Some results on the total graph of a commutative ring

Pravesh Sharma
Department of Mathematics
North-Eastern Hill University
Permanent Campus
Shillong-793022, Meghalaya
India
sharma.pravesh76@gmail.com

Sanghita Dutta*
Department of Mathematics
North-Eastern Hill University
Permanent Campus
Shillong-793022, Meghalaya
India
sanghita22@gmail.com

Abstract. In this paper, we study the total graph of a ring and determine its matching, factors and arboricity. We find the independence number of the line graph of the total graph of a ring and give complete characterization of rings for which diam $L(T(\Gamma(R))) =, <$ or $>$ diam $T(\Gamma(R))$. We find the crossing number of all local rings with some bounds in the cardinality of zero divisors and also determine the crossing number of all finite rings with the genus at most two.

Keywords: crossing number, zero divisors, matching, factors, arboricity.

1. Introduction

The idea of associating a graph to an algebraic object initiated by I. Beck [1] lead to many interesting characterization of algebraic objects. Following the same path many mathematicians have derived a number of interesting results by assigning graph to an algebraic object. All rings we consider in this paper are commutative rings with identity. In this paper, we try to obtain some characterization of a commutative ring by considering the total graph of the ring. The total graph of a ring was introduced by Anderson and Badawi in [2], defined as "an undirected graph with all elements of R as vertices and for distinct $x, y \in R$, x is adjacent to y whenever $x + y$ belongs to the set of $Z(R)$ and this graph is denoted by $T(\Gamma(R))$". In 1944, Pal. Turán [3] posed the problem of finding crossing number of a graph and still after 75 years, very little is known about this simple yet intricate nonplanarity measure. The crossing number of a complete graph is conjectured by Hill's [4] as

∗. Corresponding author

$cr(K_n) =: \frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor$ and crossing number of a complete bipartite graph by Zarankiewicz [5] as $cr(K_{m,n}) =: \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$. However, these conjectures remain open and some partial results are known. The former have been verified for $n \leq 10$, while the later holds for all n with $m \leq 6$ and for $m = 7$ with $n \leq 10$. $Reg(\Gamma(R))$ is the (induced) subgraph of $T(\Gamma(R))$ with vertex set Reg(R) and $Z(\Gamma(R))$ is the (induced) subgraph of $T(\Gamma(R))$ with vertex set Z(R). The study of a total graph breaks into two parts depending on whether R is a finite local ring or not.

In Section 2 of this paper we find the maximum matching, the perfect matching and the number of perfect matching when R is a finite ring of even order. In Section 3, we decompose the total graph into line-disjoint spanning subgraphs to find its factorisation. In Section 4, we find arboricity of the total graph of a ring. In Section 5, we determine some properties of the line graph associated with the total graph of a ring. The study of the line graph is interesting as it enables us to phrase questions on the edges of the graph which are viewed as the vertices in its corresponding line graph. In Section 6, we find all local rings up to isomorphism whose crossing number of the total graph can be computed from Hill's conjecture and Zarankiewicz conjecture with some bound on the cardinality of the set of zero divisors. At last in Section 7, we draw the total graph of all rings with the genus at most 2 to find the best possible drawing with the least crossing of edges. For basic graph-theoretic terminologies and results we refer the reader to [6] and [7].

2. Matching of total graph of ring

A matching in a graph G is a set of non-loop edges with no shared endpoints. The vertices incident to the edges of a matching M are saturated by M; the other vertices are unsaturated. A perfect matching in a graph is a matching that saturates every vertex. In a graph a maximal matching is a matching which cannot be increased by adding an extra edge. A matching with maximum cardinality is called maximum matching. A vertex cover of a graph G is a set $Q \subseteq V(Q)$ that contains at least one endpoint of every edge. The vertices in Q cover $E(Q)$. An edge cover of G is a set L of edges such that every vertex of G is incident to some edge of L. We denote "the maximum size of the independent set, the maximum size of matching, the minimum size of vertex cover and the minimum size of edge cover by $\alpha(G)$, $\alpha'(G)$, $\beta(G)$, $\beta'(G)$ respectively ".

In proposition 2.1 we find the existance of a perfect matching in the total graph of a finite commutative ring of even order.

Proposition 2.1. *If R is a finite ring of even order, then $T(\Gamma(R))$ has a perfect matching.*

Proof. First, we consider the case when $Z(R)$ is an ideal and $2 \in Z(R)$. Let $|Z(R)| = \lambda$ and $\frac{|R|}{|Z(R)|} = \beta$. Then, the total graph of the finite local ring with $2 \in Z(R)$ is $T(\Gamma(R)) = \sqcup_\beta K_\lambda$. So $|R| = 2^k$ for some k $\in \mathbb{N}$ since $2 \in Z(R)$ and

λ is a power of 2. As K_{2n} has a perfect matching and the number of perfect matching is $\frac{(2n)!}{(2^n n!)}$ by example 3.1.3 [6], $T(\Gamma(R))$ has a perfect matching. Now, we consider when $Z(R)$ is an ideal and $2 \notin Z(R)$. In this case $|R|$ is even and $\exists$ an element of order 2 in R. This implies that $\exists\ x \neq 0 \in R$ such that $2x \in Z(R)$ which is a contradiction. Next, we consider the case when $Z(R)$ is not an ideal. Let I be a maximal ideal contained in $Z(R)$. If $2 \in I$, then we construct a spanning subgraph H of the ring $\frac{R}{I}$ where the vertex set is $\frac{R}{I}$ and two vertices x and y are adjacent whenever $x + y$ belongs to an ideal I. The subgraph H is the disjoint union of complete graphs as in the proof of theorem 3.2 [8]. If the cardinality of I is even, then as the complete graph has a perfect matching, H has a perfect matching. If $|I|$ is odd, then each component of H is $|I| - 1$ regular and $|I| < |Z(R)|$, so $T(\Gamma(R))$ is obtained by adding at least 1 edge to every vertex. Now, these components are complete graphs and the addition of an extra edge makes the resultant graph connected with a perfect matching. At last, we consider I a maximal ideal with $2 \notin I$. Then, the subgraph H of the ring $\frac{R}{I}$ where the vertex set is $\frac{R}{I}$ is a disjoint union of $K_{|I|}$ and these $\frac{(\frac{|R|}{|I|}-1)}{2}$ number of disjoint $K_{|I|,|I|}$'s. Then, $\frac{|R|}{|I|}$ is odd and $|I|$ is even and hence, there is a perfect matching. □

In the following two propositions we find the maximum size of a perfect matching in the total graph of a ring.

Proposition 2.2. *If R is a finite ring of even order, then $\alpha'(T(\Gamma(R)))= \frac{|R|}{2}$. In particular, if $Z(R)$ is an ideal of R and $2 \in Z(R)$, then the number of perfect matching is equal to $((((n!)!)!)!...)!$, factorial upto l times, where l is given by $\frac{|R|}{|Z(R)|} = 2^l$ and n is the number of perfect matching in $K_{|Z(R)|}$.*

Proof. Let $|Z(R)| = \lambda$, $\frac{|R|}{|Z(R)|} = \beta$. As $2 \in Z(R)$, we have $T(\Gamma(R)) = \sqcup_\beta K_\lambda$. Clearly by proposition 2.1, $T(\Gamma(R))$ has a perfect matching and $\alpha'(T(\Gamma(R)))= \frac{|R|}{2}$. Let n be the number of perfect matching in each K_λ. Now we consider a pair of two distinct sets of perfect matchings of any K_λ say V_i, V_j having cardinality n each, where $1 \leq i, j \leq \beta$. Then, the total numbers of one-one map from V_i to V_j is $n!$. So, $K_\lambda \sqcup K_\lambda$ has $n!$ number of perfect matchings. Similarly, taking two more $K_\lambda' s$ and repeating the process we see that there are $(n!)!$ number of perfect matching of $K_\lambda \sqcup K_\lambda \sqcup K_\lambda \sqcup K_\lambda$. Continuing this process we find that the total number of perfect matchings is $((((n!)!)!)!...)!$, factorial taken l times, where l is given by $\frac{|R|}{|Z(R)|} = 2^l$. □

Proposition 2.3. *If R is a finite ring of odd order, then $\alpha'(T(\Gamma(R)))= \frac{|R|-1}{2}$.*

Proof. First, we consider the case when $Z(R)$ is an ideal and $|R|$ is odd. The total graph of the finite local ring with $2 \notin Z(R)$ is K_λ union $\frac{(\beta-1)}{2}$ number of disjoint $K_{\lambda,\lambda}$'s where $|Z(R)| = \lambda$ and $\frac{|R|}{|Z(R)|} = \beta$. Hence, we get $\alpha'(T(\Gamma(R)))=$

$(\frac{\lambda-1}{2}+(\frac{\beta-1}{2})\lambda)=\frac{\lambda\beta-1}{2}=\frac{|R|-1}{2}$. Next, we consider the case when $Z(R)$ is not an ideal and $|R|$ is odd. Let I be a maximal ideal contained in $Z(R)$. We construct a spanning subgraph H of the ring $\frac{R}{I}$ with the vertex set $\frac{R}{I}$ and the vertices x and y are adjacent whenever $x+y$ belongs to an ideal I. The subgraph H of the ring $\frac{R}{I}$ is a disjoint union of $K_{|I|}$ and $\frac{(\frac{|R|}{|I|}-1)}{2}$ number of disjoint $K_{|I|,|I|}$'s. Thus, $\alpha'(T(\Gamma(R)))=(\frac{|I|-1}{2}+(\frac{(\frac{|R|}{|I|}-1)}{2})|I|)=\frac{|R|-1}{2}$. □

Proposition 2.4. *If R is an integral domain, then $\alpha'(T(\Gamma(R)))=\beta(T(\Gamma(R)))$.*

Proof. Let $V(T(\Gamma(R)))=V_1\sqcup V_2$ such that whenever $x\in V_1$, $-x\in V_2$ and also the zero element, self inverse elements we include in any of the set V_1 or V_2. Then, $T(\Gamma(R))$ is a bipartite graph with partite sets V_1 and V_2. Now, by Köning - Egerváry theorem [9, 10] $\alpha'(T(\Gamma(R)))=\beta(T(\Gamma(R)))$. □

Remark 2.1. Converse of the above result is false. For example $\mathbb{Z}_4$, $\mathbb{Z}_2\times\mathbb{Z}_2$ are not integral domains but their total graphs are bipartite.

Proposition 2.5. $\alpha(T(\Gamma(R)))=\alpha'(T(\Gamma(R)))=\beta(T(\Gamma(R)))=\beta'(T(\Gamma(R)))=2$ *if and only if* $R\cong\mathbb{Z}_4,\ \mathbb{Z}_2\times\mathbb{Z}_2,\ \frac{\mathbb{Z}_2[X]}{(X^2)}$.

Proof. We assume, $\alpha(T(\Gamma(R)))=\alpha'(T(\Gamma(R)))=\beta(T(\Gamma(R)))=\beta'(T(\Gamma(R)))=2$. Then, $|R|=4$ and the only possible ring of order four having $Z(R)\neq\{0\}$ are $\mathbb{Z}_4,\ \mathbb{Z}_2\times\mathbb{Z}_2,\ \frac{\mathbb{Z}_2[X]}{(X^2)}$.

Conversely, suppose $R\cong\mathbb{Z}_4,\ \mathbb{Z}_2\times\mathbb{Z}_2,\ \frac{\mathbb{Z}_2[X]}{(X^2)}$. Then, $\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[X]}{(X^2)}$ are local rings of order 4 whose total graph is $2K_2$. Also $\mathbb{Z}_2\times\mathbb{Z}_2$ is a non local ring whose total graph is C_4. Hence, $\alpha(T(\Gamma(R)))=\alpha'(T(\Gamma(R)))=\beta(T(\Gamma(R)))=\beta'(T(\Gamma(R)))=2$. □

3. Factorisation of total graph of ring

A factor of a graph G is a spanning subgraph of G. A k-factor is a spanning k-regular subgraph. We say that G is the sum of factors G_i if it is line disjoint union, and such a union is called a factorization of G. If G is sum of $n-$ factors, their union is called n-factorization and G itself $n-$factorable. In this section we find when $T(\Gamma(R))$ has a factorisation. For our further computation we shall refer to [7].

Remark 3.1. $K_{2k+1,2k+1}$ is a spanning regular subgraph of the complete graph $K_{2(2k+1)}$ and we know that the complete graph K_{2n} is a sum of $1-$factor and $n-1$ spanning cycles by Theorem 9.7 [7]. As degree of a vertex in K_{2n} is $2n-1$ it is a sum of $1-$factor and $n-1$ spanning cycles. Deleting $1-$spanning cycle from K_{2n} we get a subgraph of degree $2n-3$ which is a sum of $1-$factor and $n-2$ spanning cycles. So, as we continue reducing degree by 2, number of spanning cycles is reduced by 1.

If $Z(R)$ is an ideal of R and $2 \in Z(R)$, then by Theorem 9.1 [7], we have the following:

Proposition 3.1. *Let R be a finite ring such that $Z(R)$ is an ideal of R and $2 \in Z(R)$. Then, $T(\Gamma(R))$ has a $1-$factorisation.*

Proposition 3.2. *Let R be a finite ring such that $Z(R)$ is an ideal of R and $2 \notin Z(R)$. Then, $T(\Gamma(R))$ has a factorisation.*

Proof. Let $Z(R)$ be an ideal of R. If $2 \notin Z(R)$, then the total graph of R is disjoint union of K_λ and $\frac{(\beta-1)}{2}$ number of disjoint $K_{\lambda,\lambda}$'s, where $|Z(R)| = \lambda$ and $\frac{|R|}{|Z(R)|} = \beta$. As λ is odd by Theorem 9.6 [7] K_{2k+1} is a sum of k spanning cycles. Each component $K_{2k+1,2k+1}$ is a spanning regular subgraph of the complete graph $K_{2(2k+1)}$. Therefore by remark 3.1, $K_{2k+1,2k+1}$ is a sum of $1-$factor and k spanning cycles. So, $T(\Gamma(R)) = F_1 \cup F_2 \cup F_3 ... \cup F_{k+1}$, where $F_i's$ are the set of $2-$factor in 2k+1 vertices from $Z(R)$ union $\frac{(\beta-1)}{2}$ number of 2-factor in 2(2k+1) vertices from $Reg(R)$, for all $1 \leq i \leq k$ and F_{k+1} is the graph of spanning 2k+1 vertices from $Z(R)$ union $\frac{(\beta-1)}{2}$ number of $1-$factor in 2(2k+1) vertices from $Reg(R)$. Hence, $T(\Gamma(R))$ has a factorisation. □

Proposition 3.3. *Let R be a finite ring such that $Z(R)$ is not an ideal of R and $2 \in Z(R)$. Then, $T(\Gamma(R))$ is $2-$factorable if $|Z(R)|$ is odd otherwise it is sum of a $1-$factor and spanning cycles.*

Proof. Let $2 \in Z(R)$. If $|Z(R)|$ is odd, then $T(\Gamma(R))$ is a regular graph. By Theorem 9.9 [7], it is $2-$factorable. If $|Z(R)|$ is even, then $T(\Gamma(R))$ is a regular graph of odd degree and it is the spanning subgraph of the complete graph K_{2n}. By Theorem 9.7 [7] and remark 3.1, $T(\Gamma(R))$ is a sum of $1-$factor and spanning cycles. □

Proposition 3.4. *Let R be a finite ring such that $Z(R)$ is not an ideal of R and $2 \notin Z(R)$. Then, $T(\Gamma(R)) = F_1 \cup F_2 \cup F_3 ... \cup F_{k+1}$, where $F_i's$ are factors and $k = \frac{(|Z(R)|-1)}{2}$.*

Proof. Let $2 \notin Z(R)$. Then, each element will have distinct additive inverse. For each $u \in Reg(R)$, $-u \in Reg(R)$. Therefore, $|Reg(R)|$ is even and $|Z(R)|$ is odd. Let H be a $|Z(R)|-1$ regular subgraph of $T(\Gamma(R))$ excluding the edges between regular elements with their respective additive inverse. Then, H is a connected regular graph of even degree and so it is $2-$factorable. Therefore, $H = F_1 \cup F_2 \cup F_3 ... \cup F_k$, where $k = \frac{(|Z(R)|-1)}{2}$, for all $1 \leq i \leq k$ and each $F_i's$ are spanning cycles. Next, the factor F_{k+1} is the graph with the set of spanning vertices from $Z(R)$ union $1-$factor in $Reg(R)$. Thus, $T(\Gamma(R)) = H \cup F_{k+1}$. □

4. Arboricity

Any graph G can be expressed as a sum of spanning forests. A natural problem is to determine minimum number of line-disjoint spanning forest into which G can be decomposed. This number is called arboricity of G and is denoted by $\gamma(G)$. For example, $\gamma(K_4) = 2$ and $\gamma(K_5) = 3$. Nash-Williams [11] has determined a precise formula for arboricity of graph; namely, "$\gamma(G) = \lceil \frac{e_H}{n_H - 1} \rceil$", where maximum is taken over all nontrivial subgraphs H and e_H, n_H denote the number of edges and vertices of H respectively. For complete graph $\gamma(K_n) = \lceil \frac{n}{2} \rceil$ and for complete bipartite $\gamma(K_{m,n}) = \lceil \frac{mn}{m+n-1} \rceil$.

Proposition 4.1. *Let R be a finite ring such that $Z(R)$ is an ideal of R. Then, $\gamma(T(\Gamma(R))) = \lceil \frac{\lambda}{2} \rceil$, where $|Z(R)| = \lambda$.*

Proof. If $Z(R)$ is an ideal of R and $2 \in Z(R)$, then the total graph of $T(\Gamma(R))$ is a disjoint union of the complete graphs K_λ. Thus by Nash-Williams formula [11], $\gamma(T(\Gamma(R))) = \lceil \frac{\lambda}{2} \rceil$. If $Z(R)$ is an ideal of R and $2 \notin Z(R)$, then the total graph of $T(\Gamma(R))$ is K_λ union $\frac{(\beta-1)}{2}$ number of disjoint $K_{\lambda,\lambda}$'s, where $|Z(R)| = \lambda$ and $\frac{|R|}{|Z(R)|} = \beta$. As λ is odd, say $\lambda = 2k+1$, for complete bipartite graph $\gamma(K_{\lambda,\lambda}) = \lceil \frac{\lambda\lambda}{2\lambda-1} \rceil = \lceil \frac{4k^2+4k+1}{4k+1} \rceil =$ k$+\lceil \frac{3k+1}{4k+1} \rceil =k+1 =\lceil \frac{2k+1}{2} \rceil =\lceil \frac{\lambda}{2} \rceil$. Since, arboricity of all the components are equal, $\gamma(T(\Gamma(R))) = \lceil \frac{\lambda}{2} \rceil$. □

By Nash-Williams formula we have the following:

Proposition 4.2. *Let R be a finite ring such that $Z(R)$ is not an ideal of R. Then, $\gamma(T(\Gamma(R))) = Max\{\lceil \frac{e_H}{n_H - 1} \rceil\}$, where e_H, n_H denote the number of edges and vertices of subgraph H respectively.*

5. Line Graph of the total graph of ring

The line graph of a graph G is a graph whose vertices are the set of edges of G and two vertices are adjacent if and only if the corresponding edges are adjacent in G. If G is connected, then its line graph is also connected. A connected graph is isomorphic to its line graph if and only if it is a cycle. We state here a theorem due to Ramane et.al [12] which we shall use to prove our result.

Theorem 5.1 ([12]). *For a connected graph G diam $L((G)) \leq 2$ if and only if none of the three graphs of figure* (1) *is an induced subgraph of G.*

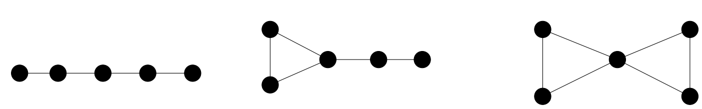

Figure 1:

The maximum matching of a graph is equal to the maximum independent set of its corresponding line graph. So from propositions 2.1 and 2.2, the following holds:

Proposition 5.1. *If R is a finite commutative ring, then*

$$\alpha(L(T(\Gamma(R))) = \begin{cases} \frac{|R|}{2}, & \textit{if } |R| \textit{ is even} \\ \frac{|R|-1}{2}, & \textit{if } |R| \textit{ is odd.} \end{cases}$$

Proposition 5.2. *$L(T(\Gamma(R))) \cong T(\Gamma(R))$ if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. We assume $L(T(\Gamma(R))) \cong T(\Gamma(R))$. Then, $T(\Gamma(R))$ is a cycle with $|Z(R)| = 3$ and $|R| \leq 9$. Thus if $|R| = 9$, then $T(\Gamma(R))$ is not regular. Next, if $|R| = 8$, then from the list of Section 5 [13] except $\mathbb{F}_8$ there exist 5 local rings with $|Z(R)| = 4$ and others are non-local rings with $|Z(R)| \geq 4$. When $|R| = 6$ we know $R \cong \mathbb{Z}_6$ with $|Z(R)| = 4$. If $|R| = 4$, then there are two local rings $\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[X]}{(X^2)}$ whose $T(\Gamma(R)) \cong 2K_2$. If R is non local, then $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $T(\Gamma(R)) \cong C_4$. Conversely if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $L(T(\Gamma(R))) \cong T(\Gamma(R))$. Hence, the proposition hold. □

Remark 5.1. A line graph is complete iff $G \cong K_3$ *or* $K_{1,n-1}$. So it follows that $L(T(\Gamma(R)))$ is complete iff $R \cong \mathbb{Z}_3$.

Proposition 5.3. *Let R be a finite ring. Then, diam $L(T(\Gamma(R)))$ = diam $T(\Gamma(R)) = 2$ if and only if $R \cong \mathbb{Z}_2^{n+1}$ or $\mathbb{Z}_2^n \times \mathbb{Z}_3$, where n is natural number.*

Proof. Let $R \cong R_1 \times R_2 ... \times R_k$, where $R_i's$ are finite local rings such that diam $L(T(\Gamma(R)))$ = diam $T(\Gamma(R)) = 2$. Then, $T(\Gamma(R))$ is connected and $Z(R)$ is not an ideal.
Case 1. Let $|Reg(R)| \geq 3$ and $u,\ v \in Reg(R)$ such that $u + v \in Z(R)$. Let $u+v = z$. As diam $T(\Gamma(R)) = 2$, $\exists\, c \in R$ and a path on five vertices $(0)-(-z)-(c)-(u)-(v)$, where 0 and $-z$ are not adjacent to u and v. If $u + v \in Reg(R)$, then $\exists\ z_1,\ z_2 \in Z(R)$ such that $z_1 + z_2 = 1$. Let $u = (u_1, u_2, ..., u_k)$, $v = (v_1, v_2, ..., v_k)$, $z_1 = (1, 0, ..., 0)$ and $z_2 = (0, 1, ..., 1)$. Then, the graph obtained by identifying a vertex of the triangle having vertices $uz_1,\ vz_1,\ 0$ with a vertex of another triangle having vertices $uz_2,\ vz_2,\ 0$ is an induced subgraph with 5 vertices. Hence by theorem 5.1, diam $L(T(\Gamma(R))) \neq 2$.
Case 2. Let $|Reg(R)| \leq 2$. Then, by Lemma 1 [14], $R \cong \mathbb{Z}_3,\ \mathbb{Z}_4, \frac{\mathbb{Z}_2[X]}{(X^2)}, \mathbb{Z}_2^{n+1}$, $\mathbb{Z}_2^n \times \mathbb{Z}_3$, $\mathbb{Z}_2^n \times \mathbb{Z}_4$, $\mathbb{Z}_2^n \times \frac{\mathbb{Z}_2[X]}{(X^2)}$. If $R \cong \mathbb{Z}_2^n \times \mathbb{Z}_4$, then the total graph of $\mathbb{Z}_2 \times \mathbb{Z}_4$ has an induced subgraph obtained by identifying a vertex of the triangle formed by the vertices $(0, 2),\ (0, 0),\ (1, 0)$ with a vertex of another triangle formed by the vertices $(1, 0),\ (1, 1),\ (1, 3)$. Hence, the total graph of $\mathbb{Z}_2^n \times \mathbb{Z}_4$ contains a graph of figure 1. as an induced subgraph. Again if $R \cong \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[X]}{(X^2)}$, then the total graph of R has an induced subgraph obtained by identifying a vertex of the triangle formed by the vertices $(0, x),\ (0, 0),\ (1, 0)$ with a vertex of another

triangle formed by the vertices $(1,0)$, $(1,1)$, $(1,1+x)$. Therefore, the total graph of $\mathbb{Z}_2^n \times \frac{\mathbb{Z}_2[X]}{(X^2)}$ contains a graph of figure 1. as an induced subgraph. Thus, diam $L(T(\Gamma(R))) \neq 2$. If $R \cong \mathbb{Z}_2^{n+1}$, then $|Reg(R)| = 1$. Suppose $\exists$ a path on 5 vertices $a,\ b,\ c,\ d,\ e$ such that a is not adjacent to d and e, then we get a contradiction as degree of each vertex is $|R|-2$. So $\nexists$ any induced subgraph of figure 1. Hence, diam $L(T(\Gamma(R))) = 2$. At last, if $R \cong \mathbb{Z}_2^n \times \mathbb{Z}_3$ and if $\exists$ an induced subgraph on 5 vertices $a,\ b,\ c,\ d,\ e$ such that a, b are not adjacent to d and e, without loss of generality we can assume $a,\ b \in \{0\} \times \{0\} \times ... \times \mathbb{Z}_6$ and $d,\ e \in \{1\} \times \{1\} \times ... \times \mathbb{Z}_6$. Let $a = (0,0,...,n)$, $b = (0,0,...,n^{'})$, $d = (1,1,...,m)$, $e = (1,1,...,m^{'})$, where $n,\ n^{'},\ m,\ m' \in \mathbb{Z}_6$. Then, $a+d,\ b+e,\ a+e,\ b+d$=$(1,1,...,5)$ or $(1,1,...,1)$. This leads us to $0 = 2$, which is a contradiction. Thus by theorem 5.1, diam $L(T(\Gamma(R))) = 2$. $\square$

Remark 5.2. From remark 5.1 and proposition 5.3 it follows that there does not exist any ring with diam $L(T(\Gamma(R))) = 1$. If $T(\Gamma(R))$ is connected and $R \ncong \mathbb{Z}_2^{n+1}$ or $\mathbb{Z}_2^n \times \mathbb{Z}_3$, then diam $L(T(\Gamma(R))) = 3$.

6. Crossing number of total graph of some rings

In the following two propositions 6.1 and 6.2 we find crossing number of the total graph of rings with $2 \in Z(R)$ and $2 \notin Z(R)$ with restriction on $|Z(R)|$.

Proposition 6.1. *Let R be a finite local ring with $2 \in Z(R)$ and $|Z(R)| \leq 15$. Then, the total graph of R with crossing number 36 or 144 is isomorphic to one of the following rings*

$$\mathbb{Z}_{16},\ \frac{\mathbb{Z}_2[X]}{(X^4)},\ \frac{\mathbb{Z}_4[X]}{(X^2+2)},$$
$$\frac{\mathbb{Z}_4[X]}{(X^2+3X)},\ \frac{\mathbb{Z}_4[x]}{(X^3-2,2X^2,2X)},\ \frac{\mathbb{Z}_2[X,Y]}{(X^3,XY,Y^2)},\ \frac{\mathbb{Z}_8[x]}{(2X,X^2)},\ \frac{\mathbb{Z}_4[x]}{(X^3,2X^2,2X)},$$
$$\frac{\mathbb{Z}_4[X]}{(X^2+2X)},\ \frac{\mathbb{Z}_8[x]}{(2X,X^2+4)},\frac{\mathbb{Z}_2[X,Y]}{(X^2,Y^2-XY)},\ \frac{\mathbb{Z}_4[X,Y]}{(X^2,Y^2-XY,XY-2,2X,2Y)},$$
$$\frac{\mathbb{Z}_4[X,Y]}{(X^2,Y^2,XY-2,2X,2Y)},$$
$$\frac{\mathbb{Z}_2[X,Y]}{(X^2,Y^2)},\ \frac{\mathbb{Z}_4[X]}{(X^2)},\ \frac{\mathbb{Z}_4[X]}{(X^3-X^2-2,2X^2,2X)},$$
$$\frac{\mathbb{Z}_2[X,Y,Z]}{((X,Y,Z)^2)},\ \frac{\mathbb{Z}_4[X,Y]}{(X^2,Y^2,XY,2X,2Y)},\ \frac{\mathbb{F}_8[X]}{(X^2)},\ \frac{\mathbb{Z}_4[X]}{(X^3+x+1)}.$$

Proof. Let $|Z(R)| = \lambda$ and $\frac{|R|}{|Z(R)|} = \beta$. Then, the total graph of the finite local ring R with $2 \in Z(R)$ is $T(\Gamma(R)) = \sqcup_\beta K_\lambda$ by Theorem 2.2 [2]. So, $|R| = 2^k$ since $2 \in Z(R)$ and λ is power of 2. Let $5 \leq \lambda \leq 15$. Then, we must have λ= 2^3 and by Theorem 1 [15] we get $|R| \leq 64$. If $|R| = 64$, then $R \cong \frac{\mathbb{F}_8[X]}{(X^2)}$ or

$\frac{\mathbb{Z}_4[X]}{(X^3+x+1)}$, by discussion in introductory section of [16]. As $T(\Gamma(R)) = \cup\beta K_8$, $cr(T(\Gamma(R))) = \beta cr(K_8) = 8 \times 18 = 144$. Next, if $|R| = 32 = 2^5$ by Theorem 2 [17] $|Z(R)| = 2^{(5-1)1} = 16$ or 1. So, $\nexists$ any local ring of order 32 with $|Z(R)| = 8$. If $|R| = 16$, by the list in Section 5 [13] we have 18 non isomorphic rings which are as follows:

$$\mathbb{Z}_{16},\ \frac{\mathbb{Z}_2[X]}{(X^4)},\ \frac{\mathbb{Z}_4[X]}{(X^2+2)},\ \frac{\mathbb{Z}_4[X]}{(X^2+3X)},$$
$$\frac{\mathbb{Z}_4[x]}{(X^3-2,2X^2,2X)},\ \frac{\mathbb{Z}_2[X,Y]}{(X^3,XY,Y^2)},\ \frac{\mathbb{Z}_8[x]}{(2X,X^2)},\ \frac{\mathbb{Z}_4[x]}{(X^3,2X^2,2X)},$$
$$\frac{\mathbb{Z}_4[X]}{(X^2+2X)},\ \frac{\mathbb{Z}_8[x]}{(2X,X^2+4)},\ \frac{\mathbb{Z}_2[X,Y]}{(X^2,Y^2-XY)},$$
$$\frac{\mathbb{Z}_4[X,Y]}{(X^2,Y^2-XY,XY-2,2X,2Y)},\ \frac{\mathbb{Z}_4[X,Y]}{(X^2,Y^2,XY-2,2X,2Y)},$$
$$\frac{\mathbb{Z}_2[X,Y]}{(X^2,Y^2)},\ \frac{\mathbb{Z}_4[X]}{(X^2)},\ \frac{\mathbb{Z}_4[X]}{(X^3-X^2-2,2X^2,2X)},$$
$$\frac{\mathbb{Z}_2[X,Y,Z]}{((X,Y,Z)^2)},\ \frac{\mathbb{Z}_4[X,Y]}{(X^2,Y^2,XY,2X,2Y)}$$

So, $T(\Gamma(R)) = \cup\beta K_8 \Rightarrow cr(T(\Gamma(R))) = \beta cr(K_8) = 2 \times 18 = 36$. □

Proposition 6.2. *Let R be a finite local ring with $2 \notin Z(R)$ and $|Z(R)| \leq 7$. Then, the total graph of R with crossing number 1 or 33 or 171 is isomorphic to one of the following rings* $\mathbb{Z}_9$, $\frac{\mathbb{Z}_3[X]}{(X^2)}$, $\mathbb{Z}_{25}$, $\frac{\mathbb{Z}_5[X]}{(X^2)}$, $\mathbb{Z}_{49}$, $\frac{\mathbb{Z}_7[X]}{(X^2)}$.

Proof. Let $|Z(R)| = \lambda$ and $\frac{|R|}{|Z(R)|} = \beta$. Then, by Theorem 2.2 [2] the total graph of the finite local ring R with $2 \notin Z(R)$ is $T(\Gamma(R)) = K_\lambda \cup \frac{(\beta-1)}{2} K_{\lambda,\lambda}$. So, if $\exists\ x(\neq 0) \in \frac{R}{Z(R)}$ such that $2x = 0$, then $2x \in Z(R)$ which is a contradiction as $Z(R)$ is a prime ideal. Thus, every non zero element has distinct inverse such that $\frac{|R|}{|Z(R)|}$ is odd and by Theorem 2 [17], $|R|$ is power of odd prime. Let $3 \leq \lambda \leq 7$. Then, we must have λ= 3 or 5 or 7. By the discussion, in Section 3 [13] for $p = 3$ or 5 or 7, R is isomorphic to $\mathbb{Z}_{p^2}$ and $\frac{\mathbb{Z}_p[x]}{(X^2)}$.

Now, $cr(T(\Gamma(R))) = cr(K_\lambda) + \frac{(\beta-1)}{2}$ cr($K_{\lambda,\lambda}$). Therefore, for λ= 3 we find that $cr(T(\Gamma(R))) = 1$, for $\lambda = 5$, $cr(T(\Gamma(R))) = 33$ and for $\lambda = 7$, $cr(T(\Gamma(R))) = 171$. □

Remark 6.1. Further, if the conjecture of the crossing number for the complete and complete bipartite graph is solved, then a generalization to the above results are as follows:

- If R is a finite local ring with $2 \in Z(R)$, then the crossing number of $T(\Gamma(R)) = \beta\ cr(K_\lambda)$ and also if R is a finite local ring with $2 \notin Z(R)$, then the crossing number of $T(\Gamma(R)) = cr(K_\lambda) + \frac{(\beta-1)}{2}\ cr(K_{\lambda,\lambda})$.

7. $T(\Gamma(R))$ with genus atmost 2

The total graph of rings upto isomorphism with genus atmost 2 are

$\mathbb{Z}_2 \times \mathbb{F}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[X]}{(X^2)}$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_{10}$, $\mathbb{Z}_3 \times \mathbb{F}_4$.

In this section we find the crossing number of $T(\Gamma(R))$ for these rings. We have $T(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4)) \cong T(\Gamma(\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[X]}{(X^2)}))$. The graph in Figure 2. is drawn by plotting $2K_4$ and adding other edges. Let $I = \{(0,0),(0,1),(0,x)(0,x+1)\}$ be the maximal ideal in $Z(R)$ with $2 \in I$. We first construct the graph by taking $x, y \in R$ as vertices and x and y are adjacent whenever $x + y$ belongs to I. Therefore, $T(\Gamma(I \cup \frac{R}{I})) = K_4 \cup K_4$ which is the spanning subgraph of $T(\Gamma(R))$. Next, we add the remaining edges to obtain the total graph. This shows that $cr(\mathbb{Z}_2 \times \mathbb{F}_4) = 2$.

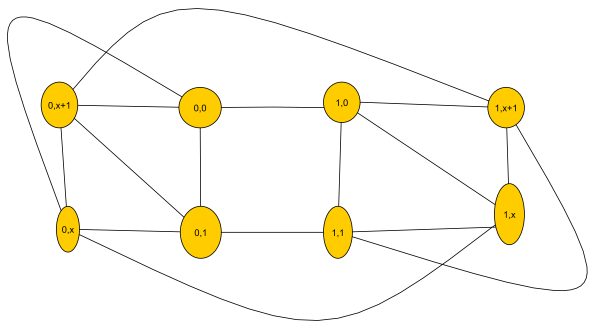

Figure 2: $cr(\mathbb{Z}_2 \times \mathbb{F}_4) = 2$

Similarly, in Figure 3. we draw $2K_4$ taking $I = \{(0,0),(0,1),(0,2),(0,3)\}$ to get the spanning subgraph of $T(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4))$ and add the remaining edges to obtain the total graph. This shows that $cr(\mathbb{Z}_2 \times \mathbb{Z}_4) = 4$.

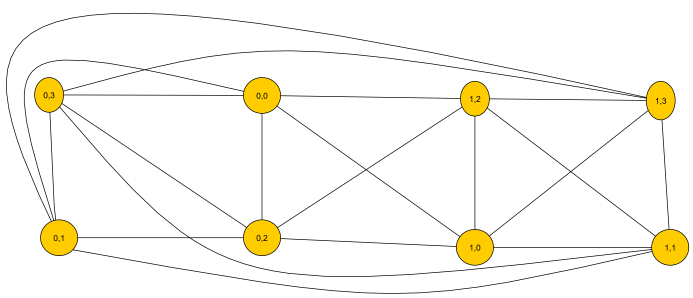

Figure 3: $cr(\mathbb{Z}_2 \times \mathbb{Z}_4) = 4$

In Figure 4, first, we draw $K_3 \cup K_{3,3}$ taking $I = \{(0,0), (0,1), (0,2)\}$ to get the spanning subgraph of $T(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3))$ and add the remaining edges to obtain the total graph. This shows that $cr(\mathbb{Z}_3 \times \mathbb{Z}_3) = 6$.

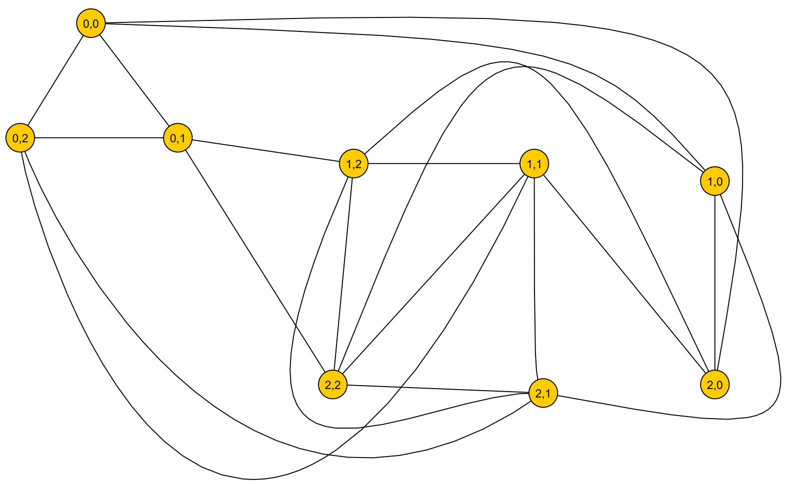

Figure 4: $cr(\mathbb{Z}_3 \times \mathbb{Z}_3) = 6$

Next, Figure 5 is a complete 4 partite graph with each partite set containing 2 elements and the resultant $cr(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) = 10$.

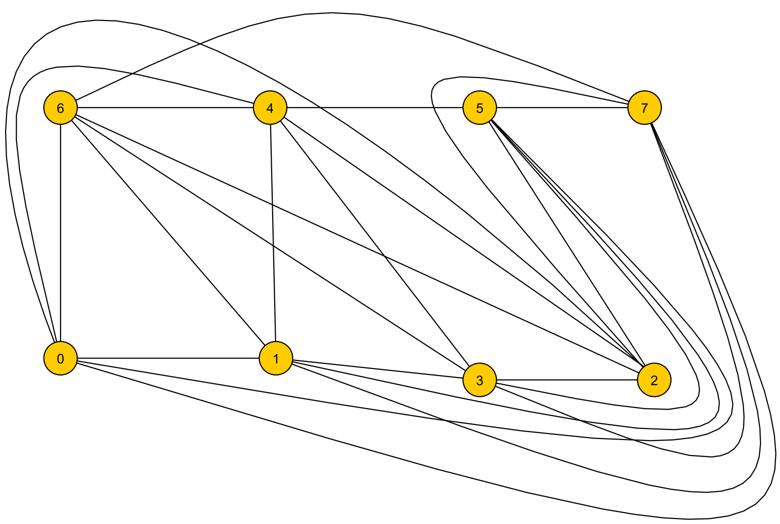

Figure 5: $cr(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) = 10$

In Figure 6, we take $I = \{0, 2, 4, 6, 8\}$ and draw $2K_5$ to get the spanning subgraph of $T(\Gamma(\mathbb{Z}_{10}))$ and add the remaining edges to obtain the total graph. This shows that $cr(\mathbb{Z}_{10}) = 8$.

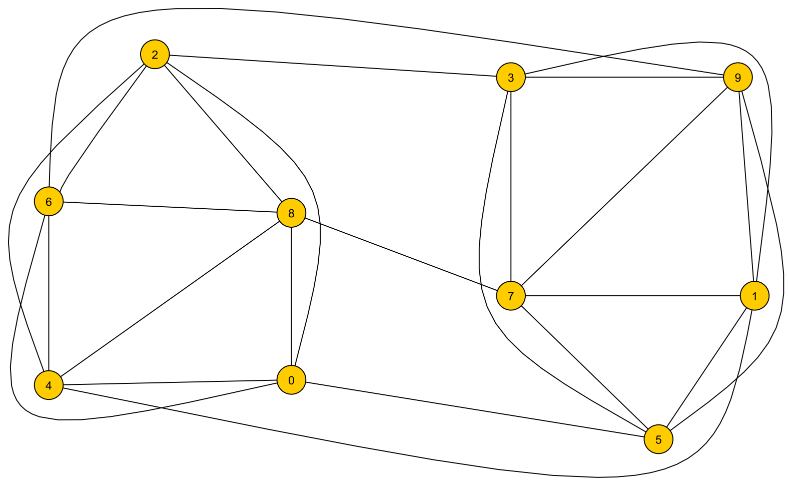

Figure 6: $cr(\mathbb{Z}_{10}) = 8$

At last, in Figure 7. we take $I = \{(0,0),(0,1),(0,x),(0,x+1)\}$ and draw $K_4 \cup K_{4,4}$ which is spanning subgraph of $T(\Gamma(\mathbb{Z}_3 \times \mathbb{F}_4))$ and then add the remaining edges to form the total graph. This shows that $cr(\mathbb{Z}_3 \times \mathbb{F}_4) = 13$.

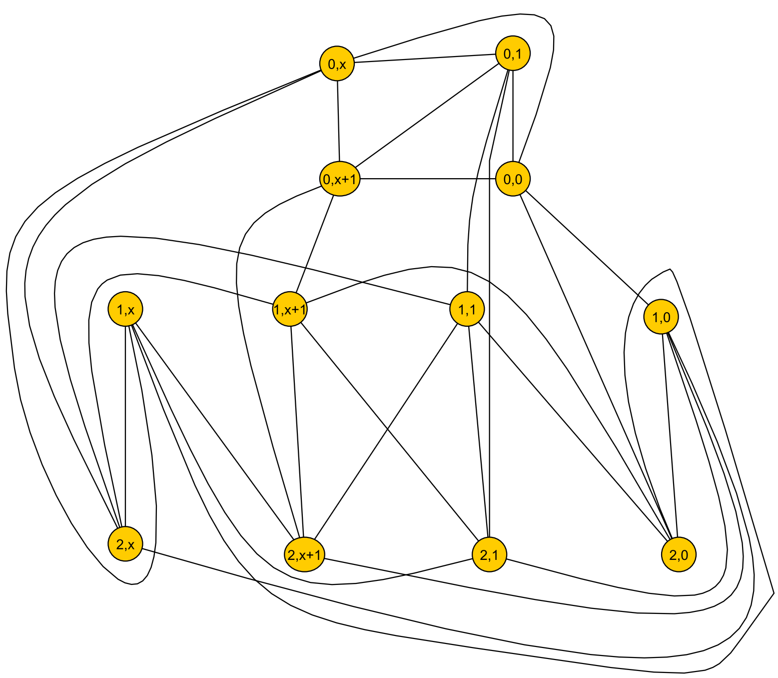

Figure 7: $cr(\mathbb{Z}_3 \times \mathbb{F}_4) = 13$

We conclude that the results may be useful for characterization of other algebraic objects such as modules over a ring, vector spaces over a field.

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