Schur power convex functions with applications to a class of conditional inequalities

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Abstract. In this paper, by the use of Schur power convexity, we prove some inequalities involving the condition that the sum of powers of variables is a fixed value. It is shown that the approach proposed is useful for establishing certain conditional inequalities.
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1. Introduction

Schur convexity was introduced by Schur in 1923, which brought out new perspectives for researchers to study the convexity of functions. In particular, the Schur-Ostrowski theorem provides a distinct characterization of Schur convex functions through local two-variables conditions (see [1]). As is known to us, Schur convexity has many important applications in linear regression, graphs,

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matrices, combinatorial optimization, special functions, stochastic orderings, reliability, and other related fields (see [2, 3]). In recent years, the applications of Schur convexity and majorization to discover and prove various kinds of analytic inequalities has been a hot research topic. Also, the Schur convexity was generalized and extended to a variety of types by adding several parameters. Among these generalization results, Schur power convexity is a class of useful convexity and has attracted great attention and interest from researchers (see [4–14]).

It is our usual practice that the conditional inequalities are processed by the Lagrangian multiplier method, but it often results in complicated computations (see [15–19]). In this paper, we provide a novel and concise method to deal with conditional inequalities, more specifically, we will use Schur power convexity to prove some inequalities involving the condition that the sum of powers of variables is a fixed value.

We begin with introducing some essential definitions and lemmas. Throughout the paper, we assume that the set of n-dimensional row vector by $\mathbb{R}^n$.

$$\mathbb{R}^n_+ = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0,\ i = 1, 2, \ldots, n\},$$

$$\mathbb{R}^n_{++} = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i > 0,\ i = 1, 2, \ldots, n\}.$$

In particular, $\mathbb{R}^1$, $\mathbb{R}^1_+$ and $\mathbb{R}^1_{++}$ denoted by $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_{++}$ respectively.

Definition 1.1 ([1, 2]). *Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.*

(*i*) *The vector $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x} \prec \boldsymbol{y}$) if $\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}$ for $k = 1, 2, \ldots, n-1$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, where $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq y_{[2]} \geq \cdots \geq y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.*

(*ii*) *A set $\Omega \subset \mathbb{R}^n$ is called a convex set if $(\alpha x_1+\beta y_1, \alpha x_2+\beta y_2, \ldots, \alpha x_n+\beta y_n) \in \Omega$ for any $\boldsymbol{x}$ and $\boldsymbol{y} \in \Omega$, where α and $\beta \in [0, 1]$ with $\alpha + \beta = 1$.*

(*iii*) *Let $\Omega \subset \mathbb{R}^n$, φ: $\Omega \to \mathbb{R}$ is said to be a Schur-convex function on Ω if $\boldsymbol{x} \prec \boldsymbol{y}$ on Ω implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. A function φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is a Schur-convex function.*

Definition 1.2 ([3]). *Let $\Omega \subset \mathbb{R}^n_+$.*

(*i*) *A set Ω is said to be a harmonically convex set if $\frac{\boldsymbol{x}\boldsymbol{y}}{\lambda\boldsymbol{x}+(1-\lambda)\boldsymbol{y}} \in \Omega$ for every $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ and $\lambda \in [0, 1]$, where $\frac{1}{\boldsymbol{x}} = \left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}\right)$ and*

$$\frac{\boldsymbol{x}\boldsymbol{y}}{\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}} = \Big(\frac{x_1 y_1}{\lambda x_1 + (1-\lambda) y_1}, \frac{x_2 y_2}{\lambda x_2 + (1-\lambda) y_2}, \cdots, \frac{x_n y_n}{\lambda x_n + (1-\lambda) y_n}\Big).$$

(*ii*) *A function $\varphi : \Omega \to \mathbb{R}_+$ is said to be a Schur harmonically convex function on Ω if $\frac{1}{\boldsymbol{x}} \prec \frac{1}{\boldsymbol{y}}$ implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. A function φ is said to be a Schur harmonically concave function on Ω if and only if $-\varphi$ is a Schur harmonically convex function.*

Definition 1.3 ([4]). *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a function defined by*

$$f(x) = \begin{cases} \dfrac{x^m - 1}{m}, & m \neq 0; \\ \ln x, & m = 0. \end{cases} \tag{1}$$

Then a function $\phi : \Omega \subset \mathbb{R}^n_+ \to \mathbb{R}$ is said to be Schur m-power convex on Ω if

$$(f(x_1), f(x_2), \ldots, f(x_n)) \prec (f(y_1), f(y_2), \ldots, f(y_n)),$$

for all $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \Omega$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \Omega$ implies $\phi(\boldsymbol{x}) \leq \phi(\boldsymbol{y})$.

If $-\phi$ is Schur m-power convex, then we say that ϕ is Schur m-power concave.

If we put $f(x) = x$ and $f(x) = \frac{1}{x}$ in Definition 1.3, respectively, then the concepts of Schur-convex function and Schur-harmonically convex function can be derived.

Lemma 1.1 ([1, 2]). *Suppose that $\Omega \subset \mathbb{R}^n$ is convex set, and has a nonempty interior set Ω°. Let $\varphi : \Omega \to \mathbb{R}$ be continuous on Ω and differentiable in Ω°. Then φ is the Schur-convex (or Schur-concave; respectively) function, if and only if it is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_2}\right) \geq 0(\text{ or} \leq 0;\ \text{respectively})$$

holds for any $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \Omega^\circ$.

Lemma 1.2 ([3]). *Let $\Omega \subset \mathbb{R}^n_+$ be a symmetric harmonically convex set with a nonempty interior Ω°. Let $\varphi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable on Ω°. Then φ is a Schur harmonically convex (or Schur harmonically concave; respectively) function if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left(x_1^2 \frac{\partial \varphi}{\partial x_1} - x_2^2 \frac{\partial \varphi}{\partial x_2}\right) \geq 0 \quad (\text{or } \leq 0;\ \text{respectively})$$

holds for any $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \Omega^\circ$.

Lemma 1.3 ([4]). *Let $\Omega \subset \mathbb{R}^n_+$ be a symmetric set with nonempty interior Ω°, and $\varphi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable in Ω°. Then φ is Schur m-power convex on Ω if and only if φ is symmetric on Ω and*

$$\frac{x_1^m - x_2^m}{m}\left[x_1^{1-m}\frac{\partial \varphi(\boldsymbol{x})}{\partial x_1} - x_2^{1-m}\frac{\partial \varphi(\boldsymbol{x})}{\partial x_2}\right] \geq 0, \quad \text{if } m \neq 0 \tag{2}$$

and

$$(\ln x_1 - \ln x_2)\left[x_1\frac{\partial \varphi(\boldsymbol{x})}{\partial x_1} - x_2\frac{\partial \varphi(\boldsymbol{x})}{\partial x_2}\right] \geq 0, \quad \text{if } m = 0 \tag{3}$$

for all $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \Omega^\circ$.

Lemma 1.4 ([1, 2]). *Let* $\boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n_+$ *and* $A_n(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^n x_i$. *Then*

$$\mathbf{u} = \Big(\underbrace{A_n(\boldsymbol{x}), A_n(\boldsymbol{x}), \cdots, A_n(\boldsymbol{x})}_{n}\Big) \prec (x_1, x_2, \cdots, x_n) = \boldsymbol{x}. \tag{4}$$

For $m \neq 0$, we have

$$\left(\frac{1}{n}\sum_{i=1}^n \frac{x_i^m - 1}{m}, \dots, \frac{1}{n}\sum_{i=1}^n \frac{x_i^m - 1}{m}\right) \prec \left(\frac{x_1^m - 1}{m}, \dots, \frac{x_n^m - 1}{m}\right)$$

$\Longleftrightarrow$

$$\left(\frac{\left(\left(\frac{1}{n}\sum_{i=1}^n x_i^m\right)^{\frac{1}{m}}\right)^m - 1}{m}, \dots, \frac{\left(\left(\frac{1}{n}\sum_{i=1}^n x_i^m\right)^{\frac{1}{m}}\right)^m - 1}{m}\right) \prec \left(\frac{x_1^m - 1}{m}, \dots, \frac{x_n^m - 1}{m}\right),$$

As a consequence of Lemma 1.4, we have the following result:

Lemma 1.5. *If* φ *is Schur* m*-power convex function (or Schur* m*-power concave function; respectively) on* Ω*, then*

$$\varphi\left(M_m(\boldsymbol{x}), M_m(\boldsymbol{x}), \dots, M_m(\boldsymbol{x})\right) \leq (or \geq; respectively)\ \varphi\left(x_1, x_2, \dots, x_n\right), \tag{5}$$

where $M_m(\boldsymbol{x}) = \left(\frac{1}{n}\sum_{i=1}^n x_i^m\right)^{\frac{1}{m}}$ *is the power mean of* $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$.

2. Main results

Theorem 2.1. *Let* n *be a natural number, and let* a, b *be positive real numbers with* $a^{-1} + b^{-1} = 1$. *Then*

$$(a+b)^n - a^n - b^n \geq 2^{2n} - 2^{n+1}. \tag{6}$$

Proof. The assumption condition, $a^{-1} + b^{-1} = 1$, implies that $a + b = ab$.

Let

$$\varphi(a, b) = (a+b)^n - a^n - b^n.$$

Then

$$\frac{\partial\varphi}{\partial a} = n(a+b)^{n-1} - na^{n-1}, \quad \frac{\partial\varphi}{\partial b} = n(a+b)^{n-1} - nb^{n-1}.$$

We have

$$\begin{aligned} &(a-b)\left(a^2\frac{\partial\varphi}{\partial a} - b^2\frac{\partial\varphi}{\partial b}\right) \\ =&n(a-b)\left[(a^2-b^2)(a+b)^{n-1} - (a^{n+1} - b^{n+1})\right] \\ =&n(a-b)^2\left[(a+b)^n - (a^n + a^{n-1}b + \cdots + ab^{n-1} + b^n)\right] \\ \geq&0. \end{aligned}$$

Thereby $\varphi(a,b)$ is a Schur-harmonically convex function on $\mathbb{R}^2_{++}$. In the light of Lemma (1.4), one has

$$\Big(\frac{a^{-1}+b^{-1}}{2},\frac{a^{-1}+b^{-1}}{2}\Big)\prec\left(a^{-1},b^{-1}\right),$$

then by $a^{-1}+b^{-1}=1$, we get $\left(\frac{1}{2},\frac{1}{2}\right)\prec\left(\frac{1}{a},\frac{1}{b}\right)$. From the Schur-harmonically convexity, it follows that $\varphi(2,2)\le\varphi(a,b)$, which implies the desired inequality (6) in Theorem 2.1. □

Similarly, we can prove the following generalization of Theorem 2.1 in the case of m-dimension.

Theorem 2.2. *Let $x_i>0$ $(i=1,2,\dots,m)$ and $\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_m}=1$. Then*

$$(x_1+x_2+\cdots+x_m)^n-x_1^n-x_2^n-\cdots-x_m^n\ge m^{2n}-m^{n+1}. \tag{7}$$

Theorem 2.3. *Let a,b,c be positive real numbers with $a^2+b^2+c^2=1$. Then*

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\ge\frac{2(a^3+b^3+c^3)}{abc}+3. \tag{8}$$

Proof. Let $\varphi(a,b,c)=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}-\frac{2(a^3+b^3+c^3)}{abc}-3$. Then

$$\frac{\partial\varphi}{\partial a}=\frac{-2}{a^3}-\frac{6a^3bc-2(a^3+b^3+c^3)bc}{a^2b^2c^2},\ \frac{\partial\varphi}{\partial b}=\frac{-2}{b^3}-\frac{6ab^3c-2(a^3+b^3+c^3)ac}{a^2b^2c^2},$$

$$\begin{aligned}\Delta:=&\frac{a^2-b^2}{2}\left(\frac{1}{a}\cdot\frac{\partial\varphi}{\partial a}-\frac{1}{b}\cdot\frac{\partial\varphi}{\partial b}\right)\\ =&(a^2-b^2)\left[\frac{a^4-b^4}{a^4b^4}+\frac{1}{a^2b^2c^2}\Big(\frac{c(a^2+b^2+c^2)(a^2-b^2)-3a^2b^2c(a-b)}{ab}\Big)\right]\\ =&(a+b)(a-b)^2\left[\frac{a^3+a^2b+ab^2+b^3}{a^4b^4}+\frac{1}{a^2b^2c}\Big(\frac{a+b-3a^2b^2}{ab}\Big)\right].\end{aligned}$$

From $a^2+b^2+c^2=1$, we get $2ab\le a^2+b^2<1$, thus $a+b-3a^2b^2\ge 2\sqrt{ab}-\frac{3}{2}ab=\sqrt{ab}(2-\frac{3}{2}\sqrt{ab})>0$. We conclude that $\Delta>0$. Therefore, φ is the Schur 2-power convex function on $\mathbb{R}^3_{++}$. Now, using inequality (5), it follows that

$$\begin{aligned}\varphi(a,b,c)\ge&\varphi\Big(\Big(\frac{a^2+b^2+c^2}{3}\Big)^{\frac{1}{2}},\Big(\frac{a^2+b^2+c^2}{3}\Big)^{\frac{1}{2}},\Big(\frac{a^2+b^2+c^2}{3}\Big)^{\frac{1}{2}}\Big)\\ =&\varphi(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})=0.\end{aligned}$$

This proves the required inequality in Theorem 2.3. □

Theorem 2.4. *Let x, y, z be positive real numbers with $x^2 + y^2 + z^2 = 1$. Then*

$$x^2yz + xy^2z + xyz^2 \le \frac{1}{3}. \tag{9}$$

Proof. Let $\varphi(x, y, z) = x^2yz + xy^2z + xyz^2$. Then

$$\begin{aligned}\Delta := &\frac{x^2-y^2}{2}\left(\frac{1}{x}\cdot\frac{\partial\varphi}{\partial x} - \frac{1}{y}\cdot\frac{\partial\varphi}{\partial y}\right)\\ =&\frac{x^2-y^2}{2}\left[\frac{1}{x}(2xyz + y^2z + yz^2) - \frac{1}{y}(x^2z + 2xyz + xz^2)\right]\\ =&\frac{x^2-y^2}{2xy}\left[-2xyz(x-y) - z(x^3-y^3) - z^2(x^2-y^2)\right].\end{aligned}$$

Note that $(x^2 - y^2)(x^k - y^k) \ge 0$ for $k = 1, 2, 3$, we obtain $\Delta \le 0$. Thus φ is the Schur 2-power concave function on $\mathbb{R}^3_{++}$. Now, utilizing inequality (5), we have

$$\varphi(x, y, z) \le \varphi\left(M_2(x, y, z), M_2(x, y, z), M_2(x, y, z)\right) = \varphi\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{1}{3}.$$

This completes the proof of Theorem 2.4. □

Theorem 2.5. *Let x, y be nonnegative real numbers with $x^2 + y^2 = 1$. Then*

$$\frac{x}{1+y} + \frac{y}{1+x} \ge 2(\sqrt{2} - 1). \tag{10}$$

Proof. Let $\varphi(x, y) = \frac{x}{1+y} + \frac{y}{1+x}$. Then

$$\begin{aligned}&\frac{x^2-y^2}{2}\left(\frac{1}{x}\cdot\frac{\partial\varphi}{\partial x} - \frac{1}{y}\cdot\frac{\partial\varphi}{\partial y}\right)\\ =&\frac{x^2-y^2}{2}\left[\frac{1}{x}\left(\frac{1}{1+y} - \frac{y}{(1+x)^2}\right) - \frac{1}{y}\left(\frac{1}{1+x} - \frac{x}{(1+y)^2}\right)\right]\\ =&\frac{(x-y)^2(x+y)}{2}\left[\frac{x^2+xy+y^2+x+y}{xy(1+y)^2(1+x)^2}\right]\\ \ge&0.\end{aligned}$$

Hence φ is the Schur 2-power convex function on $\mathbb{R}^2_+$. By means of inequality (5), we derive that

$$\varphi(x, y) \ge \varphi\left(M_2(x, y), M_2(x, y)\right) = \varphi\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 2(\sqrt{2} - 1),$$

which implies the inequality asserted by Theorem 2.5. □

Theorem 2.6. *Let x, y be positive real numbers, and n be positive integer. If $x^{2n+1} + y^{2n+1} \geq 2$, then*

$$x^{n+1} + y^{n+1} \geq x^n + y^n. \tag{11}$$

Proof. Let $\varphi(x, y) = x^{n+1} + y^{n+1} - x^n - y^n$.

Case (I). If $x^{2n+1} + y^{2n+1} = 2$, then

$$(xy)^{n+\frac{1}{2}} \leq \frac{x^{2n+1} + y^{2n+1}}{2} = 1 \iff xy \leq 1.$$

Note that

$$\frac{\partial \varphi}{\partial x} = (n+1)x^n - nx^{n-1}, \quad \frac{\partial \varphi}{\partial y} = (n+1)y^n - ny^{n-1}.$$

$$\begin{aligned}
\Delta := & \frac{x^{2n+1} - y^{2n+1}}{2n+1} \left(\frac{1}{x^{2n}} \frac{\partial \varphi}{\partial x} - \frac{1}{y^{2n}} \frac{\partial \varphi}{\partial y} \right) \\
= & \frac{x^{2n+1} - y^{2n+1}}{2n+1} \left[(n+1) \left(\frac{1}{x^n} - \frac{1}{y^n} \right) - n \left(\frac{1}{x^{n+1}} - \frac{1}{y^{n+1}} \right) \right] \\
= & \frac{x^{2n+1} - y^{2n+1}}{2n+1} \left[\frac{n(x^{n+1} - y^{n+1}) - (n+1)xy(x^n - y^n)}{x^{n+1}y^{n+1}} \right].
\end{aligned}$$

To prove $\Delta \geq 0$, it is enough to show that $n(x^{n+1} - y^{n+1}) \geq (n+1)xy(x^n - y^n)$ for $x > y$, namely,

$$\frac{1}{xy} E(n, n+1; x, y) = \frac{1}{xy} \left(\left(\frac{n}{n+1} \right) \frac{x^{n+1} - y^{n+1}}{x^n - y^n} \right)^{\frac{1}{(n+1)-n}} \geq 1.$$

On the other hand, it was proved in ([7], Theorem 1.1) that $E(n, n+1; x, y)$ is Schur convex on $\mathbb{R}^2_{++}$, therefore

$$E(n, n+1; x, y) \geq E\left(n, n+1; \frac{x+y}{2}, \frac{x+y}{2}\right) = \frac{x+y}{2},$$

this yields

$$\frac{1}{xy} E(n, n+1; x, y) \geq \frac{\frac{x+y}{2}}{xy} \geq \frac{\sqrt{xy}}{xy} = \frac{1}{\sqrt{xy}} \geq 1.$$

Thus φ is the Schur $(2n+1)$-power convex function on $\mathbb{R}^2_{++}$. By employing inequality (5), we have

$$\varphi(x, y) \geq \varphi\left(\left(\frac{x^{2n+1} + y^{2n+1}}{2} \right)^{\frac{1}{2n+1}}, \left(\frac{x^{2n+1} + y^{2n+1}}{2} \right)^{\frac{1}{2n+1}} \right) = \varphi(1, 1) = 0.$$

Case (II). If $x^{2n+1}+y^{2n+1}>2$, setting $x^{2n+1}+y^{2n+1}=a$ $(a>2)$, then

$$\left[\left(\frac{2}{a}\right)^{\frac{1}{2n+1}}x\right]^{2n+1}+\left[\left(\frac{2}{a}\right)^{\frac{1}{2n+1}}y\right]^{2n+1}=2.$$

By virtue of (I), we get

$$\left[\left(\frac{2}{a}\right)^{\frac{1}{2n+1}}x\right]^{n+1}+\left[\left(\frac{2}{a}\right)^{\frac{1}{2n+1}}y\right]^{n+1}\geq\left[\left(\frac{2}{a}\right)^{\frac{1}{2n+1}}x\right]^{n}+\left[\left(\frac{2}{a}\right)^{\frac{1}{2n+1}}y\right]^{n},$$

that is

$$x^{n+1}\left(\frac{2}{a}\right)^{\frac{n+1}{2n+1}}+y^{n+1}\left(\frac{2}{a}\right)^{\frac{n+1}{2n+1}}\geq x^{n}\left(\frac{2}{a}\right)^{\frac{n}{2n+1}}+y^{n}\left(\frac{2}{a}\right)^{\frac{n}{2n+1}},$$

this yields

$$x^{n+1}\left(\frac{2}{a}\right)^{\frac{1}{2n+1}}+y^{n+1}\left(\frac{2}{a}\right)^{\frac{1}{2n+1}}\geq x^{n}+y^{n},$$

further, we obtain $x^{n+1}+y^{n+1}>x^n+y^n$ with the help of $\frac{2}{a}<1$. The proof of Theorem 2.6 is complete. □

Theorem 2.7. *Let a,b,c be positive real numbers with $a^3+b^3+c^3=3$. Then*

$$\frac{1}{a^2+a+1}+\frac{1}{b^2+b+1}+\frac{1}{c^2+c+1}\geq 1. \tag{12}$$

Proof. Let $\varphi(a,b,c)=\frac{1}{a^2+a+1}+\frac{1}{b^2+b+1}+\frac{1}{c^2+c+1}$. Then

$$\begin{aligned}\Lambda :=&\frac{a^3-b^3}{3}\left(\frac{1}{a^2}\cdot\frac{\partial\varphi}{\partial a}-\frac{1}{b^2}\cdot\frac{\partial\varphi}{\partial b}\right)\\ =&-\frac{a^3-b^3}{3}\left[\frac{2a+1}{a^2(a^2+a+1)^2}-\frac{2b+1}{b^2(b^2+b+1)^2}\right]\\ =&\frac{(a-b)^2(a^2+ab+b^2)}{3}\Big[\frac{Q(a,b)}{a^2b^2(a^2+a+1)^2(b^2+b+1)^2}\Big],\end{aligned}$$

where

$$Q(a,b)=a+b+7ab^2+7a^2b+8ab^3+8a^3b+5ab^4+5a^4b+2ab^5+2a^5b+8a^2b^2+5a^2b^3$$
$$+5a^3b^2+2a^2b^4+2a^3b^3+2a^4b^2+4ab+2a^2+3a^3+2a^4+2b^2+a^5+3b^3+2b^4+b^5.$$

Hence, $\Lambda\geq 0$, we deduce that $\varphi(a,b,c)$ is Schur 3-power convex on $\mathbb{R}^3_{++}$. Using inequality (5) gives

$$\varphi(a,b,c)\geq\varphi\Big(M_3(a,b,c),M_3(a,b,c),M_3(a,b,c)\Big)=\varphi(1,1,1)=1.$$

This completes the proof of Theorem 2.7. □

Theorem 2.8. *If* $a, b, c > 0$ *and* $a^2 + b^2 + c^2 = 1$, *then*

$$a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 4\sqrt{3}. \tag{13}$$

Proof. Let $\varphi(a, b, c) = a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. Then

$$\begin{aligned}\Lambda :&= \frac{a^2 - b^2}{2}\left(\frac{1}{a} \cdot \frac{\partial\varphi}{\partial a} - \frac{1}{b} \cdot \frac{\partial\varphi}{\partial b}\right) \\ &= \frac{a^2 - b^2}{2}\left[\left(\frac{1}{a} - \frac{1}{a^3}\right) - \left(\frac{1}{b} - \frac{1}{b^3}\right)\right] \\ &= \frac{(a-b)^2(a+b)}{2a^3b^3}\left(a^2 + ab + b^2 - a^2b^2\right).\end{aligned}$$

From the condition $a^2 + b^2 + c^2 = 1$, it follows that $2ab \leq a^2 + b^2 < 1$. Hence $a^2 + ab + b^2 - a^2b^2 \geq 3ab - a^2b^2 = ab(3 - ab) > 0$. Then we get $\Lambda \geq 0$. Therefore, φ is the Schur 2-power convex function on $\mathbb{R}^3_{++}$. Applying inequality (5), we obtain

$$\varphi(a, b, c) \geq \varphi\left(M_2(a, b, c), M_2(a, b, c), M_2(a, b, c)\right) = \varphi\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = 4\sqrt{3},$$

which implies the required inequality in Theorem 2.8. □

Theorem 2.9. *Let* $x_i > 0$ $(i = 1, 2, \ldots, n)$ *with* $x_1^2 + x_2^2 + \cdots + x_n^2 = n$. *Then*

$$\frac{x_1}{x_1 + n} + \frac{x_2}{x_2 + n} + \cdots + \frac{x_n}{x_n + n} \leq \frac{n}{n+1}. \tag{14}$$

Proof. Let $\varphi(x_1, x_2, \ldots, x_n) = \frac{x_1}{x_1+n} + \frac{x_2}{x_2+n} + \cdots + \frac{x_n}{x_n+n}$. Then

$$\begin{aligned}\Lambda :&= \frac{x_1^2 - x_2^2}{2}\left(\frac{1}{x_1} \cdot \frac{\partial\varphi}{\partial x_1} - \frac{1}{x_2} \cdot \frac{\partial\varphi}{\partial x_2}\right) \\ &= \frac{x_1^2 - x_2^2}{2}\left[\frac{n}{x_1(x_1+n)^2} - \frac{n}{x_2(x_2+n)^2}\right] \\ &= \frac{-n(x_1 - x_2)^2(x_1 + x_2)(2nx_1 + 2nx_2 + x_1^2 + x_2^2 + x_1x_2 + n^2)}{2x_1x_2(x_1+n)^2(x_2+n)^2} \\ &\leq 0.\end{aligned}$$

Thus φ is the Schur 2-power concave function on $\mathbb{R}^n_{++}$. By virtue of inequality (5), it follows that

$$\begin{aligned}\varphi(x_1, x_2, \ldots, x_n) &\leq \varphi\left(M_2(x_1, x_2, \ldots, x_n), \ldots, M_2(x_1, x_2, \ldots, x_n)\right) \\ &= \varphi(1, 1, \ldots, 1) = \frac{n}{n+1}.\end{aligned}$$

The Theorem 2.9 is proved. □

Theorem 2.10. *Let a, b, c, m, n be positive real numbers with $\sqrt{a}+\sqrt{b}+\sqrt{c}=3$ and $m \geq 2n$. Then*

$$(ab)^{\frac{n}{m}} + (bc)^{\frac{n}{m}} + (ca)^{\frac{n}{m}} \leq 3. \tag{15}$$

Proof. Setting $a = x^{2m}, b = y^{2m}, c = z^{2m}$, then $x^m + y^m + z^m = 3$. We just need to prove that

$$x^{2n}y^{2n} + y^{2n}z^{2n} + z^{2n}x^{2n} \leq 3. \tag{16}$$

Let $\varphi(x, y, z) = x^{2n}y^{2n} + y^{2n}z^{2n} + z^{2n}x^{2n}$. Since

$$\begin{aligned}\Lambda :&= \frac{x^m - y^m}{m}\left(\frac{1}{x^{m-1}} \cdot \frac{\partial \varphi}{\partial x} - \frac{1}{y^{m-1}} \cdot \frac{\partial \varphi}{\partial y}\right) \\ &= \frac{-2n(x^m - y^m)}{m}\left[\frac{(x^m - y^m) + z^{2n}(x^{m-2n} - y^{m-2n})}{x^{m-2n}y^{m-2n}}\right] \leq 0.\end{aligned}$$

We conclude that φ is the Schur m-power concave function on $\mathbb{R}^3_{++}$. Applying inequality (5), it follows that

$$\varphi(x, y, z) \leq \varphi\left(M_m(x, y, z), M_m(x, y, z), M_m(x, y, z)\right) = \varphi(1, 1, 1) = 3.$$

The proof of Theorem 2.10 is complete. □

Theorem 2.11. *If x, y, z are all positive numbers, and $x^2 + y^2 + z^2 = 8$, then*

$$x^3 + y^3 + z^3 \geq \frac{16}{3}\sqrt{6}. \tag{17}$$

Proof. Let $\varphi(x, y, z) = x^3 + y^3 + z^3$. Note that

$$\Lambda := \frac{x^2 - y^2}{2}\left(\frac{1}{x} \cdot \frac{\partial \varphi}{\partial x} - \frac{1}{y} \cdot \frac{\partial \varphi}{\partial y}\right) = \frac{3(x^2 - y^2)(x - y)}{2} \geq 0.$$

So φ is the Schur 2-power convex function on $\mathbb{R}^3_{++}$, by using inequality (5), we obtain

$$\varphi(x, y, z) \geq \varphi(M_2(x, y, z), M_2(x, y, z), M_2(x, y, z)) = \varphi\Big(\sqrt{\frac{8}{3}}, \sqrt{\frac{8}{3}}, \sqrt{\frac{8}{3}}\Big) = \frac{16}{3}\sqrt{6}.$$

This completes the proof of Theorem 2.11. □

Theorem 2.12. *Let x, y be positive real numbers, $x^2 + y^2 = 1$ and $p > 2$. Then*

$$x^{p-1}y^{p-1}(x^{-p} + y^{-p}) \geq 2^{2-\frac{p}{2}}. \tag{18}$$

Proof. Let $\varphi(x,y) = x^{p-1}y^{p-1}(x^{-p}+y^{-p})$. Since

$$\begin{aligned}\Lambda :&= \frac{x^2-y^2}{2}\left(\frac{1}{x}\cdot\frac{\partial\varphi}{\partial x}-\frac{1}{y}\cdot\frac{\partial\varphi}{\partial y}\right)\\ &= \frac{x^2-y^2}{2}\left[(p-1)x^{-1}y^{-1}(x^{p-2}-y^{p-2})+x^{-3}y^{-3}(x^{2+p}-y^{2+p})\right]\geq 0,\end{aligned}$$

we conclude that φ is the Schur 2-power convex function on $\mathbb{R}^2_{++}$. From the inequality (5), we deduce that

$$\varphi(x,y)\geq\varphi\left(M_2(x,y),M_2(x,y)\right)=\varphi(2^{-\frac{1}{2}},2^{-\frac{1}{2}})=2^{2-\frac{p}{2}}.$$

The Theorem 2.12 is proved. □

Theorem 2.13. *Let $a_i>0$ $(i=1,2,\ldots,n)$ with $\sum_{i=1}^n a_i^2=S$. Then*

$$\begin{aligned}&\frac{a_1^3}{a_2+a_3+\cdots+a_n}+\frac{a_2^3}{a_1+a_3+\cdots+a_n}\\ &+\cdots+\frac{a_n^3}{a_1+a_2+\cdots+a_{n-1}}\geq\frac{S}{n-1}.\end{aligned}\tag{19}$$

Proof. Let

$$p=\sum_{i=1}^n a_i,\quad q=\sum_{i=1}^n\frac{a_i^3}{(p-a_i)^2},\quad \varphi(a_1,a_2,\ldots,a_n)=\sum_{i=1}^n\frac{a_i^3}{p-a_i}.$$

Then

$$\frac{\partial\varphi}{\partial a_1}=\frac{a_1^2(3p-2a_1)}{(p-a_1)^2}-q,\quad \frac{\partial\varphi}{\partial a_2}=\frac{a_2^2(3p-2a_2)}{(p-a_2)^2}-q.$$

We have

$$\begin{aligned}\Lambda :&= \frac{a_1^2-a_2^2}{2}\left(\frac{1}{a_1}\cdot\frac{\partial\varphi}{\partial a_1}-\frac{1}{a_2}\cdot\frac{\partial\varphi}{\partial a_2}\right)\\ &= \frac{a_1^2-a_2^2}{2}\left(\frac{a_1(3p-2a_1)}{(p-a_1)^2}-\frac{a_2(3p-2a_2)}{(p-a_2)^2}-q\Big(\frac{1}{a_1}-\frac{1}{a_2}\Big)\right)\\ &= \frac{a_1^2-a_2^2}{2}\left[\frac{p(a_1-a_2)(a_1a_2+p(a_1+a_2+3a_3))}{(p-a_1)^2(p-a_2)^2}+\frac{q(a_1-a_2)}{a_1a_2}\right]\\ &\geq 0.\end{aligned}$$

Thus φ is the Schur 2-power convex function on $\mathbb{R}^n_{++}$. By utilizing inequality (5), we get

$$\begin{aligned}&\varphi(a_1,a_2,\ldots,a_n)\geq\varphi\left(M_2(a_1,a_2,\ldots,a_n),\ldots,M_2(a_1,a_2,\ldots,a_n)\right)\\ &\geq\varphi\Big(\sqrt{\frac{S}{n}},\sqrt{\frac{S}{n}},\ldots,\sqrt{\frac{S}{n}}\ \Big)=\frac{S}{n-1}.\end{aligned}$$

This completes the proof of Theorem 2.13. □

Theorem 2.14. *If* $x_i > 0$ $(i = 1, 2, \ldots, n)$ *and* $k > m > 0$*, then*

$$\left(\frac{\sum_{i=1}^{n} x_i^k}{n}\right)^{\frac{1}{k}} \geq \left(\frac{\sum_{i=1}^{n} x_i^m}{n}\right)^{\frac{1}{m}}. \tag{20}$$

Proof. The inequality (20) can be rewritten as a conditional inequalities:

$$\sum_{i=1}^{n} x_i^k \geq n\left(\frac{\ell}{n}\right)^{\frac{k}{m}} \quad \text{where} \quad \sum_{i=1}^{n} x_i^m = \ell.$$

Let $\varphi(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i^k$. Note that

$$\frac{x_1^m - x_2^m}{m}\left(\frac{1}{x_1^{m-1}} \cdot \frac{\partial\varphi}{\partial x_1} - \frac{1}{x_2^{m-1}} \cdot \frac{\partial\varphi}{\partial x_2}\right) = \frac{k}{m}(x_1^m - x_2^m)(x_1^{k-m} - x_2^{k-m}) \geq 0.$$

Hence φ is the Schur m-power convex function on $\mathbb{R}_{++}^n$. By making use of the inequality (5), we obtain

$$\varphi(x_1, x_2, \ldots, x_n) \geq \varphi\left(M_m(x_1, x_2, \ldots, x_n), \ldots, M_m(x_1, x_2, \ldots, x_n)\right) = n\left(\frac{\ell}{n}\right)^{\frac{k}{m}},$$

which is equivalent to the inequality (20). The proof of Theorem 2.14 is complete. □

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References

[1] A. M. Marshall, I. Olkin, *Inequalities: the theory of majorization and its application,* Academic Press, New York, 1979.

[2] B.-Y. Wang, *Foundations of majorization inequalities*, Beijing Normal University Press, Beijing, 1990 (Chinese).

[3] H.-N. Shi, *Schur convex functions and inequalities*, Harbin Institute of Technology Press, Harbin, 2017 (Chinese).

[4] Z.-H. Yang, *Schur power convexity of Stolarsky means*, Publ. Math. Debrecen, 80 (2012), 43-66.

[5] Z.-H. Yang, *Schur power convexity of Gini means*, Bull. Korean Math. Soc., 50 (2013), 485-498.

[6] Z.-H. Yang, *Schur power comvexity of the Daróczy means*, Math. Inequal. Appl., 16 (2013), 751-762.

[7] Y.-M Chu, X.-M. Zhang, *Necessary and sufficient conditions such that extended mean values are Schur-convex or Schur-concave*, J. Math. Kyoto Univ., 48 (2008), 229-238.

[8] W. Wang, S.-G. Yang, *Schur m-power convexity of a class of multiplicatively convex functions and applications*, Abstr. Appl. Anal., 2014 (2014), Article ID 258108, 12 pages.

[9] H.-P. Yin, H.-N. Shi, F. Qi, *On Schur m-power convexity for ratios of some means*, J. Math. Inequal., 9 (2015), 145-153.

[10] Q. Xu, *Reserch on Schur power-convexity of the quotient of arithmetic mean and geometric mean*, J. Fudan Univ. (Nat. Sci.), 54 (2015), 299-295.

[11] Y.-P. Deng, S.-H. Wu, D. He, *The Schur power convexity for the generalized Muirhead mean*, Math. Pract. Theory, 44 (2014), 255-268.

[12] H.-P. Yin, H.-N. Shi, F. Qi, *On Schur m-power convexity for ratios of some means*, J. Math. Inequal., 9 (2015), 145-153.

[13] S.-H. Wu, Y.-M. Chu. *Schur m-power convexity of generalized geometric Bonferroni mean involving three parameters,* J. Inequal. Appl., 2019 (2019), Article ID 57, 11 pages.

[14] D.-S. Wang, C.-R. Fu, H.-N Shi, *Schur-m power convexity of Cauchy means and its application*, Rocky Mountain J.Math., 50 (2020), 1859-1869.

[15] S.-J. Li, S. Li, *Questing for the secrets of inequality exploration*, Harbin Institute of Technology Press, Harbin, 2017 (Chinese).

[16] Z.-P. An, *Inequality explore*, Harbin Institute of Technology Press, Harbin, 2015 (Chinese).

[17] S.-C Deng, *Artiele about the Mathematical Olympic inequality*, Harbin Institute of Technology Press, Harbin, 2011 (Chinese).

[18] F. Lu, X.-Y Dong, S. Liang, *100 cases of application of Higher Mathematics in middle school mathematics*, Northeast Normal University Press, Changchun, 1995 (Chinese).

[19] X.-G Huang, *Convex functions and Jenson inequalities*, China University of science and Technology Press, Hefei, 2014 (Chinese).

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