Multi-monad graph

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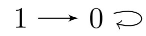
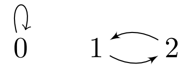
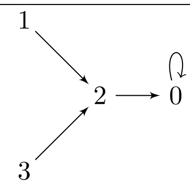
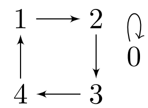
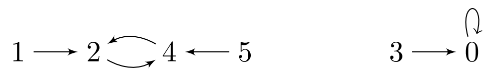
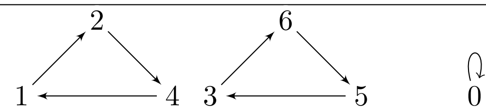
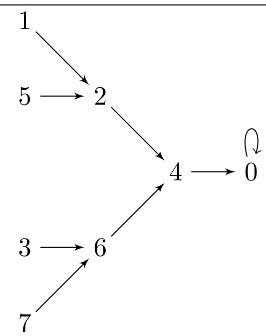
Abstract. Monad graph is a directed graph related to finite group G where every vertex of the elements of the correspond group G is adjacent with it's image by directed connected edge under the action of given map. In this paper, we introduce multi-monad graph, which is a joint union of finite number of monad graphs.

Keywords: discrete dynamical systems, Euler function, quadratic residues.

1. Introduction

Dynamics group theory is one of the most active area in group theory, which allowing understand the behavior of each elements of the given group. In [1], 2003 was introduced by V.I. Arnold a very interested phenomena termed by **monad.** Let G be finite group. **A monad** is a mapping of each element from G into itself, i.e. $f : G \to G$ for all $g \in G$. **The monad graph** $\Gamma(G)$ is adjacent every vertex of elements of the correspond G with it's image by directed connected edge under the action of f. For more details, we refer to [2], [4], [5], [6], [7]. Below, by using additive notation for the group operation, we show some monad graphs with "squaring" doubles the residue of simplest cyclic groups $n \leq 8$:

*. Corresponding author

$\Gamma(\mathbb{Z}_2)$		A_1
$\Gamma(\mathbb{Z}_3)$		$O_1 + O_2$
$\Gamma(\mathbb{Z}_4)$		T_4
$\Gamma(\mathbb{Z}_5)$		$O_1 + O_4$
$\Gamma(\mathbb{Z}_6)$		$A_1 + A_2$
$\Gamma(\mathbb{Z}_7)$		$O_1 + 2O_3$
$\Gamma(\mathbb{Z}_8)$		T_8

Let us remind about the meaning of the following symbols:

O_n : a directed cycle on n vertices;

D_n : the $4n$-vertex graph consisting of the cycle O_n of length n equipped, at each vertex, with three edges entering the vertex;

T_{2^n}, : the binary rooted tree with $2n$ vertices and n levels double branching at the levels $1, \ldots, n-1$ (exactly two edges enter each vertex at such a level); the root forms the zeroth level of the tree, and two edges enter the root as well: one from itself, and the other from the unique vertex of the first level.

In short, monad is dynamical system contains the following elements: finite action group G, monad function f and monad graph $\Gamma(G)$. In [1] was considered a special case where G is a finite group and f is squaring map. Result of this kind is the following:

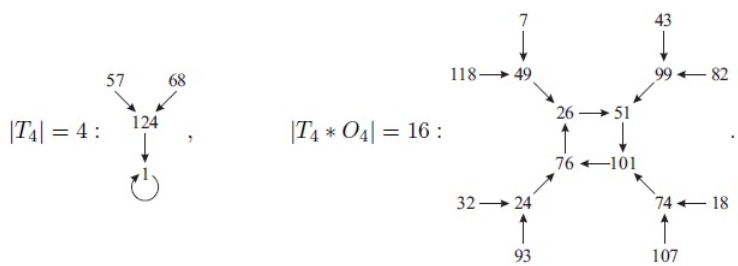
Theorem 1.1 ([1]). *Each connected component of a monad graph is a forest of rooted trees directed towards their roots, which lie on a directed cycle formed by the edges connecting the roots.*

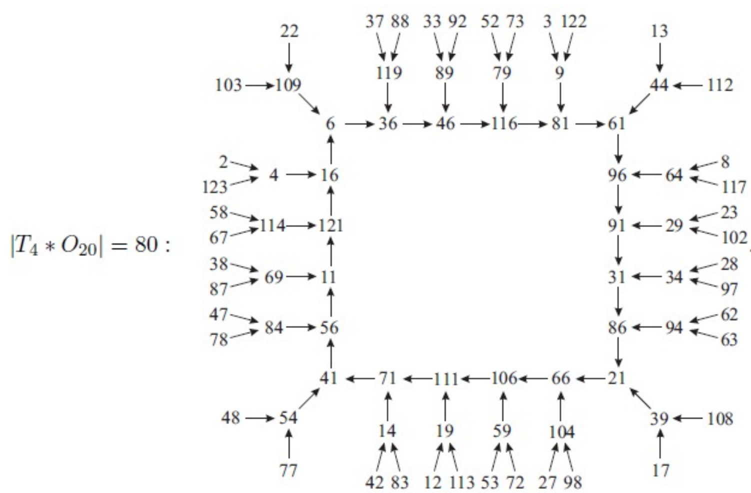
Moreover, Arnold showed that if $\gamma > 0$ and η is odd number, then the monad graphs defined by a finite commutative groups G_n of order $n = 2^\gamma \times \eta$, are sums of products of monads of the form T_{2^γ} and O_η. Notice that the number of cycles does not exceed n.

To show particular details, let's consider $\mathbb{Z}_{100} = \mathbb{Z}_4 \times \mathbb{Z}_{25}$ (see, [1]). The monad graph of $\Gamma(\mathbb{Z}_4) = T_{2^2}$. For $\mathbb{Z}_{25}$, we begin from calculating the graphs of the factors. The monad graph of $\mathbb{Z}_{25}$ of odd order splits into cycles of orders 1, 4 and 20. Hence the monad of $\mathbb{Z}_{25}$ is $O_1 + O_4 + O_{20}$. Therefore, the product has the form

$$\Gamma(\mathbb{Z}_{100}) = T_4 + (T_4 * O_4) + (T_4 * O_{20}) \tag{1}$$

and





V.I. Arnold also proved that in the case of non commutative finite, the trees framing each attracting cycle are homogeneous, but it is unknown if the framing trees have some special structure:

Homogeneity theorem. Given a component of the squaring monad $g \to g^2$ of any finite group, the framing trees of the attracting cycle are homogeneous (the rooted trees attracted by the vertices of the cycle are all isomorphic – as directed graphs – along that component).

Now, let n be a positive integer, the Euler group $E(n)$ is the multiplicative group of coprime residue classes modulo n. Nontrivial Euler groups begins from

$$E(2) = \{1\},\ E(3) \approx \mathbb{Z}_2,\ E(5) \approx \mathbb{Z}_4,\ E(6) \approx \mathbb{Z}_2,$$
$$E(7) \approx \mathbb{Z}_6,\ E(8) \approx \mathbb{Z}_2 \times \mathbb{Z}_2\ E(9) \approx \mathbb{Z}_6, E(13) \approx \mathbb{Z}_{12}.$$

In [3], was obtained that he Euler group $E(n)$ is cyclic whenever n is a prime or a degree of q prime such as

$$E(p^q) \approx \mathbb{Z}_{\alpha(p^q)},$$

where $\alpha(p^q) = (p-1)p^{q-1}$.

In the following table, we show the monad graphs defined by the Euler groups $E(n)$, $n \leq 22$ with multiplication operation:

n	$\Gamma(E(n))$	n	$\Gamma(E(n))$
2	O_1	12	D_1
3	A_1	13	$T_4 + (T_4 * O_2)$
4	A_1	14	$A_1 + A_2$
5	T_4	15	$T_4 * A_1$
6	A_1	16	$T_4 * A_1$
7	$A_1 + A_2$	17	T_{16}
8	D_1	18	$A_1 + A_2$
9	$A_1 + A_2$	19	$A_1 + A_2 + A_6$
10	T_4	20	$T_4 * A_1$
11	$A_1 + A_4$	21	$A_1 + A_2 + D_1 + D_2$

Table 2.2. The monad graph defined by Euler group

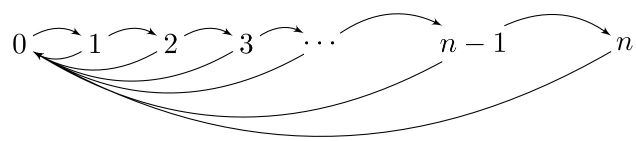
After brief introduction to monad graphs, let us introduce the following system:

Definition 1.1. *Let G be finite group and $f : G \to G$ for $g \in G$. The graph generated by the following system:*

$$\Gamma(G_1, f) \cup \Gamma(G_2, f) \cup \cdots \cup \Gamma(G_n, f) = \bigcup_{i=1}^{n} \Gamma(G_i, f) := \Psi_{n,f}$$

we called **multi-monad graph** *and denoted by* $\Gamma(\Psi_{n,f})$.

Obviously, if f is a shift function, i.e. $f_+(g) = g + 1$ for all $g \in G$, then the multi-monad $\Gamma(\Psi_{n,f_+})$ can be represented by $\bigcup_{i=2}^{n} O_i$ such as:



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In this paper, we are interested in studying the properties of multi-monad graph with respect to a squaring map.

2. Main result

To obtain our result we will review some supportive material:

Definition 2.1. *The graph of the product monad will be called the product of the graphs of the factors; the multiplication of graphs will be denoted by the same symbol* $*$*:*

$$[graph(A * B)] = [graph(A)] * [graph(B)].$$

Lemma 2.1 ([1])**.** *The graph of multiplication in an additive cyclic group of odd order is a disjoint union of cycles* O_n*.*

Lemma 2.2 ([1])**.** *The graph of multiplication in the additive cyclic group of residues modulo* 2^n *is the binary rooted tree* T_{2^n}*.*

The combinatorics of squaring monad graph showed that even the simplest algebraic theories has complicated topological structure while the combinatorics of squaring multi-monad graph shows more complicated structure and its dynamical behavior is unpredictable (chaotic behavior), as we show in next example:

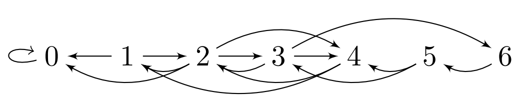
Example 2.1. Let f be a squaring map and

$$\Psi_{7,f} = \Gamma(Z_2, f) \cup (Z_3, f) \cup \cdots \cup \Gamma(Z_7, f).$$

Then, we have

$$\Gamma(\Psi_{7,f}) = A_1 \cup (O_1 + O_2) \cup T_4 \cup (O_1 + O_4) \cup (A_1 + A_2) \cup (O_1 + 2O_3)$$

and $\Gamma(\Psi_{7,f})$ is pertaining as:



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By $T_{2^r,\tau,\zeta}$ we denote the binary rooted trees T_{2^r}, which contain a troubled branches, i.e. if there are some edges connecting the branches at levels $1, \ldots, n-1$ and there are some edges from levels $n > 1$ entering the root except one from itself, and the other from the unique vertex of the first level. In other words, troubled binary rooted trees $T_{2^r,\tau,\zeta}$ contain τ, which is the number of edges connecting between the tree branches and ζ which is the number of edges entering the root (except one from itself, and the other from the unique vertex of the first level).

By T_{2^r,τ,ζ,O_i} we denote troubled binary rooted tree which included a cycle O_i of finite order.

2.1 Algorithm of multi-monad graphs

In particular, we will compute the algorithm of multi-monad graphs generated by the following system

$$\Psi_{2^r,f} = \bigcup_{m=1}^{r} \Gamma(\mathbb{Z}_{2^m}), \quad r > 0. \tag{2}$$

Theorem 2.1. *Let $\Psi_{n,f}$ is given by (2). Then, the multi-monad graph pertaining to system $\Psi_n(\mathbb{Z}_n)$ is $T_{2^r,\tau,\zeta}$.*

Proof. From Lemma 2, we have that each group of $\Psi_n(\mathbb{Z}_n) = \bigcup_{m=1}^{r} \mathbb{Z}_{2^m}$ pertaining a binary tree T_{2^n}. According to joint union of them, one can see that each element belongs to $\mathbb{Z}_{2^{r-1}} \bigcap \mathbb{Z}_{2^r}$ has two action under the considered map with one image obtained by the residue of 2^{r-1} and another image, which belongs to the group of 2^r. This implies the following algorithm:

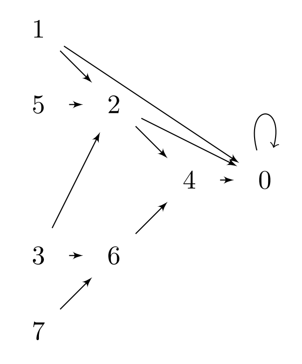
- each vertex of form 2^k for $k = 1, 2, \ldots, r$ have a set of entering edges defined as:
$$E(\Gamma|_{2^k}) = \bigcup_{k=1}^{r-1}\{ \bigcup_{j=k-1}^{r-2} \{2^k, 2^k + \sum_{d=k-1}^{j} 2^d\}\}.$$

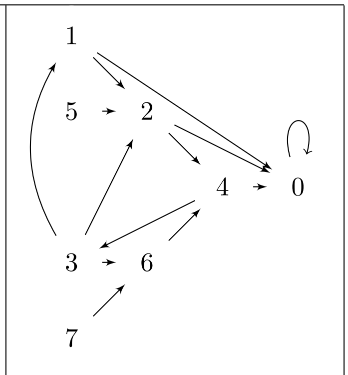
- each vertex of form $2^k m$ for $k = 1, 2, \ldots, r$ and m is odd, have two sets of outgoing edges defined as:
$$E(\Gamma|_{2^k m}) = \{\{2^k m, 2^k m + 2^{k-1}\}, \{2^k m, 2^{k-1} * m\}\}, \ k, m > 1.$$

Its easy to check that some vertices has two outgoing edges belong to $E(\Gamma|_{2^k m})$, which connect to vertices in groups of 2^k and 2^k+1 for $k = 1, 2, \ldots, r$, i.e. connecting the branches of T_{2^k} and $T_{2^{k+1}}$. Moreover, each vertex of form 2^k have zero residue in each group of (2), so they has outgoing edge to zero, which proves the theorem. $\square$

Theorem 2.2. *Let $\Psi_{n,f} = \left[\bigcup_{m=1}^{r} \Gamma(\mathbb{Z}_{2^m})\right] \cup \Gamma(\mathbb{Z}_p)$ where p is odd number. Then the multi-monad graph pertaining to system $\Psi_n(\mathbb{Z}_n)$ is T_{2^r,τ,ζ,O_i}, where O_i be cyclic of order $i \leq p$.*

Directly from Lemma 2.1 and Theorem 2.1





$\Gamma(\Psi(\bigcup_{m=1}^{3}\mathbb{Z}_{2^m}))$	$\Gamma(\Psi([\bigcup_{m=1}^{3}\mathbb{Z}_{2^m}]\cup\mathbb{Z}_5))$
$T_{2^3,1,2}$	$T_{2^3,1,2,O_4}$

Table 2. The multi-monad graphs of Theorems 2.1. and 2.2

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