V), \mathrm{Fuz}(S))$ be defined as follows:

Given $x \in S$ and $\delta \in v^{-1}(V)$, define

$$[\Gamma(\widetilde{\mathcal{G}})]_\delta(x) = \widetilde{\mathcal{G}}_{v(\delta)}(u(x)).$$

Let $\Sigma(\widetilde{\mathcal{G}},V) := (\Gamma(\widetilde{\mathcal{G}}), v^{-1}(S))$.

Remark 4.1. Let $\widetilde{\mathcal{FS}}(S,U)$ and $\widetilde{FS}(T,V)$ be two inverse int-fuzzy soft classes over semigroups S and T, respectively. Suppose that $u : S \to T$ and $v : U \to V$ are two mappings and $\Psi : \widetilde{FS}(S,U) \to \widetilde{FS}(T,V)$ and $\Sigma : \widetilde{FS}(T,V) \to \widetilde{FS}(S,U)$ are also two mappings. If u and v are bijections, then $\Sigma \circ \Psi = I$ and $\Psi \circ \Sigma = I$, where $\circ$ is a composite mapping and I is an identity mapping.(Hence Ψ and Σ are bijections.)

The following theorem shows that the image of an inverse int-fuzzy soft subsemigroup over a semigroup is an inverse int-fuzzy soft subsemigroup over the semigroup under some conditions.

Theorem 4.1. *Let $\widetilde{FS}(S,U)$ and $\widetilde{FS}(T,V)$ be two inverse int-fuzzy soft classes over semigroups S and T, respectively. Suppose that $\Psi : \widetilde{FS}(S,U) \to \widetilde{FS}(T,V)$ is a mapping, $v : U \to V$ is an injection and u is an isomorphism mapping from S to T. If $(\widetilde{\mathcal{F}},U)$ is an inverse int-fuzzy soft subsemigroup over S then $\Psi(\widetilde{\mathcal{F}},U)$ is an inverse int-fuzzy soft subsemigroup over T.*

Proof. Suppose that $(\widetilde{\mathcal{F}},U)$ is an inverse int-fuzzy soft subsemigroup over S. By Theorem 3, $(\widetilde{\mathcal{F}},U)^\alpha$ is an inverse int-soft subsemigroup over S for all $\alpha \in [0,1]$. We shall show that $\Psi(\widetilde{\mathcal{F}},U)^\alpha = (\Omega(\widetilde{\mathcal{F}}), v(U))^\alpha$ is an inverse int-soft subsemigroup over T for all $\alpha \in [0,1]$. Let $p \in v(U)$. There exists $q \in U$ such that $p = v(q)$. Let $z_1, z_2 \in ([\Omega(\widetilde{\mathcal{F}})]_p)^\alpha$. Thus

$$\alpha \leq [\Omega(\widetilde{\mathcal{F}})]_p(z_1) = \bigvee_{s\in u^{-1}(z_1)} [\bigvee_{\beta\in v^{-1}(p)\cap U} \widetilde{\mathcal{F}}_\beta(s)] = \bigvee_{s\in u^{-1}(z_1)} \widetilde{\mathcal{F}}_q(s),$$

and

$$\alpha \leq [\Omega(\widetilde{\mathcal{F}})]_p(z_2) = \bigvee_{s\in u^{-1}(z_2)} [\bigvee_{\beta\in v^{-1}(p)\cap U} \widetilde{\mathcal{F}}_\beta(s)] = \bigvee_{s\in u^{-1}(z_2)} \widetilde{\mathcal{F}}_q(s).$$

This implies that there exist $s_1, s_2 \in S$ such that $u(s_1) = z_1, u(s_2) = z_2$ and $\widetilde{\mathcal{F}}_q(s_1) \geq \alpha, \widetilde{\mathcal{F}}_q(s_2) \geq \alpha$. Then,

$$\begin{aligned}[\Omega(\widetilde{\mathcal{F}})]_p(z_1 \cdot z_2) &= \bigvee_{s\in u^{-1}(z_1\cdot z_2)} [\bigvee_{\beta\in v^{-1}(p)\cap U} \widetilde{\mathcal{F}}_\beta(s)] \\ &= \bigvee_{s\in u^{-1}(z_1\cdot z_2)} \widetilde{\mathcal{F}}_q(s) \\ &\geq \widetilde{\mathcal{F}}_q(s_1 \cdot s_2) \\ &\geq \widetilde{\mathcal{F}}_q(s_1) \wedge \widetilde{\mathcal{F}}_q(s_2) \geq \alpha.\end{aligned}$$

Thus $z_1 \cdot z_2 \in ([\Omega(\widetilde{\mathcal{F}})]_p)^\alpha$. Hence $([\Omega(F)]_p)^\alpha$ is an inverse int-soft subsemigroup of T. Therefore $\Psi(\widetilde{\mathcal{F}}, U)^\alpha = (\Omega(\widetilde{\mathcal{F}}), v(U))^\alpha$ is an inverse int-fuzzy soft subsemigroup over T for all $\alpha \in [0, 1]$. By Theorem 3.2, $\Psi(\widetilde{\mathcal{F}}, U)$ is an inverse int-fuzzy soft subsemigroup over T. □

The following theorem shows that $\Sigma(\widetilde{\mathcal{G}}, V)$ of an inverse int-fuzzy soft subsemigroup over a semigroup is an inverse int-fuzzy soft subsemigroup over the semigroup under some conditions.

Theorem 4.2. *Let $\widetilde{FS}(S, U)$ and $\widetilde{FS}(T, V)$ be two inverse int-fuzzy soft classes over semigroups S and T, respectively. Suppose that $\Sigma : \widetilde{FS}(T, V) \to \widetilde{FS}(S, V)$ is a mapping, $v : U \to V$ is a mapping and u is a homomorphism from S to T. If $(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft subsemigroup over T then $\Sigma(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft subsemigroup over S.*

Proof. Assume that $(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft subsemigroup over T. Let $\delta \in v^{-1}(V)$. We then have $v(\delta) \in V$. This implies that $\widetilde{\mathcal{G}}_{v(\delta)}$ is an inverse int-fuzzy subsemigroup on T. Let $x, y \in S$. Then

$$\begin{aligned}[\Gamma(\widetilde{\mathcal{G}})]_\delta(x \cdot y) &= \widetilde{\mathcal{G}}_{v(\delta)}(u(x \cdot y)) \\ &= \widetilde{\mathcal{G}}_{v(\delta)}(u(x) \cdot u(y)) \\ &\geq \widetilde{\mathcal{G}}_{v(\delta)}(u(x)) \wedge \widetilde{\mathcal{G}}_{v(\delta)}(u(y)) \\ &= [\Gamma(\widetilde{\mathcal{G}})]_\delta(x) \wedge [\Gamma(\widetilde{\mathcal{G}})]_\delta(y).\end{aligned}$$

This implies that $[\Gamma(\widetilde{\mathcal{G}})]_\delta$ is an inverse int-fuzzy subsemigroup on S. Hence $\Sigma(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft subsemigroup over S. □

The following corollary shows that $(\Sigma\circ\Psi)(\widetilde{\mathcal{F}}, U) = (\widetilde{\mathcal{F}}, U)$ and $(\Psi\circ\Sigma)(\widetilde{\mathcal{G}}, V) = (\widetilde{\mathcal{G}}, V)$ for every inverse int-fuzzy soft subsemigroups $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$, where $\circ$ is a composite mapping, under certain conditions.

Corollary 4.1. *Let $\widetilde{FS}(S,U)$ and $\widetilde{FS}(T,V)$ be two inverse int-fuzzy soft classes over semigroups S and T, respectively. Suppose that $u : S \to T$ and $v : U \to V$ are two mappings and $\Psi : \widetilde{FS}(S,U) \to \widetilde{FS}(T,V)$ and $\Sigma : \widetilde{FS}(T,V) \to \widetilde{FS}(S,U)$ are also two mappings. If u and v are bijections, then $(\Sigma \circ \Psi)(\widetilde{\mathcal{F}},U) = (\widetilde{\mathcal{F}},U)$ and $(\Psi \circ \Sigma)(\widetilde{\mathcal{G}},V) = (\widetilde{\mathcal{G}},V)$ for every inverse int-fuzzy soft subsemigroups $(\widetilde{\mathcal{F}},U)$ and $(\widetilde{\mathcal{G}},V)$, where $\circ$ is a composite mapping.*

Proof. It follows from Remark 4.1. □

5. Conclusions

In this paper, we defined inverse int-soft sets, inverse int-fuzzy soft sets, inverse int-soft subsemigroups, inverse int-fuzzy soft subsemigroups and inverse int-soft (left, right) ideals, inverse int-fuzzy soft (left, right) ideal over a semigroup. We obtained some of their properties such as AND operation, OR operation, union and intersection of them over semigroups with supported examples. We also proved that the images of inverse int-fuzzy soft (left, right) ideals over semigroups are the inverse int-fuzzy soft (left, right) ideals over semigroups under some conditions.

Acknowledgment

The authors wish to express their sincere thanks to the referees for the valuable suggestions which lead to an improvement of this paper. This research project was supported by the thailand science research and innovation fund and the University of Phayao (Grant No. FF64-UoE036).

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Accepted: January 27, 2021