# Frames and topological spaces in MSet

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**Abstract.** In this paper we give a definition for frames in the category **MSet** of actions of a monoid $M$ on sets. We then show that, like in the classical case, frames and (internally) complete Heyting algebras are the same. Further, we give a definition of a topological space in **MSet** and study the relation between frames and topological spaces in this category. We show that the well known adjunction between the two categories still exists in **MSet**.

**Keywords:** frame, topological space, topos, $M$-set.

## 1. Introduction

Algebraic study of topological spaces through their lattice of open sets also known as *pointless* topology or theory of *frames* was initiated by M. Stone's representation theorem for Boolean algebras where a distinguished class of topological spaces known as *Stone* spaces arise as the representation of Boolean algebras. Further there was established a duality between frames and topological spaces in general and a class of spaces namely *sober* spaces were classified which could be studied algebraically through the lattice of their open sets and regardless of their geometric entity. In this paper frames and topological objects are defined in the context of the topos of actions of a monoid on sets and it is shown that such an adjunction exists in this context.

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## 2. Preliminaries

In this section we recall some basic facts and definitions needed in the sequel from [2], [3], and [5].

**Definition 2.1.** *A topos is a cartesian closed category (a category with finite limits and exponentiation) $\mathcal{E}$ together with a subobject classifier which is an object $\Omega \in \mathcal{E}$ together with a morphism $true : \mathbf{1} \to \Omega$ such that for every monomorphism $f : A \to B$, there exists a unique morphism $\chi_f : B \to \Omega$ making the diagram below a pullback square:*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \big\downarrow{\scriptstyle !} & & \big\downarrow{\scriptstyle \chi_f} \\ \mathbf{1} & \xrightarrow{true} & \Omega \end{array}$$

**Definition 2.2.** *Let $M$ be a monoid with the identity element $e$. A (left) $M$-Set $X$ is a set equipped with a map $\lambda : M \times X \to X$, called the (left) action of $M$ on $X$, such that, denoting $\lambda(s, x) = sx$,*

- $ex = x \quad (\forall x \in X)$,
- $(st)x = s(tx) \quad (\forall s, t \in M, \forall x \in X)$.

*A map $f : X \to Y$ between $M$-sets $X$ and $Y$ is said to be action-preserving or an equivariant map if $f(sx) = sf(x)$ for all $s \in M$ and $x \in X$. The category so obtained is denoted by* **MSet**.

Recall (from, for example, [3, p.100] or [2]) that, **MSet** is a topos. Limits in **MSet** are computed just as they are in **Set**. For $X, Y \in \mathbf{MSet}$, $X^Y = \{f : M \times Y \to X \mid f \text{ is equivariant}\}$ is the exponential object with the evaluation arrow $ev : X^Y \times Y \to X$ defined by $ev(f, y) = f(e, y)$ and with the action of $M$ on $X^Y$ defined as $(sf)(m, y) = f(ms, y)$ for all $s, m \in M$ and $y \in Y$. Also, the subobject classifier $\Omega$ is the set $L_M$ of all left ideals of $M$ ($I \subseteq M$ is a left ideal if for all $s \in M$ and $x \in I$ we have $sx \in I$) with the action $sI = \{m \in M : ms \in I\}$, for every $s \in M$ and $I \in \Omega$. Further, notice that monomorphisms and epimorphisms in **MSet** are, respectively, one-one and onto equivariant maps.

**Remark 2.1.** Every $f \in X^Y$ can be equivalently seen as $f = (f_s)_{s \in M}$ where $f_s : Y \to X$ is a map, $f$ being equivariant imposes a compatibility rule on the set $\{f_s : s \in M\}$ such that for all $t \in M$ and for all $y \in Y$, $tf_s(y) = f_{ts}(ty)$ or for short, $tf_s = f_{ts}t$. We will use this interpretation of elements of $X^Y$ throughout this paper. Also, as a matter of convenience and in some definitions and proofs, for every $X \in \mathbf{MSet}$ and every $Y \in \Omega^X$, regarding the fact that in **MSet**, $Sub_{\mathbf{MSet}}(M \times X) \simeq hom_{\mathbf{MSet}}(M \times X \to \Omega) \simeq \Omega^X$, $Y$ will be equivalently regarded as $Y = (Y_s)_{s \in M}$ where $Y_s \subseteq X$ and for every $t \in M$, $tY_s = \{ty : y \in Y_s\} \subseteq Y_{ts}$ (see, for example, [1]).

**Definition 2.3.** *A frame is a complete lattice $A$ satisfying the infinite distributive law. That is, for every $X \subseteq A$ and every $a \in A$ we have that*

$$a \wedge \bigvee X = \bigvee \{a \wedge x : x \in X\}.$$

*We denote the category of frames and maps preserving infinite joins and finite meets between them by* **Frm**.

**Remark 2.2.** We recall that the category **Frm** is related to the category **Top** of topological spaces and continuous maps in the following way (for more information see, for example, [4] or [7]).

There exists an adjunction

$$pt : \mathbf{Frm} \leftrightarrows \mathbf{Top} : O$$

in which $O$ takes every topological space to its frame of open subsets and every continuous map $f : X \to Y$ to the frame homomorphism $f^{-1} : O(Y) \to O(X)$. The functor $pt$ takes every frame $A$ to the set of so called "points" of $A$ which are frame homomorphisms from $A$ to the two-element frame **2**. The set of points of $A$ comes with a natural topology $\{pt(a) : a \in A\}$ where $pt(a) = \{p \in pt(A) : p^{-1}(a) = 1\}$. The functor $pt$ takes every frame homomorphism $\gamma : A \to B$ to the continuous map $pt(\gamma) : pt(B) \to pt(A)$ by composition.

## 3. Frames in MSet

In this section we introduce the notion of frames and Heyting algebras in the category **MSet** and show that frames and complete Heyting algebras coincide in this category, just as in the classical context.

**Definition 3.1.** *A lattice in the category* **MSet** *is a lattice with action preserving maps $\vee, \wedge : A \times A \to A$ satisfying the equalities of a lattice. A homomorphism of lattices in* **MSet** *is that of a lattice which is also action preserving. We denote the category of lattices and lattice homomorphisms in* **Set** *by* **Latt** *and the category of lattices in* **MSet** *and action-preserving lattice homomorphisms (or $M$-lattice homomorphisms) by* **MLatt**. *We also denote the (full) subcategory of* **MLatt** *of bounded lattices in* **MSet** *by* $\mathbf{MLatt_{01}}$.

**Definition 3.2** ([1])**.** *A lattice $A$ in* **MSet** *is said to be internally complete if there exists an equivariant, order-preserving map $\bigvee : \Omega^A \to A$ such that for every $X = (X_s)_{s \in M} \in \Omega^A$ and every $a \in A$ we have that*

$$\bigvee X \leq a \Leftrightarrow X \leq [a],$$

*where $[a]_s = \{b \in A : b \leq sa\} = \downarrow sa$.*

**Definition 3.3.** *A frame in* **MSet** *is an internally complete lattice $A$ in* **MSet** *such that*

$$a \wedge \bigvee X = \bigvee (\{sa \wedge x : x \in X_s\})_{s \in M},$$

*for every $a \in A$ and $X \in \Omega^A$.*

**Remark 3.1** ([6])**.** Let $A$ be an internally complete lattice in **MSet**. Then, for each $X \in \Omega^A$, $\bigvee X = \bigvee \langle X \rangle$, where $\langle X \rangle \in \Omega^A$ is such that for every $s \in M$, $\langle X \rangle_s$ is the ideal of $A$ generated by $X_s$.

**Lemma 3.1.** *Let $A \in$ **MFrm**. Then, for $a \in A$ we have*

$$\bigvee [a] = \bigvee (sa)_{s \in M} = a.$$

**Proof.** Since $[a] \leq [a]$, by definition of internal supremum, we have that $\bigvee [a] \leq a$. On the other side we have that $\bigvee [a] \leq \bigvee [a]$ so again by the definition of the internal supremum we have that $[a] \leq [\bigvee [a]]$. So, for every $s \in M$ we have that $\downarrow sa \subseteq \downarrow s \bigvee [a]$ and therefore by substituting $s$ with $e$, we have $a \leq \bigvee [a]$. So, $\bigvee [a] = a$. Also, $\bigvee (sa)_{s \in M} = \bigvee [a]$ follows from the previous remark. $\square$

**Definition 3.4.** *An $M$-frame homomorphism between two $M$-frames $A$ and $B$ is an internally complete $M$-lattice homomorphism. That is $h : A \to B$ is said to be internally complete if it commutes with the internal join as in the diagram*

$$\begin{array}{ccc} \Omega^A & \xrightarrow{\exists h} & \Omega^B \\ \downarrow \bigvee & & \downarrow \bigvee \\ A & \xrightarrow{h} & B \end{array}$$

*where $(\exists h)(X) = (h(X_s))_{s \in M}$ for every $X \in \Omega^A$.*

The category **MFrm** is consisted of $M$-frames and $M$-frame homomorphisms between them.

**Definition 3.5.** *Let $A \in$ $\mathbf{MLatt_{01}}$. Then, $A$ is called an $M$-Heyting algebra if there exists an equivariant map $\to : A \times A \to A$ such that*

$$x \wedge a \leq b \quad \Leftrightarrow \quad x \leq a \to b \quad \forall x, a, b \in A.$$

*We denote the category of Heyting algebras in* **MSet** *and $\to$-preserving equivariant lattice homomorphisms by* **MHeyt***.*

The following is a straightforward consequence of the definition of Heyting algebras in **MSet** just as in the classical case. See [5, p.200].

**Lemma 3.2.** **MHeyt** *is the subcategory of* $\mathbf{MLatt_{01}}$ *the objects of which satisfy the equations*

- $a \to a = 1$,
- $a \wedge (a \to b) = a \wedge b$,
- $b \wedge (a \to b) = b$,
- $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$.

Just as the case for frames in **Set**, (internal) completeness in **MFrm** makes defining pseudocomplements and implication operation possible. Hence making a connection with Heyting algebras.

**Lemma 3.3.** *Let $A \in$ **MFrm** be given with the internal join $\bigvee : \Omega^A \to A$. Define $\to : A \times A \to A$ as*

$$a \to b = \bigvee(\{x : x \wedge sa \leq sb\})_{s\in M}.$$

*Then, $\to$ is equivariant and it satisfies the condition stated in 3.5.*

**Proof.** First observe that if $x \wedge sa \leq sb$ for $s \in M$ and $a, b \in A$, then for every $t \in M$ we have that $tx \wedge tsa \leq tsb$ and therefore $(\{x : x \wedge sa \leq sb\})_{s\in M} \in \Omega^A$. Now, let $m \in M$. We have that

$$\begin{aligned} m(a \to b) &= m\bigvee(\{x : x \wedge sa \leq sb\})_{s\in M} \\ &= \bigvee m(\{x : x \wedge sa \leq sb\})_{s\in M} \\ &= \bigvee(\{x : x \wedge sma \leq smb\})_{s\in M} \\ &= ma \to mb. \end{aligned}$$

So, $\to$ is equivariant. Now, suppose that $x, a, b \in A$ and $x \wedge a \leq b$. Then, we have that

$$\begin{aligned} &(\forall s \in M)(sx \wedge sa \leq sb) \\ &\Rightarrow (\forall s \in M)(\downarrow sx \subseteq \{x \in A : x \wedge sa \leq sb\}) \\ &\Rightarrow \bigvee(\downarrow sx)_{s\in M} \leq \bigvee(\{x :\in A : x \wedge sa \leq sb\})_{s\in M} \\ &\Rightarrow x \leq (a \to b). \end{aligned}$$

Now, conversely suppose that $x \leq a \to b$. Then

$$\begin{aligned} &a \wedge x \leq a \wedge (a \to b) \\ \Rightarrow\ &a \wedge x \leq a \wedge \bigvee(\{x : x \wedge sa \leq sb\})_{s\in M} \\ \Rightarrow\ &a \wedge x \leq \bigvee(\{sa \wedge x : x \wedge sa \leq sb\})_{s\in M} \\ \Rightarrow\ &a \wedge x \leq \bigvee(\downarrow sb)_{s\in M} \\ \Rightarrow\ &a \wedge x \leq b. \end{aligned}$$

□

**Remark 3.2.** Let $A \in$ **MFrm** and $x \in A$. Then

$$x^* = \bigvee(\{y \in A : y \wedge sx = 0\})_{s\in M}$$

is the largest element of $A$ such that $x^* \wedge x = 0$.

**Proof.** We have that $x \wedge x^* = 0$. For

$$\begin{aligned} x \wedge x^* &= x \wedge \bigvee(\{y \in A : y \wedge sx = 0\})_{s\in M} \\ &= \bigvee(\{sx \wedge y : y \wedge sx = 0\})_{s\in M} \\ &= \bigvee \mathbf{0} \\ &= 0, \end{aligned}$$

where by $\mathbf{0}$ we imply $(0_s)_{s\in M} \in \Omega^A$ with $0_s = \{0_A\}$ for every $s \in M$. The last line being due to the fact that $0_s = s\{0_A\}$ for every $s \in M$. Now, let $a \in A$ and $a \wedge x = 0$. Then, for every $s \in M$, we have $sa \wedge sx = 0$, and so $\downarrow sa \subseteq \{y \in A : y \wedge sx = 0\}$ for every $s \in M$. Thus $\bigvee[a] = \bigvee(\downarrow sa)_{s\in M} \leq \bigvee\{y \in A : y \wedge sx = 0\} = x^*$. But by Lemma 3.1, $a = \bigvee[a]$, and therefore $a \leq x^*$. □

**Lemma 3.4.** *An object $A \in$ **MSet** is an $M$-frame if and only if it is an internally complete $M$-Heyting algebra.*

**Proof.** Necessity. By 3.3, it is evident.

Sufficiency. Suppose $A \in$ **MHeyt** and is internally complete. We need to show that for all $a \in A$ and $X \in \Omega^A$,

$$a \wedge \bigvee X = \bigvee(\{ma \wedge x : x \in X_m\})_{m\in M}.$$

We have that

$$\begin{aligned} & a \wedge \bigvee X \leq \bigvee(\{ma \wedge x : x \in X_m\})_{m\in M} \\ & \Leftrightarrow \bigvee X \leq (a \to \bigvee(\{ma \wedge x : x \in X_m\})_{m\in M}) \\ & \Leftrightarrow (\forall s \in M)(X_s \subseteq \downarrow s(a \to \bigvee(\{ma \wedge x : x \in X_m\})_{m\in M})) \\ & \Leftrightarrow (\forall s \in M)(X_s \subseteq \downarrow (sa \to \bigvee(\{msa \wedge x : x \in X_{ms}\})_{m\in M})) \\ & \Leftrightarrow (\forall s \in M)(\forall x \in X_s)(x \leq sa \to (\bigvee(\{msa \wedge x : x \in X_{ms}\})_{m\in M})) \\ & \Leftrightarrow (\forall s \in M)(\forall x \in X_s)(x \wedge sa \leq \bigvee(\{msa \wedge x : x \in X_{ms}\})_{m\in M}) \\ & \Leftrightarrow (\forall s \in M)(\forall x \in X_s)(\bigvee(\downarrow (mx \wedge msa))_{m\in M} \leq \bigvee(\{msa \wedge x : x \in X_{ms}\})_{m\in M}). \end{aligned}$$

the last line being due to lemma 3.1 and the fact that

$$(\forall s, m \in M)(mX_s \subseteq X_{ms}).$$

For the other side we have that $\{ma \wedge x : x \in X_m\} \subseteq \downarrow ma$ for every $m \in M$. So, $\bigvee(\{ma \wedge x : x \in X_m\})_{m\in M} \leq \bigvee[a] = a$. Since $\{ma \wedge x : x \in X_m\} \subseteq \langle X_m \rangle$ for every $m \in M$, we conclude that

$$\bigvee(\{ma \wedge x : x \in X_m\})_{m\in M} \leq \bigvee \langle X \rangle = \bigvee X.$$

So, $\bigvee(\{ma \wedge x : x \in X_m\})_{m\in M} \leq a \wedge \bigvee X$. □

**Lemma 3.5.** *For every* $X \in \mathbf{MSet}$, $\Omega^X$ *is a frame with the internal join*

$$\bigvee : \Omega^{\Omega^X} \to \Omega^X,$$
$$\bigvee \Theta = (\bigcup \{X_e : X = (X_s)_{s\in M} \in \Theta_m\})_{m\in M}.$$

*for every* $\Theta = (\Theta_s)_{s\in M} \in \Omega^{\Omega^X}$.

**Proof.** It is proven in [6] that $\Omega^X$ is an internally complete lattice with the stated internal join. So, we just need to show that for every $\Theta \in \Omega^{\Omega^X}$ and every $U \in \Omega^X$,

$$U \wedge \bigvee \Theta = \bigvee (\{sU \wedge \Theta_s\})_{s\in M}.$$

We have

$$\begin{aligned} U \wedge \bigvee \Theta &= U \wedge (\bigcup \{X_e : X = (X_s)_{s\in M} \in \Theta_m\})_{m\in M} \\ &= (U_m \cap \bigcup \{X_e : X = (X_s)_{s\in M} \in \Theta_m\})_{m\in M}. \end{aligned}$$

On the other hand

$$\begin{aligned} \bigvee (\{sU \wedge \Theta_s\})_{s\in M} &= (\bigcup \{X_e : X = (X_s)_{s\in M} \in mU \cap \Theta_m\})_{m\in M} \\ &= (\bigcup \{X_e : X = (X_s)_{s\in M} \in U_m \cap \Theta_m\})_{m\in M} \\ &= (U_m \cap \bigcup \{X_e : X = (X_s)_{s\in M} \in \Theta_m\})_{m\in M}. \end{aligned}$$

So, both sides are equal and $\Omega^{\Omega^X}$ is an $M$-frame as required. □

## 4. Topological object in the category MSet

In this section we introduce the notion of a *topological object* in the category **MSet** using the concept of internal completeness. For a general notion of a topological space in a topos one can refer to [8].

**Remark 4.1.** For every $X \in \mathbf{MSet}$, $\Omega^X$ is an externally complete distributive lattice with arbitrary join defined as

$$(\bigvee_{i\in I} (\{f_i : i \in I)\})_m(x) = \bigcup_{i\in I} (f_i)_m(x),$$

for every family $\{f_i : i \in I\} \subseteq \Omega^X$. Meet is also defined as

$$(\bigwedge_{i\in I} (\{f_i : i \in I\}))_m(x) = \bigcap_{i\in I} (f_i)_m(x),$$

or, by the alternative interpretation of $\Omega^X$,

$$\bigvee_{i\in I} X^i = (\bigcup_{i\in I} X^i_s)_{s\in M},$$

for every family $\{X^i : i \in I\}$ in $\Omega^X$ and

$$\bigwedge_{i\in I} X^i = (\bigcap_{i\in I} X^i_s)_{s\in M}.$$

$\Omega^X$ is also bounded with the least member being $f^{\varnothing} = (f^{\varnothing}_m)_{m\in M}$, where for every $m \in M$ and every $x \in X$, $f_m(x) = \varnothing$, and with the greatest element $f^M = (f^M_m)_{m\in M}$, where for every $m \in M$ and for every $x \in X$, $f^M_m(x) = M$.

**Definition 4.1.** *A topological object in the category* **MSet** *is a pair* $(X, O(X))$ *such that* $X \in$ **MSet** *and* $O(X) \subseteq \Omega^X$ *is a sub* $M$*-frame of* $\Omega^X$ *or, equivalently, is a sub* $M$*-set of* $\Omega^X$ *containing the least and the greatest elements of* $\Omega^X$ *and is closed under finite meets (intersections) and internal joins in* $\Omega^X$. *This is summerized as:*

- $f^{\varnothing}$ *and* $f^M$ *belong to* $O(X)$.
- *If* $\Theta = (\Theta_s)_{s\in M} \in \Omega^{O(X)}$ *then* $\bigvee \Theta$ *as defined in lemma 3.5 belongs to* $O(X)$.
- *Any binary (external) meet of elements of* $O(X)$ *belongs to* $O(X)$.

*We call every element of* $O(X)$ *an open* $M$*-subset of the topology which can be regarded both as some* $U = (U_s)_{s\in M}, U_s \subseteq X$ *or, as used in this paper, as some* $f^U = (f^U_s)_{s\in M}, f^U_s : X \to \Omega$.

**Definition and Lemma 4.1.** *Let* $(X, O(X))$ *and* $(Y, O(Y))$ *be two topological spaces in* **MSet**. *Every equivariant map* $f : X \to Y$ *induces an* $M$*-frame homomorphism* $f^{-1} : \Omega^Y \to \Omega^X$ *such that for every* $(f^U) \in \Omega^Y$, $f^{-1}((f^U)) = (f^U_s \circ f)_{s\in M}$. *Also* $f$ *is said to be* $M$*-continuous if* $f^{-1}((f^U_s)_{s\in M}) \in O(X)$ *for every* $(f^U_s)_{s\in M} \in O(Y)$.

*Note that considering the isomorphism* $\Omega^X \simeq Sub_{\mathbf{MSet}}(M \times X)$, $f^{-1}$ *can be alternatively viewed as* $f^{-1}((Y_s)_{s\in M}) = (f^{-1}(Y_s))_{s\in M}$ *for* $(Y_s)_{s\in M} \in \Omega^Y$ *where the right-hand side* $f^{-1}$ *is the set-theoretical inverse image.*

**Proof.** It is well-known that $f^{-1}$ is a morphism of Heyting algebras in any topos (see, [5, p. 201]). We just need to show that it is internally complete. Let $\Theta \in \Omega^{\Omega^Y}$. We need to show that $\mathbf{f}^{-1}(\bigvee \Theta) = \bigvee (f^{-1}(\Theta_s))_{s\in M}$. We have

$$\begin{aligned}
\mathbf{f}^{-1}(\bigvee \Theta) &= (f^{-1}(\bigvee \Theta)_s)_{s\in M} \\
&= (f^{-1}(\cup\{Y_e : Y = (Y_m)_{m\in M} \in \Theta_s\}))_{s\in M} \\
&= (\cup\{X_e : (X_m)_{m\in M} \in f^{-1}(\Theta_s)\})_{s\in M} \\
&= \bigvee (f^{-1}(\Theta_s))_{s\in M}.
\end{aligned}$$

□

We denote the category of topological $M$-sets and $M$-continuous maps as **MTop**. In the classical setting, all maps from the one-point topological space (with the obvious topology) to a topological space $(X, O(X))$ are continuous. These maps are in one-one correspondence with actual points of the set $X$. We show that in **MSet**, the topological object $(M, \Omega^M)$ has such property.

**Lemma 4.1.** *For every monoid $M$, $(M, \Omega^M)$ is a topology in* **MSet**. *Moreover, for every topology $(X, O(X))$ in* **MSet**, *every equivariant map $f : M \to X$ is continuous. In fact, there exists a one-one correspondence between actual points of $X$ and continuous maps from $(M, \Omega^M)$ to $(X, O(X))$.*

**Proof.** Let $f : M \to X$ be equivariant. In fact every equivariant map from $M$ to $X$ is identified with some $x \in X$ where $x = f(e)$. Then, by Lemma 4.1, $f$ induces the $M$-frame homomorphism $f^{-1} : \Omega^X \to \Omega^M$, taking $(f_s^U)_{s\in M} \in \Omega^X$ to $(f_s^U \circ f)_{s\in M}$. It is trivial that $f$ is continuous, since $M$ is equipped with the largest topology. □

## 5. Adjunction between MTop and MFrm

As we know for topological spaces and frames in the classical setting, there exists an adjunction between the two categories. In this section we show the existence of such an adjunction between the two categories in **MSet**.

**Lemma 5.1.** *Let $A \in$* **MFrm** *and put $pt(A) = \{\gamma : A \to \Omega^M : \gamma$ is equivariant$\}$. Then $pt(A)$ is an $M$-set with the action of $M$ on it defined as*

$$((t\gamma)(a))_m(n) = (\gamma(a))_m(nt), \forall a \in A, \forall t, m, n \in M.$$

**Proof.** First we need to show that $(t\gamma)(a) \in \Omega^M$ for every $t \in M$ and every $a \in A$ or, equivalently, for every $s, m, n \in M$,

$$s((t\gamma)(a))_m(n) = ((t\gamma)(a))_{sm}(sn).$$

But, we have

$$\begin{aligned} s((t\gamma)(a))_m(n) &= s(\gamma(a))_m(nt) \\ &= (\gamma(a))_{sm}(snt) \\ &= ((t\gamma)(a))_{sm}(sn). \end{aligned}$$

Now, to show that $t\gamma$ is equivariant, for every $s, t, m, n \in M$ we have

$$\begin{aligned} ((t\gamma)(sa))_m(n) &= (\gamma(sa))_m(nt) \\ &= (s\gamma(a))_m(nt) \\ &= (\gamma(a))_{ms}(nt) \\ &= (t\gamma(a))_{ms}(n) \\ &= (s(t\gamma(a)))_m(n). \end{aligned}$$

So, $t\gamma \in pt(A)$. Also,

$$\begin{aligned}(((st)\gamma)(a))_m(n) &= (\gamma(a))_m(n(st))\\ &= (\gamma(a))_m((ns)t)\\ &= (t\gamma(a))_m(ns)\\ &= (s(t\gamma(a)))_m(n).\end{aligned}$$

□

**Lemma 5.2.** *For every $A \in$ **MFrm** and $a \in A$, put $pt(a) = (\{\gamma \in pt(A) : (\gamma(a))_m(e) = M\})_{m\in M}$. Then $pt(a) \in \Omega^{pt(A)}$ and $\{pt(a) : a \in A\}$ is a sub $M$-set of $\Omega^{pt(A)}$ and is an $M$-frame.*

**Proof.** Let $a \in A$ and $\gamma \in (pt(a))_s$. So, we have $(\gamma(a))_s(e) = M$. Then, for every $t \in M$

$$\begin{aligned}((t\gamma)(a))_{ts}(e) &= (\gamma(a))_{ts}(t)\\ &= t(\gamma(a))_s(e),\end{aligned}$$

and therefore $t\gamma \in (pt(a))_{ts}$ and $pt(a) \in \Omega^{pt(A)}$. Now, we need to show that $\{pt(a) : a \in A\}$ is an $M$-set. Let $t \in M$. We have

$$\begin{aligned}tpt(a) &= (\{\gamma \in pt(A) : (\gamma(a))_{mt}(e) = M\})_{m\in M}\\ &= (\{\gamma \in pt(A) : (t\gamma(a))_m(e) = M\})_{m\in M}\\ &= (\{\gamma \in pt(A) : (\gamma(ta))_m(e) = M\})_{m\in M}\\ &= pt(ta).\end{aligned}$$

Now, we want to show that $\{pt(a) : a \in A\}$ is an $M$-frame. Suppose $\Theta \in \Omega^{\{pt(a):a\in A\}}$. Then, by Lemma 3.5,

$$\begin{aligned}(\bigvee \Theta)_m &= \bigcup\{(pt(a))_e : pt(a) \in \Theta_m\}\\ &= \{\gamma \in pt(A) : \exists a \in A : pt(a) \in \Theta_m, (\gamma(a))_e(e) = M\}.\end{aligned}$$

Now, put $Y = (\{a \in A : pt(a) \in \Theta_s\})_{s\in M}$. It is easy to verify that $Y \in \Omega^A$. Now, since $\gamma$ is an internally complete lattice homomorphism, we have that

$$\gamma(\bigvee Y) = \bigvee(\gamma(Y)),$$

where $\gamma(Y) = (\gamma(Y_s))_{s\in M}$ and $\gamma(Y) \in \Omega^{\Omega^M}$. So, again, by Lemma 3.5, for every $m \in M$ we have

$$\begin{aligned}(\bigvee \gamma(Y))_m &= \bigcup\{X_e : X \in (\gamma(Y))_m\}\\ &= \bigcup\{X_e : X \in \{\gamma(a) : pt(a) \in \Theta_m\}\}\\ &= \bigcup\{(\gamma(a))_e : pt(a) \in \Theta_m\}.\end{aligned}$$

And

$$\begin{aligned}(pt(\gamma(\bigvee Y)))_m &= \{\gamma \in pt(A) : (\bigvee \gamma(Y))_m(e) = M\}\\ &= \{\gamma \in pt(A) : \cup\{(\gamma(a))_e : pt(a) \in \Theta_m\} = M\}\\ &= \{\gamma \in pt(A) : \exists a \in A : pt(a) \in \Theta_m, (\gamma(a))_e(e) = M\}\\ &= (\bigvee \Theta)_m.\end{aligned}$$

So, $\{pt(a) : a \in A\}$ is closed under internal join and therefore is a frame. □

**Remark 5.1.** Considering that for a monoid $M$ and an $M$-set $X$, $Sub_{\mathbf{MSet}}(M \times X) \simeq \Omega^X$, $pt(a)$ can be equivalently regarded as

$$(pt(a))_s(\gamma) = \{x \in M : x \in (\gamma(a))_s(e)\}.$$

**Lemma 5.3.** *Let $A \in$ **MFrm**. Then $(pt(A), \{pt(a) : a \in A\})$ is a topology in* **MSet**.

**Proof.** By Lemma 5.1, $pt(A)$ is an $M$-set and, by Lemma 5.2, $\{pt(a) : a \in A\}$ is a frame in **MSet** and a subobject of $\Omega^{pt(A)}$. □

**Lemma 5.4.** *The mapping $A \mapsto pt(A)$, is functorial from* $\mathbf{MFrm}^{op}$ *to* **MTop**. *In fact, it takes an $M$-frame $A$ to the $M$-topological space $(pt(A), \{pt(a) : a \in A\})$, and an $M$-frame homomorphism $\gamma : A \to B$ to $M$-continuous map $pt(\gamma) : pt(B) \to pt(A)$ taking $f : B \to \Omega^M$ to $f \circ \gamma$.*

**Proof.** For every $M$-frame homomorphism $\gamma : A \to B$, $pt(\gamma)$ is equivariant. Let $m \in M$. Then

$$\begin{aligned}((pt\gamma(mf))(a))_t(n) &= ((mf)(\gamma(a)))_t(n)\\ &= (f(\gamma(a)))_t(nm)\\ &= ((m(f \circ \gamma))(a))_t(m)\\ &= (m(pt\gamma f))_t(n).\end{aligned}$$

We also have that $pt\gamma$ is continuous. Let $\Theta \in \Omega^{\{pt(a):a\in A\}}$ and $f \in pt(B)$. Then

$$\begin{aligned}((pt\gamma)^{-1}(\bigvee \Theta))_s(f) &= (\bigvee \Theta)_s(pt\gamma(f))\\ &= (\bigvee \Theta)_s(f \circ \gamma)\\ &= \{x \in M : \exists a \in A : pt(a) \in \Theta_{xs}, ((f \circ \gamma)(a))_e(x) = M\}.\end{aligned}$$

Now, let $\Gamma = (\{pt(\gamma(a)) : pt(a) \in \Theta_m\})_{m\in M}$. It is straightforward to see that $\Gamma \in O(pt(B))$. Then

$$\begin{aligned}(\bigvee \Gamma)_s(f) &= \{x \in M : \exists b \in B : pt(b) \in \Gamma\}\\ &= \{x \in M : \exists b \in B : pt(b) \in \Gamma_{xs}, (f(b))_e(x) = M\}\\ &= \{x \in M : \exists a \in A : pt(a) \in \Theta_{xs}, ((f \circ \gamma)(a))_e(x) = M\}.\end{aligned}$$

So, $pt\gamma(\bigvee \Theta) = \bigvee \Gamma$ and $pt\gamma$ is continuous. □

**Proposition 5.1.** *There exists an adjunction between the categories* $\mathbf{MFrm^{op}}$ *and* **MTop**.

**Proof.** Let $X \in \mathbf{MTop}$. First note that, by Lemma 4.1, for every nonempty $X$, $pt(O(X))$ is nonempty. Define $\alpha : X \to pt(O(X))$ as $((\alpha(x))(f^U)))_m(n) = f^U_m(nx)$.

- For every $x \in X$ and every $f^U \in O(X)$, $(\alpha(x))(f^U) \in \Omega^M$. For, let $t, m, n \in M$. Then

$$\begin{aligned} t((\alpha(x))(f^U))_m(n) &= t(f^U)_m(nx) \\ &= (f^U)_{tm}(tnx) \\ &= ((\alpha(x))(f^U))_{tm}(tn). \end{aligned}$$

- For every $x \in X$, $\alpha(x) \in pt(O(X))$ or $\alpha(x) : O(X) \to \Omega^M$ is equivariant. For let $t, m, n \in M$ and $f^U \in O(X)$. Then

$$\begin{aligned} ((\alpha(x))(tf^U))_m(n) &= (tf^U)_m(nx) \\ &= (f^U)_{mt}(nx) \\ &= (\alpha(x)(f^U))_{mt}(n) \\ &= (t\alpha(x)(f_U))_m(n). \end{aligned}$$

- $\alpha$ is equivariant. Let $t, m, n \in M$ and $f^U \in O(X)$. Then

$$\begin{aligned} ((\alpha(tx))(f^U))_m(n) &= f^U_m(ntx) \\ &= ((\alpha(x))(f^U))_m(nt) \\ &= ((t\alpha(x))(f^U))_m(n). \end{aligned}$$

- $\alpha$ is continuous. Let $f^U = (f^U_s)_{s\in M} \in O(X)$. Then $pt(f^U) \in O(pt(O(X)))$. For every $x \in X$ we have

$$\begin{aligned} (\alpha^{-1}(pt(f^U)))_s(x) &= ((pt(f^U))_s(\alpha(x))_s \\ &= \{y \in M : y \in (\alpha(x)(f^U))_s(e)\} \\ &= \{y \in M : y \in f^U_s(x)\} \\ &= f^U_s(x). \end{aligned}$$

Now, we show that for every $A \in \mathbf{MFrm}$, and every continuous $\beta : X \to pt(A)$, there exists a unique $M$-frame homomorphism $\gamma : A \to O(X)$ such that $pt(\gamma) \circ \alpha = \beta$. For every $m \in M, x \in X$, $a \in A$ put

$$(\gamma(a))_m(x) = ((\beta(x))(a))_m(e).$$

- $\gamma(a) \in O(X)$. We have that

$$\begin{aligned}(\beta^{-1}(pt(a)))_m(x) &= (pt(a))_m(\beta(x)) \\ &= \{y \in M : y \in ((\beta(x))(a))_m(e)\} \\ &= (\gamma(a))_m(x).\end{aligned}$$

So, we have that $\gamma(a) = \beta^{-1}(pt(a))$ and by continuity of $\beta$, $\gamma(a) \in O(X)$.

- $\gamma$ is equivariant. Let $m, n \in M, a \in A$ and $x \in X$. Then

$$\begin{aligned}(\gamma(na))_m(x) &= ((\beta(x))(na))_m(e) \\ &= (n(\beta(x))(a))_m(e) \\ &= ((\beta(x))(a))_{mn}(e) \\ &= (n\gamma(a))_m(x).\end{aligned}$$

- $pt\gamma \circ \alpha = \beta$. Let $x \in X$ and $a \in A$. Then

$$\begin{aligned}(pt\gamma(\alpha(x))(a))_m(n) &= ((\alpha(x))(\gamma(a)))_n(m) \\ &= (\gamma(a))_m(nx) \\ &= ((\beta(nx))(a))_m(e) \\ &= ((n\beta(x))(a))_m(e) \\ &= ((\beta(x))(a))_m(n).\end{aligned}$$

- $\gamma$, as defined above, is the unique $M$-frame homorphism from $A$ to $O(X)$ such that $pt\gamma \circ \alpha = \beta$. Suppose $\lambda : A \to O(X)$ is an $M$-frame homomorphism such that $pt\lambda(\alpha(x)) = \beta(x)$. So, we have that $(\alpha(x))(\lambda) = \beta(x)$, which yields $(\lambda(a))_m(nx) = (\beta(x)(a))_m(n)$ and therefore $(\lambda(a))_m(x) = (\beta(x)(a))_m(e)$. □

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