Fixed point results in a complex-valued generalized G_b-metric space

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Abstract. In this paper, we introduce the concept of a complex $G_b^{\alpha\beta}$-metric space and prove the Banach fixed point principle and Kannan's fixed point theorem. Our results generalize the idea of a complex valued G_b-metric space.
Keywords: G_b- metric, partial order, Kannan's fixed point theorem.

1. Introduction

Bakhtin [3] presented b-metric spaces as a generalization of metric spaces. In the frame work of b-metric spaces in [7, 6], authors Mebawondu et al., introduced some classes of mappings called TAC-Suzuki Berinde Type F-contractions and TAC-Suzuki type rational F-contractions. Azam et al. [2] defined a complex valued metric space. Rao et al. [10] then introduced the concept of a complex valued b-metric space. Similarly, G-metric spaces were defined by Mustafa and Sims [8]. Thereafter followed the concept of G_b-metric spaces by Aghajani et al. [1]. Recently as an application Mebawondu et al. [5] discussed the existence of solutions for integral equations for Z-contraction mappings in the frame work of a complete G_b metric spaces. Ozgur [9] extended this idea to define a complex valued G_b-metric space. Likewise, Singh [11] introduced the idea of a α, β b-metric space and thereafter a $G_b^{\alpha\beta}$-metric space. Here we extend this concept to a complex valued $G_b^{\alpha\beta}$-metric space and prove the Banach contraction principle and Kannan's [4] fixed point theorem in this space.

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2. Preliminaries

Here we state some properties of $G_b^{\alpha\beta}$-metric spaces.

Definition 2.1 ([11]). *Let $X \neq \emptyset$ be a set and $\alpha, \beta \geq 1$ be real numbers. A function $G_b^{\alpha\beta} : X \times X \times X \to \mathbb{R}^+$ is called a $G_b^{\alpha\beta}$-metric if the following hold, for every $x, y, z, a \in X$:*

(i) $G_b^{\alpha\beta}(x, y, z) = 0$ *if* $x = y = z$;

(ii) $G_b^{\alpha\beta}(x, x, y) > 0$ *with* $x \neq y$;

(iii) $G_b^{\alpha\beta}(x, x, y) \leq G_b^{\alpha\beta}(x, y, z)$ *with* $y \neq z$;

(iv) $G_b^{\alpha\beta}(x, y, z) = G_b^{\alpha\beta}(\rho\{x, y, z\})$, *where* ρ *is any permutation of* $\{x, y, z\}$;

(v) $G_b^{\alpha\beta}(x, y, z) \leq \alpha G_b^{\alpha\beta}(x, a, a) + \beta G_b^{\alpha\beta}(a, y, z)$.

Then, $(X, G_b^{\alpha\beta})$ is a $G_b^{\alpha\beta}$-metric space, when $\alpha = \beta$, $G_b^{\alpha\beta}$ is a G_b metric.

Proposition 2.1. *Let X be a $G_b^{\alpha\beta}$-metric space, then for $x, y, z \in X$ it follows that:*

(i) *if* $G_b^{\alpha\beta}(x, y, z) = 0$ *then* $x = y = z$;

(ii) $G_b^{\alpha\beta}(x, y, z) \leq \alpha G_b^{\alpha\beta}(x, x, y) + \beta G_b^{\alpha\beta}(x, x, z)$;

(iii) $G_b^{\alpha\beta}(x, y, y) \leq (\alpha + \beta) G_b^{\alpha\beta}(y, x, x)$;

(iv) $G_b^{\alpha\beta}(x, y, z) \leq \alpha G_b^{\alpha\beta}(x, a, z) + \beta G_b^{\alpha\beta}(a, y, z)$.

Definition 2.2. *Let X be a $G_b^{\alpha\beta}$-metric space. A sequence $\{x_n\}$ is said to be:*

(i) $G_b^{\alpha\beta}$*-Cauchy if for each* $\epsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that for all* $m, n, l \geq N$, $G_b^{\alpha\beta}(x_n, x_m, x_l) < \epsilon$;

(ii) $G_b^{\alpha\beta}$*-convergent to* $x^\star \in X$ *if for each* $\epsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that for all* $m, n \geq N$, $G_b^{\alpha,\beta}(x_m, x_n, x^\star) < \epsilon$.

Proposition 2.2. *Let X be a $G_b^{\alpha\beta}$-metric space and $\{x_n\}$ a sequence in X. Then, the following are equivalent:*

(i) *The sequence* $\{x_n\}$ *is* $G_b^{\alpha\beta}$*-Cauchy;*

(ii) *For any* $\epsilon > 0$, *there exists* $N \in \mathbb{N}$ *such that* $G_b^{\alpha\beta}(x_n, x_m, x_m) < \epsilon$, *for all* $m, n \geq N$.

Proposition 2.3. *Let X be a $G_b^{\alpha\beta}$-metric space. The the following are equivalent:*

(i) $\{x_n\}$ *is* $G_b^{\alpha\beta}$*-convergent to* $x^\star$*;*

(ii) $G_b^{\alpha\beta}(x_n, x_n, x^\star) \to 0$ *as* $n \to \infty$*;*

(iii) $G_b^{\alpha\beta}(x_n, x^\star, x^\star) \to 0$ *as* $n \to \infty$*.*

Definition 2.3. *A* $G_b^{\alpha\beta}$*-metric space* X *is complete if every* $G_b^{\alpha\beta}$*-Cauchy sequence is* $G_b^{\alpha\beta}$*-convergent in* X*.*

Definition 2.4. *Let* $\mathbb{C}$ *be the set of complex numbers and for* $z_1, z_2 \in \mathbb{C}$ *define a partial order* $\precsim$ *on* $\mathbb{C}$ *by*

$$z_1 \precsim z_2 \text{ if and only if } \Re(z_1) \leq \Re(z_2) \text{ and } \Im(z1) \leq \Im(z_2).$$

Proposition 2.4. *If* $z_1 \precsim z_2$ *then one of the following are true:*

(i) $\Re(z_1) = \Re(z_2)$ *and* $\Im(z_1) = \Im(z_2)$*;*

(ii) $\Re(z_1) < \Re(z_2)$ *and* $\Im(z_1) = \Im(z_2)$*;*

(iii) $\Re(z_1) = \Re(z_2)$ *and* $\Im(z_1) < \Im(z_2)$*;*

(iv) $\Re(z_1) < \Re(z_2)$ *and* $\Im(z_1) < \Im(z_2)$*.*

We write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (ii),(iii) or (iv) holds. If only (iv) is satisfied then we write $z_1 \prec z_2$.

Proposition 2.5. *The following statements are true:*

(i) *if* $a, b \in \mathbb{R}$ *with* $a \leq b$ *then* $az \precsim bz$*;*

(ii) *if* $0 \precsim z_1 \precnsim z_2$ *then* $|z_1| < |z_2|$*;*

(iii) *if* $z_1 \precsim z_2$ *and* $z_2 \prec z_3$*, then* $z_1 \prec z_3$*.*

3. Complex-valued $G_b^{\alpha\beta}$-metric spaces

Definition 3.1. *Let* $X \neq \emptyset$ *be a set and* $\alpha, \beta \geq 1$ *be real numbers. A function* $G_b^{\alpha\beta} : X \times X \times X \to \mathbb{C}$ *is called a* $G_b^{\alpha\beta}$*-complex metric if the following hold, for every* $x, y, z, a \in X$

(i) $G_b^{\alpha\beta}(x, y, z) = 0$ *if* $x = y = z$*;*

(ii) $0 \prec G_b^{\alpha\beta}(x, x, y)$ *with* $x \neq y$*;*

(iii) $G_b^{\alpha\beta}(x, x, y) \precsim G_b^{\alpha\beta}(x, y, z)$ *with* $y \neq z$*;*

(iv) $G_b^{\alpha\beta}(x, y, z) = G_b^{\alpha\beta}(\rho\{x, y, z\})$ *where* ρ *is any permutation of* $\{x, y, z\}$*;*

(v) $G_b^{\alpha\beta}(x, y, z) \precsim \alpha G_b^{\alpha\beta}(x, a, a) + \beta G_b^{\alpha\beta}(a, y, z)$*.*

Then $(X, G_b^{\alpha\beta})$ *is a* $G_b^{\alpha\beta}$*-complex metric space. When* $\alpha = \beta$*,* $G_b^{\alpha\beta}$ *is a* G_b*-complex metric.*

Example 3.1. Let $X = (1, 3)$ then define $G_b^{\alpha\beta} : X \times X \times X \to \mathbb{C}$ by

$$G_b^{\alpha\beta}(x, y, z) = \begin{cases} (1+i)e^{|x-y|+|y-z|+|z-x|}, & x \neq y \neq z \\ 0, & x = y = z. \end{cases}$$

To show that $G_b^{\alpha\beta}(x, y, z)$ is a $G_b^{\alpha\beta}$-metric we verify properties (i)-(v) of definition 3.1. Properties (i)-(iv) are easily verified. We only verify property (v) of definition 3.1. Let $x, y, z \in X$ such that $x \neq y \neq z$ then from proposition 2.5 it follows that

$$\begin{aligned} & G_b^{\alpha\beta}(x, y, z) \\ & \precsim (1+i)e^{|x-a|+|y-a|+|y-z|+|z-a|+|x-a|} \\ & = (1+i)e^{2|x-a|+|y-a|+|y-z|+|z-a|} \\ & \precsim \sup_{x,y,z\in X} e^{\frac{2}{3}|2(x-a)|+\frac{1}{3}[|y-a|+|y-z|+|z-a|]}(1+i)e^{\frac{1}{3}|2(x-a)|+\frac{2}{3}[|y-a|+|y-z|+|z-a|]} \\ & \precsim \frac{e^{\frac{14}{3}}}{3}(1+i)e^{|2(x-a)|} + \frac{2e^{\frac{14}{3}}}{3}(1+i)e^{[|y-a|+|y-z|+|z-a|]} \\ & = \frac{e^{\frac{14}{3}}}{3} G_b^{\alpha\beta}(x, a, a) + \frac{2e^{\frac{14}{3}}}{3} G_b^{\alpha\beta}(a, y, z) \end{aligned}$$

with $\alpha = \frac{e^{\frac{14}{3}}}{3} \geq 1$ and $\beta = \frac{2e^{\frac{14}{3}}}{3} \geq 1$, $\alpha \neq \beta$ and $\alpha < \beta$.

Proposition 3.1. *Let* $(X, G_b^{\alpha\beta})$ *be a* $G_b^{\alpha\beta}$*-complex metric space, then:*

1. $G_b^{\alpha\beta}(x, y, z) \precsim \alpha G_b^{\alpha\beta}(x, x, y) + \beta G_b^{\alpha\beta}(x, x, z)$*;*
2. $G_b^{\alpha\beta}(x, y, y) \precsim (\alpha + \beta) G_b^{\alpha\beta}(y, x, x)$*.*

Proof. 1. Interchanging variables x and y in property (v) of definition 3.1 we get $G_b^{\alpha\beta}(y, x, z) \precsim \alpha G_b^{\alpha\beta}(y, a, a) + \beta G_b^{\alpha\beta}(a, x, z)$. Setting $a = x$, we get $G_b^{\alpha\beta}(y, x, z) \precsim \alpha G_b^{\alpha\beta}(y, x, x) + \beta G_b^{\alpha\beta}(x, x, z)$. Finally, using property (iv) of definition 3.1 we get $G_b^{\alpha\beta}(x, y, z) \precsim \alpha G_b^{\alpha\beta}(x, x, y) + \beta G_b^{\alpha\beta}(x, x, z)$.

2. Interchanging variables x and y in property (v) of definition 3.1, we get $G_b^{\alpha\beta}(y, x, z) \precsim \alpha G_b^{\alpha\beta}(y, a, a) + \beta G_b^{\alpha\beta}(a, x, z)$ for all $x, y, z \in X$. Taking $z = y$, we get $G_b^{\alpha\beta}(y, x, y) \precsim \alpha G_b^{\alpha\beta}(y, a, a) + \beta G_b^{\alpha\beta}(a, x, y)$. Finally, setting $a = x$ we get $G_b^{\alpha\beta}(y, x, y) \precsim \alpha G_b^{\alpha\beta}(y, x, x) + \beta G_b^{\alpha\beta}(x, x, y)$. Using property (iv) of definition 3.1 we obtain $G_b^{\alpha\beta}(x, y, y) \precsim (\alpha + \beta) G_b^{\alpha\beta}(y, x, x)$. □

Definition 3.2. *Let* X *be a* $G_b^{\alpha\beta}$*-complex metric space. A sequence* $\{x_n\}$ *is said to be:*

1. *$G_b^{\alpha\beta}$-Cauchy if for each $0 \prec \epsilon$ there exists $N \in \mathbb{N}$ such that for all $m, n, l \geq N$, $G_b^{\alpha\beta}(x_n, x_m, x_l) \prec \epsilon$;*

2. *$G_b^{\alpha\beta}$-convergent to $x^\star \in X$ if for each $0 \prec \epsilon$ there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $G_b^{\alpha\beta}(x_m, x_n, x^\star) \prec \epsilon$.*

Proposition 3.2. *Let $(X, G_b^{\alpha\beta})$ be a $G_b^{\alpha\beta}$-complex metric space and $\{x_n\}$ a sequence in X. Then $\{x_n\}$ is complex valued $G_b^{\alpha\beta}$-convergent to $x^\star$ if and only if $|G_b^{\alpha\beta}(x_m, x_n, x^\star)| \to 0$ as $n, m \to \infty$.*

Proof. ($\Rightarrow$) Let $\epsilon' > 0$ be given and consider $\epsilon = \frac{\epsilon'}{\sqrt{2}}(1+i)$, then $0 \prec \epsilon \in \mathbb{C}$ and there exists a $N \in \mathbb{N}$ such that for all $n, m > N$,

$$0 \prec G_b^{\alpha\beta}(x_n, x_n, x^\star) \preceq G_b^{\alpha\beta}(x_m, x_n, x^\star) \prec \epsilon.$$

Hence, $|G_b^{\alpha\beta}(x_m, x_n, x^\star)| < |\epsilon| = \epsilon'$, which implies that $|G_b^{\alpha\beta}(x_m, x_n, x^\star)| \to 0$ as $n, m \to \infty$.

($\Leftarrow$) Suppose $|G_b^{\alpha\beta}(x_m, x_n, x^\star)| \to 0$ as $n, m \to \infty$. Given $\epsilon \in \mathbb{C}$ with $0 \prec \epsilon$, it follows that $0 < |\epsilon|$. By the convergence of $|G_b^{\alpha\beta}(x_m, x_n, x^\star)|$ to 0 there exists $N \in \mathbb{N}$ such that for all $n, m > N$, $|G_b^{\alpha\beta}(x_m, x_n, x^\star)| < |\epsilon|$, which implies that $G_b^{\alpha\beta}(x_m, x_n, x^\star) \prec \epsilon$ for all $n, m > N$. □

Theorem 3.1. *Let $(X, G_b^{\alpha,\beta})$ be a $G_b^{\alpha\beta}$-complex metric space and $\{x_n\}$ a sequence in X. Then the following are equivalent:*

(i) $\{x_n\}$ is complex valued $G_b^{\alpha\beta}$-convergent to $x^\star$;

(ii) $|G_b^{\alpha\beta}(x_n, x_n, x^\star)| \to 0$ as $n \to \infty$;

(iii) $|G_b^{\alpha\beta}(x_n, x^\star, x^\star)| \to 0$ as $n \to \infty$;

(iv) $|G_b^{\alpha\beta}(x_m, x_n, x^\star)| \to 0$ as $n, m \to \infty$.

Proof. (i) $\Rightarrow$ (ii) Replace m by n in Proposition 3.2.

(ii) $\Rightarrow$ (iii) By Proposition 3.1 (ii) it follows that

$$G_b^{\alpha\beta}(x_n, x^\star, x^\star) \precsim (\alpha+\beta) G_b^{\alpha\beta}(x_n, x_n, x^\star).$$

Hence, $|G_b^{\alpha\beta}(x_n, x^\star, x^\star)| \leq (\alpha+\beta)|G_b^{\alpha\beta}(x_n, x_n, x^\star)|$ and taking the limit as $n \to \infty$ the result is proved.

(iii) $\Rightarrow$ (iv) By Proposition 3.1 (i) it follows that

$$G_b^{\alpha,\beta}(x_m, x_n, x^\star) = G_b^{\alpha\beta}(x^\star, x_m, x_n) \precsim \alpha G_b^{\alpha\beta}(x^\star, x^\star, x_m) + \beta G_b^{\alpha\beta}(x^\star, x^\star, x_n),$$

which implies that

$$|G_b^{\alpha\beta}(x_m, x_n, x^\star)| \leq |G_b^{\alpha\beta}(x^\star, x_m, x_n)| \leq \alpha |G_b^{\alpha\beta}(x^\star, x^\star, x_m)| + \beta |G_b^{\alpha\beta}(x^\star, x^\star, x_n)|.$$

Taking the limit as $m, n \to \infty$ the result is obtained.

(iv)⇒(i) This is by Proposition 3.2 (ii). □

Theorem 3.2. *Let $(X, G_b^{\alpha\beta})$ be a complete complex valued $G_b^{\alpha\beta}$ metric space and $T : X \to X$ be a mapping that satisfies*

$$G_b^{\alpha\beta}(Tx, Ty, Tz) \precsim \lambda G_b^{\alpha\beta}(x, y, z), \tag{1}$$

for all $x, y, z \in X$, where $\lambda \in [0, \frac{1}{\beta})$. Then T has a unique fixed point.

Proof. Consider the sequence $\{x_n\}$ generated by $x_n = Tx_{n-1}$, where $x_0 \in X$ is arbitrary. Using (1) recursively it may be shown that

$$\begin{aligned} G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) &= G_b^{\alpha,\beta}(Tx_{n-1}, Tx_n, Tx_n) \\ &\precsim \lambda G_b^{\alpha\beta}(x_{n-1}, x_n, x_n) \\ &\precsim \lambda G_b^{\alpha\beta}(Tx_{n-2}, Tx_{n-1}, Tx_{n-1}) \\ &= \lambda^2 G_b^{\alpha\beta}(x_{n-2}, x_{n-1}, x_{n-1}) \\ &\precsim \lambda^n G_b^{\alpha\beta}(x_0, x_1, x_1). \end{aligned} \tag{2}$$

For $m, n \in \mathbb{N}$ with $m > n$ we have

$$\begin{aligned} G_b^{\alpha\beta}(x_n, x_m, x_m) &\precsim \alpha G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) + \beta G_b^{\alpha\beta}(x_{n+1}, x_m, x_m) \\ &\precsim \alpha G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) \\ &+ \beta \left[\alpha G_b^{\alpha\beta}(x_{n+1}, x_{n+2}, x_{n+2}) + \beta G_b^{\alpha\beta}(x_{n+2}, x_m, x_m)\right] \\ &= \alpha G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) + \alpha\beta G_b^{\alpha\beta}(x_{n+1}, x_{n+2}, x_{n+2}) \\ &+ \beta^2 G_b^{\alpha\beta}(x_{n+2}, x_m, x_m) \\ &\precsim \alpha G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) + \alpha\beta G_b^{\alpha\beta}(x_{n+1}, x_{n+2}, x_{n+2}) \\ &+ \alpha\beta^2 G_b^{\alpha\beta}(x_{n+2}, x_{n+3}, x_{n+3}) \\ &+ \cdots + \alpha\beta^{m-n-2} G_b^{\alpha\beta}(x_{m-2}, x_{m-2}, x_{m-1}) \\ &+ \beta^{m-n-1} G_b^{\alpha\beta}(x_{m-1}, x_m, x_m). \end{aligned} \tag{3}$$

Using (2) in (3) we obtain

$$\begin{aligned} G_b^{\alpha\beta}(x_n, x_m, x_m) &\precsim \left(\alpha\lambda^n + \alpha\beta\lambda^{n+1} + \cdots + \alpha\beta^{m-n-2}\lambda^{m-2} + \beta^{m-n-1}\lambda^{m-1}\right) \\ &\times G_b^{\alpha\beta}(x_0, x_1, x_1) \\ &\precsim \alpha\lambda^n \left(1 + \beta\lambda + \cdots + (\beta\lambda)^{m-n-2} + (\beta\lambda)^{m-n-1}\right) \\ &\times G_b^{\alpha\beta}(x_0, x_1, x_1) \\ &\precsim \frac{\alpha\lambda^n}{1 - \beta\lambda} G_b^{\alpha\beta}(x_0, x_1, x_1). \end{aligned} \tag{4}$$

Thus, we have

$$|G_b^{\alpha\beta}(x_n, x_m, x_m)| \leq \frac{\alpha\lambda^n}{1-\beta\lambda}|G_b^{\alpha\beta}(x_0, x_1, x_1)|. \tag{5}$$

Taking the limit as $n \to \infty$ we obtain $G_b^{\alpha\beta}(x_n, x_m, x_m) \to \infty$. From Proposition 3.1 we have

$$G_b^{\alpha\beta}(x_n, x_m, x_l) \precsim \alpha G_b^{\alpha\beta}(x_l, x_m, x_m) + \beta G_b^{\alpha\beta}(x_n, x_m, x_m), \tag{6}$$

for $n, m, l \in \mathbb{N}$, which yields

$$|G_b^{\alpha\beta}(x_n, x_m, x_l)| \leq \alpha|G_b^{\alpha\beta}(x_n, x_m, x_m)| + \beta|G_b^{\alpha\beta}(x_l, x_m, x_m)|. \tag{7}$$

Taking the limit as $n, m, l \to \infty$ we obtain $G_b^{\alpha\beta}(x_n, x_m, x_l) = 0$, so $\{x_n\}$ is $G_b^{\alpha\beta}$-complex Cauchy. By the completion of $(X, G_b^{\alpha\beta})$ there is $x^\star \in X$ such that $\{x_n\}$ is $G_b^{\alpha\beta}$-convergent to $x^\star$. To show that $x^\star$ is a fixed point of T consider

$$G_b^{\alpha\beta}(x_{n+1}, Tx^\star, Tx^\star) \precsim \lambda G_b^{\alpha\beta}(x_n, x^\star, x^\star),$$

which implies that

$$|G_b^{\alpha\beta}(x_{n+1}, Tx^\star, Tx^\star)| \leq \lambda|G_b^{\alpha\beta}(x_n, x^\star, x^\star)|.$$

Taking the limit as $n \to \infty$, we have

$$|G_b^{\alpha\beta}(x^\star, Tx^\star, Tx^\star)| \leq \lambda G_b^{\alpha\beta}(x^\star, x^\star, x^\star) = 0,$$

hence $G_b^{\alpha\beta}(x^\star, Tx^\star, Tx^\star) = 0$ and $Tx^\star = x^\star$. To prove uniqueness of the fixed point assume that $x^{\star\star}$ is another fixed point of T then

$$\begin{aligned} G_b^{\alpha\beta}(x^\star, x^{\star\star}, x^{\star\star}) &= G_b^{\alpha\beta}(Tx^\star, Tx^{\star\star}, Tx^{\star\star}) \\ &\precsim \lambda G_b^{\alpha\beta}(x^\star, x^{\star\star}, x^{\star\star}). \end{aligned} \tag{8}$$

Hence,

$$|G_b^{\alpha\beta}(x^\star, x^{\star\star}, x^{\star\star})| \leq \lambda|G_b^{\alpha\beta}(x^\star, x^{\star\star}, x^{\star\star})|,$$

and as $\lambda < 1$ this implies that $G_b^{\alpha,\beta}(x^\star, x^{\star\star}, x^{\star\star)} = 0$ or equivalently $x^\star = x^{\star\star}$. □

We prove Kannan's fixed point theorem for $G_b^{\alpha\beta}$-complex metric spaces.

Theorem 3.3. *Let $(X, G_b^{\alpha\beta})$ be a complete $G_b^{\alpha\beta}$-complex metric space and $T : X \to X$ satisfies*

$$G_b^{\alpha\beta}(Tx, Ty, Ty) \precsim \lambda[G_b^{\alpha\beta}(x, Tx, Tx) + G_b^{\alpha\beta}(y, Ty, Ty)], \tag{9}$$

for all $x, y \in X$, where $0 < \lambda < \frac{1}{\alpha+\beta}$. Then T has a unique fixed point.

Proof. Define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$. We shall show that $\{x_n\}$ is $G_b^{\alpha\beta}$-Cauchy. We also assume that $x_{n+1} \neq x_n$ for $n \geq 0$, otherwise x_n is a fixed point of T. It follows from (9) that

$$\begin{aligned} G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) &= G_b^{\alpha\beta}(Tx_{n-1}, Tx_n, Tx_n) \\ &\precsim \lambda[G_b^{\alpha,\beta}(x_{n-1}, Tx_{n-1}, Tx_{n-1}) + G_b^{\alpha\beta}(x_n, Tx_n, Tx_n)] \\ &= \lambda[G_b^{\alpha\beta}(x_{n-1}, x_n, x_n) + G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}]. \end{aligned} \tag{10}$$

Then

$$G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) \precsim \frac{\lambda}{1-\lambda} G_b^{\alpha\beta}(x_{n-1}, x_n, x_n). \tag{11}$$

Repeating the process we have

$$G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) \precsim \left(\frac{\lambda}{1-\lambda}\right)^n G_b^{\alpha\beta}(x_0, x_1, x_1). \tag{12}$$

Now,

$$\begin{aligned} G_b^{\alpha\beta}(x_n, x_m, x_m) &= G_b^{\alpha\beta}(T^n x_0, T^m x_0, T^m x_0) \\ &\precsim G_b^{\alpha\beta}(T^{n-1}x_0, T^n x_0, T^n x_0) + G_b^{\alpha\beta}(T^{m-1}x_0, T^m x_0, T^m x_0) \\ &= G_b^{\alpha\beta}(x_{n-1}, x_n, x_n) + G_b^{\alpha\beta}(x_{m-1}, x_m, x_m) \\ &\precsim \left[\left(\frac{\lambda}{1-\lambda}\right)^{n-1} + \left(\frac{\lambda}{1-\lambda}\right)^{m-1}\right] G_b^{\alpha\beta}(x_0, x_1, x_1). \end{aligned} \tag{13}$$

Hence, $|G_b^{\alpha\beta}(x_n, x_m, x_m)| \to 0$ as $n, m \to \infty$, which implies that $\{x_n\}$ is $G_b^{\alpha\beta}$-Cauchy in X. Since $(X, G_b^{\alpha\beta})$ is complete there exists $x^\star \in X$ such that $x_n \to x$. Consider

$$\begin{aligned} G_b^{\alpha\beta}(Tx^\star, x^\star, x^\star) &\precsim \alpha G_b^{\alpha\beta}(Tx^\star, Tx_n, Tx_n) + \beta G_b^{\alpha\beta}(Tx_n, x^\star, x^\star) \\ &\precsim \alpha\lambda[G_b^{\alpha\beta}(x^\star, Tx^\star, Tx^\star) + G_b^{\alpha\beta}(x_n, Tx_n, Tx_n)] \\ &\quad + \beta G_b^{\alpha\beta}(Tx_n, x^\star, x^\star) \\ &= \alpha\lambda[G_b^{\alpha\beta}(x^\star, Tx^\star, Tx^\star) + G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1})] \\ &\quad + \beta G_b^{\alpha\beta}(x_{n+1}, x^\star, x^\star). \end{aligned} \tag{14}$$

Taking the limit as $n \to \infty$ we have $|G_b^{\alpha\beta}(x^\star, Tx^\star, Tx^\star)| = 0$ and hence $G_b^{\alpha\beta}(x^\star, Tx^\star, Tx^\star) = 0$ as $\alpha\lambda < 1$, proving that $x^\star$ is a fixed point of T. To prove uniqueness assume that $x^{\star\star}$ is another fixed point of T, then

$$\begin{aligned} G_b^{\alpha\beta}(x^\star, x^{\star\star}, x^{\star\star}) &= G_b^{\alpha\beta}(Tx^\star, Tx^{\star\star}, Tx^{\star\star}) \\ &\precsim \lambda[G_b^{\alpha\beta}(x^\star, Tx^\star, Tx^\star) + G_b^{\alpha\beta}(x^{\star\star}, Tx^{\star\star}, Tx^{\star\star})] \\ &\precsim \lambda[G_b^{\alpha,\beta}(x^\star, x^\star, x^\star) + G_b^{\alpha\beta}(x^{\star\star}, x^{\star\star}, x^{\star\star})] \\ &\precsim 0. \end{aligned} \tag{15}$$

Hence, we conclude that $x^\star = x^{\star\star}$. □

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