Certain fixed point theorems on partial metric spaces

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Abstract. The existence and uniqueness of common fixed point theorems for two pairs of weakly compatible self mappings satisfying some contractive condition on partial metric space is given. Further proved a theorem for a pair of self mappings satisfying rational type contraction condition without using weakly compatibility. Some examples are also given to support our results.

Keywords: common fixed point, weakly compatible mappings, partial metric space.

1. Introduction

Contraction is one of the main tool, to prove the existence and uniqueness of fixed point. Banach [2] contraction principle gives an answer on is the existence and uniqueness of the solution of an equation $Tx = x$. This principle is one of the most useful tool in the study of nonlinear equations. There are many

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generalizations of Banach contraction mappings in the literature (see [5]-[9],[13]-[14] etc.).

Matthews [10] generalize the Banach contraction principle and introduced the concept of partial metric space. The fixed point result given by matthews [10] is generalized by several authors (see, [11], [3]-[4], [8] and others). It is widely recognized that the partial metric spaces play an important role in constructing models in the theory of computation. Partial metric has applications in the branch of science, where the size of the data point is represented by its self distance.

It is known that "Every metric space is a partial metric space with zero self distance that is partial metric spaces are the generalization of metric spaces." In this direction recently, using rational contraction, Arshed et. al [1] proved some fixed point theorems in the setting of partially ordered metric space. Motivated by [1] in this paper, some coincidence and common fixed point results in partial metric space are given.

The definition of partial metric space is given by Matthews [10] as follows:

Definition 1.1. *Let X be a nonempty set and let $p : X \times X \to \mathbb{R}_0^+$ satisfy:*

1. $x = y \iff p(x,x) = p(y,y) = p(x,y)$;
2. $p(x,x) \leq p(x,y)$;
3. $p(x,y) = p(y,x)$;
4. $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$,

for all $x, y \in X$, where $\mathbb{R}_0^+ = [0,\infty)$. Then, the pair (X,p) is called a partial metric space and p is called a partial metric on X. It is clear that if $p(x,y) = 0$, then from (1) and (2) $x = y$. But if $x = y$, $p(x,y)$ may not be zero.

Example 1.1. Let $X = \mathbb{R}_0^+$ and $p : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ defined by $p(x,y) = \max\{x,y\}$ for all $x, y \in \mathbb{R}_0^+$, then $(\mathbb{R}_0^+, p)$ is a partial metric space.

Example 1.2. Let $X = \{[a,b] : a, b \in \mathbb{R}, a \leq b\}$ and define $p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$. Then (X,p) is a partial metric space.

Let (X,p) be a partial metric space. Then, the function $d_p : X \times X \to \mathbb{R}_0^+$ given by $d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ is a (usual) metric on X.Each partial metric p on X generates a T_0 topology τ_p on X with a base of the family of open p- balls $\{B_p(x,\epsilon) : x \in X, \epsilon > 0\}$, where $B_p(x,\epsilon) = \{y \in X : p(x,y) < p(x,x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$.

Definition 1.2. *Let (X,p) be a partial metric space.*

(1) *A sequence $\{x_n\}$ in a partial metric space (X,p) converges to a point $x \in X$ if $\lim_{n\to\infty} p(x,x_n) = p(x,x)$.*

(2) *A sequence $\{x_n\}$ in a partial metric space (X,p) is called a Cauchy sequence if and only if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and finite.*

(3) *A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p, to a point $x \in X$ such that $p(x,x) = lim_{n,m\to\infty} p(x_n,x_m)$.*

Lemma 1.1. *Let (X,p) be a partial metric space. Then:*

(1) *A sequence $\{x_n\}$ in a partial metric space (X,p) is a Cauchy sequence if and only if it is a Cauchy sequence in the metric space (X,d_p).*

(2) *A partial metric space (X,p) is complete if and only if the metric space (X,d_p) is complete. Moreover, $\lim_{n\to\infty} d_p(x_n,x) = 0$ iff $\lim_{n\to\infty} p(x,x_n) = \lim_{n,m\to\infty} p(x_n,x_m) = p(x,x)$.*

(3) *A subset E of a partial metric space (X,p) is closed if whenever $\{x_n\}$ is a sequence in E such that $\{x_n\}$ converges to some $x \in X$, then $x \in E$.*

Definition 1.3. *Let f and g are self mappings on a set X. A point $x \in X$ is called a coincidence point of f and g if $fx = gx = w$, where w is called a point of coincidence of f and g.*

Definition 1.4. *Two self mappings f and g on a set X are said to be weakly compatible if f and g commute at their coincidence points, that is, if $fx = gx$ for some $x \in X$, then $fgx = gfx$.*

Now, we present our main results.

2. Main results

Theorem 2.1. *Let (X,p) be a complete partial metric space. Suppose that f, g, F and G are self mappings satisfying the following conditions:*

(*i*) *$f(X) \subseteq g(X)$ and $F(X) \subseteq G(X)$;*

(*ii*) *there exist α, β, and L in $(0,1)$ with $2\alpha + \beta + 2L < 1$ such that*

$$p(Fx,fy) \le \alpha \frac{\max\{p(gx,Fx).p(gy,Fy), p(gx,Gy).p(fy,Fx), p(gx,fy).p(Gy,Fx)\}}{p(gx,Gy)} + \beta p(gx,Gy) + L\min\{p(gx,Fx), p(fy,Gy), p(gx,fy), p(Fx,Gy)\},$$

for all $x, y \in X$;

(*iii*) *$f(X)$ or $g(X)$ is closed.*

If the pairs (f,G) and (g,F) are weakly compatible, then f, g, F and G have unique common fixed point in X.

Proof. Suppose $x_0 \in X$ be an arbitrary element of X. Since $f(X) \subseteq g(X)$ and $F(X) \subseteq G(X)$. Construct a sequence $\{y_n\}$ in X satisfying $y_n = Fx_n = Gx_{n+1}$ and $y_{n+1} = fx_{n+1} = gx_{n+2}$, for all $n \in \mathbb{N} \cup \{0\}$. Taking $x = x_n$ and $y = x_{n+1}$ and Using condition (ii), we have

$$\begin{aligned}
p(Fx_n, fx_{n+1}) &\leq \alpha \frac{1}{p(gx_n, Gx_{n+1})} \max\{p(gx_n, Fx_n).p(fx_{n+1}, Gx_{n+1}),\\
&\quad p(gx_n, Gx_{n+1}).p(fx_{n+1}, Fx_n), p(gx_n, fx_{n+1}).p(Gx_{n+1}, Fx_n)\}\\
&\quad + \beta p(gx_n, Gx_{n+1}) + L\min\{p(gx_n, Fx_n), p(Gx_{n+1}, fx_{n+1}),\\
&\qquad p(gx_n, fx_{n+1}), p(Fx_n, Gx_{n+1})\}\\
&= \alpha \frac{\max\{p(y_{n-1}, y_n).p(y_n, y_{n+1}), p(y_{n-1}, y_n).p(y_n, y_{n+1}), p(y_{n-1}, y_{n+1}).p(y_n, y_n)\}}{p(y_{n-1}, y_n)}\\
&\quad + \beta p(y_{n-1}, y_n)\\
&\quad + L\min\{p(y_{n-1}, y_n), p(y_n, y_{n+1}), p(y_{n-1}, y_{n+1}), p(y_n, y_n)\}\\
&= \alpha \frac{\max\{p(y_{n-1}, y_n).p(y_n, y_{n+1}), p(y_{n-1}, y_{n+1}).p(y_n, y_n)\}}{p(y_{n-1}, y_n)}\\
&\quad + \beta p(y_{n-1}, y_n) + Lp(y_n, y_n).
\end{aligned}$$

Case I. If

$$\max\{p(y_{n-1}, y_n).p(y_n, y_{n+1}), p(y_{n-1}, y_{n+1}).p(y_n, y_n)\} = p(y_{n-1}, y_n).p(y_n, y_{n+1})$$

and

$$\min\{p(y_{n-1}, y_{n+1}), p(y_n, y_n)\} = p(y_n, y_n),$$

then we have

$$p(y_n, y_{n+1}) \leq \alpha \frac{p(y_{n-1}, y_n).p(y_n, y_{n+1})}{p(y_{n-1}, y_n)} + \beta p(y_{n-1}, y_n) + Lp(y_n, y_n).$$

Since $p(y_n, y_n) \leq p(y_n, y_{n+1})$, we find that

$$p(y_n, y_{n+1}) \leq \alpha p(y_n, y_{n+1}) + \beta p(y_{n-1}, y_n) + Lp(y_n, y_{n+1}).$$

which implies

$$p(y_n, y_{n+1}) \leq \frac{\beta}{(1 - \alpha - L)} p(y_{n-1}, y_n).$$

Therefore,

$$p(y_n, y_{n+1}) \leq K_1 p(y_{n-1}, y_n).$$

where $K_1 = \frac{\beta}{(1-\alpha-L)}$.

Case II. If

$$\max\{p(y_{n-1}, y_n).p(y_n, y_{n+1}), p(y_{n-1}, y_{n+1}).p(y_n, y_n)\} = p(y_{n-1}, y_{n+1}).p(y_n, y_n)$$

and $\min\{p(y_{n-1},y_{n+1}),p(y_n,y_n)\}=p(y_n,y_n)$ then, we have

$$p(y_n,y_{n+1})\leq\alpha\frac{p(y_{n-1},y_{n+1}).p(y_n,y_n)}{p(y_{n-1},y_n)}+\beta p(y_{n-1},y_n)+Lp(y_n,y_n).$$

Since $p(y_n,y_n)\leq p(y_n,y_{n-1})$ and $p(y_n,y_n)\leq p(y_n,y_{n+1})$ we obtain

$$\begin{aligned}&p(y_n,y_{n+1})\leq\alpha p(y_{n-1},y_{n+1})+\beta p(y_{n-1},y_n)+Lp(y_n,y_{n+1})\\&\leq\alpha p(y_{n-1},y_n)+\alpha p(y_n,y_{n+1})-\alpha p(y_n,y_n)+\beta p(y_{n-1},y_n)+Lp(y_n,y_{n+1})\\&\leq\alpha p(y_{n-1},y_n)+\alpha p(y_n,y_{n+1})+\beta p(y_{n-1},y_n)+Lp(y_n,y_{n+1}).\end{aligned}$$

Consequently, we get

$$p(y_n,y_{n+1})\leq\frac{\alpha+\beta}{(1-\alpha-L)}p(y_{n-1},y_n)$$

which in turn implies that $p(y_n,y_{n+1})\leq K_2p(y_{n-1},y_n)$, where $K_2=\frac{\alpha+\beta}{(1-\alpha-L)}$.

Case III. If

$$\max\{p(y_{n-1},y_n).p(y_n,y_{n+1}),p(y_{n-1},y_{n+1}).p(y_n,y_n)\}=p(y_{n-1},y_n).p(y_n,y_{n+1})$$

and

$$\min\{p(y_{n-1},y_{n+1}),p(y_n,y_n)\}=p(y_{n-1},y_{n+1}),$$

then, we have

$$\begin{aligned}&p(y_n,y_{n+1})\leq\alpha\frac{p(y_{n-1},y_n).p(y_n,y_{n+1})}{p(y_{n-1},y_n)}+\beta p(y_{n-1},y_n)+Lp(y_{n-1},y_{n+1})\\&\leq\alpha p(y_n,y_{n+1})+\beta p(y_{n-1},y_n)+Lp(y_{n-1},y_n)+Lp(y_n,y_{n+1})-Lp(y_n,y_n)\\&\leq\alpha p(y_n,y_{n+1})+\beta p(y_{n-1},y_n)+Lp(y_{n-1},y_n)+Lp(y_n,y_{n+1}).\end{aligned}$$

We obtain

$$p(y_n,y_{n+1})\leq\frac{\beta+L}{1-\alpha-L}p(y_{n-1},y_n)=K_3p(y_{n-1},y_n),$$

where $K_3=\frac{\beta+L}{1-\alpha-L}$.

Case IV. If

$$\max\{p(y_{n-1},y_n).p(y_n,y_{n+1}),p(y_{n-1},y_{n+1}).p(y_n,y_n)\}=p(y_{n-1},y_{n+1}).p(y_n,y_n)$$

and $\min\{p(y_{n-1},y_{n+1}),p(y_n,y_n)\}=p(y_{n-1},y_{n+1})$, then, we have

$$p(y_n,y_{n+1})\leq\alpha\frac{p(y_{n-1},y_{n+1}).p(y_n,y_n)}{p(y_{n-1},y_n)}+\beta p(y_{n-1},y_n)+Lp(y_{n-1},y_{n+1}).$$

Since $p(y_n,y_n)\leq p(y_{n-1},y_n)$, we have

$$\begin{aligned}p(y_n,y_{n+1})&\leq\alpha p(y_{n-1},y_n)+\alpha p(y_n,y_{n+1})-\alpha p(y_n,y_n)+\beta p(y_{n-1},y_n)\\&\quad+Lp(y_{n-1},y_n)+Lp(y_n,y_{n+1})-Lp(y_n,y_n)\\&\leq(\alpha+L)p(y_n,y_{n+1})+(\alpha+\beta)p(y_{n-1},y_n).\end{aligned}$$

From this inequality we obtain

$$p(y_n, y_{n+1}) \leq \frac{\beta + L}{1 - \alpha - L} p(y_{n-1}, y_n) = K_4 p(y_{n-1}, y_n),$$

where $K_4 = \frac{\beta+L}{1-\alpha-L}$.

Let $K = \max\{K_1, K_2, K_3, K_4\}$, so, $0 < K < 1$. Therefore, for each $n \in \mathbb{N}$, we obtain that $p(y_n, y_{n+1}) \leq K^n p(y_0, y_1)$. Now, we shall show that $\{y_n\}$ is a Cauchy sequence in (X, d_p). Let $m, n \in \mathbb{N}$ with $m > n$, we have $p(y_n, y_m) \leq p(y_n, y_{n+1}) + p(y_{n+1}, y_m) - p(y_{n+1}, y_{n+1}) \leq p(y_n, y_{n+1}) + p(y_{n+1}, y_m)$. Also, $p(y_{n+1}, y_m) \leq p(y_{n+1}, y_{n+2}) + p(y_{n+2}, y_m) - p(y_{n+2}, y_{n+2}) \leq p(y_{n+1}, y_{n+2}) + p(y_{n+2}, y_m)$. Therefore, we have $p(y_n, y_m) \leq p(y_n, y_{n+1}) + p(y_{n+1}, y_{n+2}) + ... + p(y)m - 1, y_m) \leq (K^n + K^{n+1} + ... + K^{m-1}) p(y_0, y_1)$. By straight forward calculations, we have

$$\lim_{n,m\to\infty} p(y_n, y_m) = 0. \tag{1}$$

We observe that

$$d_p(y_n, y_m) = 2p(y_n, y_m) - p(y_m, y_m) - p(y_n, y_n) \leq 2p(y_n, y_m).$$

By (1), we conclude that $d_p(y_n, y_m) = 0$. Hence $\{y_n\}$ is a Cauchy sequence in (X, d_p). Since X is complete, we have $\lim_{n\to\infty} y_n = z$ for some $z \in X$. By Lemma 1.1, we obtain

$$p(z, z) = \lim_{n\to\infty} p(y_n, z) = \lim_{n,m\to\infty} p(y_n, y_m).$$

we conclude that

$$p(z, z) = 0. \tag{2}$$

Assume that $g(X)$ is closed, therefore there exists a point $u \in X$ such that $z = gu$. Now,

$$\begin{aligned}
p(z, Fu) &\leq p(z, y_{n+1}) + p(y_{n+1}, Fu) - p(y_{n+1}, y_{n+1}) \\
&\leq p(z, y_{n+1}) + p(y_{n+1}, Fu) = p(z, y_{n+1}) + p(Fu, fx_{n+1}) \leq p(z, y_{n+1}) \\
&+ \alpha \frac{\max\{p(gu, Fu).p(Gx_{n+1}, fx_{n+1}), p(gu, Gx_{n+1}).p(fx_{n+1}, Fu), p(gu, fx_{n+1}).p(Fu, Gx_{n+1})\}}{p(gu, Gx_{n+1})} \\
&+ \beta p(gu, Gx_{n+1}) + L \min\{p(gu, Fu), p(Gx_{n+1}, fx_{n+1}), \\
&\quad p(gu, fx_{n+1}), p(Fu, Gx_{n+1})\} \\
&= p(z, y_{n+1}) + \alpha \frac{\max\{p(gu, Fu).p(y_n, y_{n+1}), p(gu, gu).p(Fu, y_{n+1}), p(gu, y_{n+1}).p(Fu, y_n)\}}{p(gu, y_n)} \\
&+ \beta p(gu, y_n) + L \min\{p(gu, Fu), p(y_n, y_{n+1}), p(gu, y_{n+1}), p(Fu, y_n)\}.
\end{aligned}$$

Taking $n \to \infty$ and using (2), we have $p(z, Fu) = 0$. So, $Z = Fu$, and hence $z = Fu = gu$. Since F and g are weakly compatible, we obtain $gFu = Fgu$. Hence $gz = Fz$. Therefore, z is a coincidence point of F and g. Again $F(X) \subseteq G(X)$, there exists a point $v \in X$ such that $z = Gv$.

$$\begin{aligned}
p(z, fv) &= p(Fu, fv) \\
&\leq \alpha \frac{\max\{p(gu, Fu).p(Gv, fv), p(gu, Gu).p(fv, Fu), p(gu, fv).p(Fu, Gv)\}}{p(gu, Gv)} \\
&+ \beta p(gu, Gv) + L \min\{p(gu, Fu), p(Gv, fv), p(gu, fv), p(Fu, Gv)\}.
\end{aligned}$$

Taking $n \to \infty$, we have $p(z, fv) \leq Lp(z, fv)$ implies $z = fv$. Hence, $fv = z = Gv$. Since G and f are weakly compatible, we obtain $fGv = Gfv$ implies $fz = Gz$. Therefore, z is a coincidence point of f, g, F and G.

Now, we show that z is a fixed point of F and g.

$$\begin{aligned}
p(Fz, z) &= p(Fz, fv) \\
&\leq \alpha \frac{\max\{p(gz, Fz).p(Gv, fv), p(gz, Gv).p(fv, Fz), p(gz, fv).p(Fz, Gv)\}}{p(gu, Gv)} \\
&+ \beta p(gz, Gv) + L \min\{p(gz, Fz), p(Gv, fv), p(gz, fv), p(Fz, Gv)\} \\
&= \alpha \frac{\max\{p(Fz, Fz).p(z, z), p(z, Fz).p(z, Fz), p(Fz, z).p(Fz, z)\}}{p(Fz, z)} \\
&+ \beta p(Fz, z) + L \min\{p(Fz, Fz), p(z, z), p(Fz, z), p(Fz, z)\} \\
&= (\alpha + \beta) p(Fz, z),
\end{aligned}$$

which gives $p(Fz, z) = 0$ implies $Fz = z$, and so $gz = Fz = z$. Thus, z is a common fixed point of F and g. Again

$$\begin{aligned}
p(z, fz) &= p(Fz, fz) \\
&\leq \alpha \frac{\max\{p(gz, Fz).p(Gz, fz), p(gz, Gz).p(fz, Fz), p(gz, fz).p(Fz, Gz)\}}{p(gz, Gz)} \\
&+ \beta p(gz, Gz) + L \min\{p(gz, Fz), p(Gz, fz), p(gz, fz), p(Fz, Gz)\}. \\
&= \alpha \frac{\max\{p(z, Fz).p(fz, fz), p(z, fz).p(z, fz), p(z, fz).p(z, fz)\}}{p(z, fz)} \\
&+ \beta p(z, fz) + L \min\{p(z, z), p(fz, z), p(z, fz), p(z, fz)\} \\
&= (\alpha + \beta) p(z, fz),
\end{aligned}$$

implies that $p(z, fz) = 0$, so $fz = z = Gz$.

Next, we shall show that z is unique common fixed point of $f, g, F$and G. On contrary, assume that w is another common fixed point in X.

$$\begin{aligned}
p(z, w) &= p(Fz, fw) \\
&\leq \alpha \frac{\max\{p(gz, Fz).p(Gw, fw), p(gz, Gw).p(fw, Fz), p(gz, fw).p(Fz, Gw)\}}{p(gz, Gw)}
\end{aligned}$$

$$
\begin{aligned}
&+\beta p(gz,Gw)+L\min\{p(gz,Fz),p(Gw,fw),p(gz,fw),p(Fz,Gw)\}\\
&=\alpha\frac{\max\{p(z,z).p(w,w),p(z,w).p(z,w),p(z,w).p(z,w)\}}{p(z,w)}\\
&+\beta p(z,w)+L\min\{p(z,z),p(w,w),p(z,w),p(z,w)\}\\
&=(\alpha+\beta)p(z,w),
\end{aligned}
$$

implies that $p(z,w)=0$ implies $z=w$. Hence z is a unique common fixed point of f,g,F and G. □

Corollary 2.1. *Let (X,p) be a complete partial metric space. Suppose f and g be two self mappings on X, satisfying the following conditions:*

(i) $f(X)\subseteq g(X)$;

(ii) there exist α,β and L in $(0,1)$ with $2\alpha+\beta+2L<1$ such that

$$
\begin{aligned}
p(fx,fy)\le{}&\alpha\frac{\max\{p(gx,fx).p(gy,fy),p(gx,gy).p(fx,fy),p(gx,fy).p(gy,fx)\}}{p(gx,gy)}\\
&+\beta p(gx,gy)+L\min\{p(fx,gx),p(fy,gy),p(gx,fy),p(fx,gy)\},
\end{aligned}
$$

for all $x,y\in X$;

(iii) $f(X)$ or $g(X)$ is complete.

If f and g are weakly compatible, then f and g have a unique common fixed point in X.

Corollary 2.2. *Let (X,p) be a complete partial metric space. Suppose f be a self mapping on X. Suppose that there exist α,β and L in $(0,1)$ with $2\alpha+\beta+2L<1$ such that*

$$
\begin{aligned}
p(fx,fy)\le{}&\alpha\frac{\max\{p(x,fx).p(y,fy),p(x,y).p(fx,fy),p(x,fy).p(y,fx)\}}{p(x,y)}\\
&+\beta p(x,y)+L\min\{p(x,fx),p(y,fy),p(x,fy),p(y,fx)\},
\end{aligned}
$$

for all $x,y\in X$. If $f(X)$ is complete, then f has a unique fixed point in X.

Example 2.1. Let $X=[0,6]$ and $p:X\times X\to\mathbb{R}$ be defined by $p(x,y)=\max\{x,y\}$. Then (X,p) is a complete partial metric space. Let $f,g:X\to X$ be defined by

$$
fx=\begin{cases}2x-1, & x\in[1,3],\\ 3, & x\in\,]3,6].\end{cases}
$$

and

$$
gx=\begin{cases}1, & x=1,\\ 2, & x\in\,]1,\frac{4}{3}[\\ 2x, & x\in[\frac{4}{3},3]\\ 6, & x\in\,]3,6].\end{cases}
$$

Now, consider the following cases:
Case I. When $x = y = 1$, we have

$$\begin{aligned} p(fx, fy) &= 1 = p(gx, gy) \\ &\leq \alpha \frac{\max\{p(gx, fx).p(gy, fy), p(gx, gy).p(fx, fy), p(gx, fy).p(gy, fx)\}}{p(gx, gy)} \\ &+ \beta p(gx, gy) + L \min\{p(fx, gx), p(fy, gy), p(gx, fy), p(fx, gy)\}. \end{aligned}$$

Case II. For $x, y \in [1, \frac{4}{3}[$ and $x < y$, we have

$$\begin{aligned} p(fx, fy) &= 2y - 1 \leq \frac{5}{6}2 = \frac{5}{6}p(gx, gy) \\ &\leq \alpha \frac{\max\{p(gx, fx).p(gy, fy), p(gx, gy).p(fx, fy), p(gx, fy).p(gy, fx)\}}{p(gx, gy)} \\ &+ \beta p(gx, gy) + L \min\{p(fx, gx), p(fy, gy), p(gx, fy), p(fx, gy)\}. \end{aligned}$$

Case III. If $x \in [1, 3]$,$y \in [\frac{4}{3}, 3]$ and $x < y$, then we have

$$\begin{aligned} p(fx, fy) &= 2y - 1 \leq \frac{5}{6}2y = \frac{5}{6}p(gx, gy) \\ &\leq \alpha \frac{\max\{p(gx, fx).p(gy, fy), p(gx, gy).p(fx, fy), p(gx, fy).p(gy, fx)\}}{p(gx, gy)} \\ &+ \beta p(gx, gy) + L \min\{p(fx, gx), p(fy, gy), p(gx, fy), p(fx, gy)\}. \end{aligned}$$

Case IV. For $x \in [1, 3]$,$y \in [3, 6]$ and $x < y$, we have

$$\begin{aligned} p(fx, fy) &= \max\{2x - 1, 3\} \leq \frac{5}{6}6 = \frac{5}{6}p(gx, gy) \\ &\leq \alpha \frac{\max\{p(gx, fx).p(gy, fy), p(gx, gy).p(fx, fy), p(gx, fy).p(gy, fx)\}}{p(gx, gy)} \\ &+ \beta p(gx, gy) + L \min\{p(fx, gx), p(fy, gy), p(gx, fy), p(fx, gy)\}. \end{aligned}$$

Case V. $x, y \in [3, 6]$ and $x < y$, we have

$$\begin{aligned} p(fx, fy) &= 3 \leq \frac{5}{6}6 = \frac{5}{6}p(gx, gy) \\ &\leq \alpha \frac{\max\{p(gx, fx).p(gy, fy), p(gx, gy).p(fx, fy), p(gx, fy).p(gy, fx)\}}{p(gx, gy)} \\ &+ \beta p(gx, gy) + L \min\{p(fx, gx), p(fy, gy), p(gx, fy), p(fx, gy)\}. \end{aligned}$$

Thus, in all cases the contractive conditions of Corollary 2.1 are satisfied and (f, g) weakly compatible, by Corollary 2.1 $f \& g$ have a unique common fixed point $x = 1$ in X.

Theorem 2.2. *Let (X,p) be a complete partial metric space and let f and g be self mappings on X. If there exist α,β,γ and δ in $(0,1)$ with $\alpha+\beta+2\gamma+3\delta<1$ such that*

$$p(fx,gy)\leq \alpha p(x,y)+\beta\frac{p(x,fx).p(y,gy)}{1+p(x,y)}+\gamma\frac{p(x,gy).p(y,fx)}{1+p(1+p(x,y))} + \delta\frac{p(x,fx).p(x,gy)+p(y,fx).p(y,gy)}{1+p(x,y)}, \tag{3}$$

for all $x,y\in X$. Then, there exist $z\in X$ such that $fz=gz=z$.

Proof. Let $x_0\in X$ be arbitrary element of X. Define a sequence in X such that $x_{2k+2}=fx_{2k+1}$ and $x_{2k+1}=gx_{2k}$ for $k=0,1,2,3,...$ If there exist a positive integer N such that $x_{2N}=x_{2N+1}$, then x_{2n} is a fixed point of g and hence fixed point of f. Since $x_{2N}=x_{2N+1}=Sx_{2N}$, using equation (3), we have

$$\begin{aligned}
&p(x_{2N+2},x_{2N+1})\\
&=p(fx_{2N+1},gx_{2N})\\
&\leq \alpha p(x_{2N+1},x_{2N})+\beta\frac{p(x_{2N+1},fx_{2N+1}).p(x_{2N},gx_{2N})}{1+p(x_{2N+1},x_{2N})}\\
&+\gamma\frac{p(x_{2N+1},gx_{2N}).p(x_{2N},fx_{2N+1})}{1+p(x_{2N+1},x_{2N})}\\
&+\delta\frac{p(x_{2N+1},fx_{2N+1}).p(x_{2N+1},gx_{2N})+p(x_{2N},fx_{2N+1}).p(x_{2N},gx_{2N})}{1+p(x_{2N+1},x_{2N})}\\
&\leq \alpha p(x_{2N+1},x_{2N})+\beta\frac{p(x_{2N+1},x_{2N+2}).p(x_{2N},x_{2N+1})}{1+p(x_{2N+1},x_{2N})}\\
&+\gamma\frac{p(x_{2N+1},x_{2N+1}).p(x_{2N},x_{2N+2})}{1+p(x_{2N+1},x_{2N})}\\
&+\delta\frac{p(x_{2N+1},x_{2N+2}).p(x_{2N+1},x_{2N+1})+p(x_{2N},x_{2N+2}).p(x_{2N},x_{2N+1})}{1+p(x_{2N+1},x_{2N})}.
\end{aligned}$$

Using the fact $p(x_{2N+1},x_{2N+1})\leq p(x_{2N},x_{2N+1})$ and $\frac{p(x_{2N},x_{2N+1})}{1+p(x_{2N},x_{2N+1})}<1$, we obtain

$$\begin{aligned}
p(x_{2N+2},x_{2N+1})&\leq \alpha p(x_{2N+1},x_{2N})+\beta p(x_{2N+1},x_{2N+2})+\gamma p(x_{2N},x_{2N+2})\\
&+\delta(p(x_{2N+1},x_{2N+2})+p(x_{2N},x_{2N+2}))\\
&\leq(\alpha+\gamma+\delta)p(x_{2N},x_{2N+1})+(\beta+\gamma+2\delta)p(x_{2N+1},x_{2N+2})\\
&-(\gamma+\delta)p(x_{2N+1},x_{2N+1})\\
&\leq(\alpha+\gamma+\delta)p(x_{2N},x_{2N+1})+(\beta+\gamma+2\delta)p(x_{2N+1},x_{2N+2})
\end{aligned}$$

implies that

$$p(x_{2N+2},x_{2N+1})\leq\Big(\frac{\alpha+\gamma+\delta}{1-\beta-\gamma-2\delta}\Big)p(x_{2N+1},x_{2N})=Kp(x_{2N+1},x_{2N}),$$

where $K = (\frac{\alpha+\gamma+\delta}{1-\beta-\gamma-2\delta}) < 1$. Since $x_{2N} = x_{2N+1}$, we have $p(x_{2N+2}, x_{2N+1}) \leq Kp(x_{2N+1}, x_{2N+1}) \leq Kp(x_{2N+1}, x_{2N+2})$, which given $p(x_{2N+1}, x_{2N+1}) = 0$, implies $x_{2N+1} = x_{2N+2} = Tx_{2N+1}$.

Note that, $Sx_{2N} = x_{2N} = x_{2N+1} = Tx_{2N+1}$. Thus, in this case $x_{2N} = x_{2N+1}$ is a fixed point of S and T. A similar conclusion holds if $x_{2N+1} = x_{2N+2}$ for some $N \in \mathbb{Z}^+$. Therefore, we may assume that $x_k \neq x_{k+1}$ for all k.

Case I. If k is odd, using (3), we have

$$\begin{aligned}
p(x_{k+1}, x_{k+2}) &= p(fx_K, gx_{k+1}) \\
&\leq \alpha p(x_k, x_{k+1}) + \beta \frac{p(x_k, fx_k).p(x_{k+1}), gx_{k+1}}{1 + p(x_k, x_{k+1})} \\
&+ \gamma \frac{p(x_k, gx_{k+1}).p(x_{k+1}), fx_k}{1 + p(x_k, x_{k+1})} \\
&+ \delta \frac{p(x_k, fx_k).p(x_k, gx_{k+1}) + p(x_{k+1}, fx_k).p(x_{k+1}, gx_{k+1})}{1 + p(x_k, x_{k+1}} \\
&= \alpha p(x_k, x_{k+1}) + \beta \frac{p(x_k, x_{k+1}).p(x_{k+1}, x_{k+2})}{1 + p(x_k, x_{k+1})} \\
&+ \gamma \frac{p(x_k, x_{k+2}).p(x_{k+1}, x_{k+1})}{1 + p(x_k, x_{k+1})} \\
&+ \delta \frac{p(x_k, x_{k+1}).p(x_k, x_{k+2}) + p(x_{k+1}, x_{k+1}).p(x_{k+1}, x_{k+2})}{1 + p(x_k, x_{k+1})}.
\end{aligned}$$

Using the fact $p(x_{k+1}, x_{k+1}) \leq p(x_k, x_{k+1})$ and $\frac{p(x_k, x_{k+1})}{1+p(x_k, x_{k+1})} < 1$, we have

$$\begin{aligned}
p(x_{k+1}, x_{k+2}) &\leq \alpha p(x_k, x_{k+1}) + \beta p(x_{k+1}, x_{k+2}) + \gamma p(x_k, x_{k+2}) \\
&+ \delta p(x_k, x_{k+2}) + \delta p(x_{k+1}, x_{k+2}) \\
&= \alpha p(x_k, x_{k+1}) + (\beta + \delta) p(x_{k+1}, x_{k+2}) + (\gamma + \delta) p(x_k, x_{k+2}) \\
&\leq \alpha p(x_k, x_{k+1}) + (\beta + \delta) p(x_{k+1}, x_{k+2}) + (\gamma + \delta)(p(x_k, x_{k+1}) \\
&+ p(x_k, x_{k+1}) - p(x_{k+1}, x_{k+1}))
\end{aligned}$$

implies

$$p(x_{k+1}, x_{k+2}) \leq \Big(\frac{\alpha + \gamma + \delta}{1 - \beta - \gamma - 2\delta}\Big) p(x_k, x_{k+1}) = rp(x_k, x_{k+1}), \tag{4}$$

where $r = \frac{\alpha+\gamma+\delta}{1-\beta-\gamma-2\delta}$.

Case II. If k is even, using (2.3), we have

$$\begin{aligned}
p(x_{k+1}, x_{k+2}) &= p(gx_k, fx_{k+1}) \\
&\leq \alpha p(x_k, x_{k+1}) + \beta \frac{p(x_{k+1}, fx_{k+1}).p(x_k), gx_k}{1 + p(x_k, x_{k+1})} \\
&+ \gamma \frac{p(x_{k+1}, gx_k).p(x_k), fx_{k+1}}{1 + p(x_k, x_{k+1})}
\end{aligned}$$

$$+\,\delta\frac{p(x_{k+1},fx_{k+1}).p(x_{k+1},gx_k)+p(x_k,fx_{k+1}).p(x_k,gx_k)}{1+p(x_k,x_{k+1}}$$
$$=\alpha p(x_k,x_{k+1})+\beta\frac{p(x_{k+1},x_{k+2}).p(x_k,x_{k+1})}{1+p(x_k,x_{k+1})}$$
$$+\,\gamma\frac{p(x_{k+1},x_{k+1}).p(x_k,x_{k+2})}{1+p(x_k,x_{k+1})}$$
$$+\,\delta\frac{p(x_{k+1},x_{k+2}).p(x_{k+1},x_{k+1})+p(x_k,x_{k+2}).p(x_k,x_{k+1})}{1+p(x_k,x_{k+1})}.$$

Using the fact $p(x_{k+1},x_{k+1})\le p(x_k,x_{k+1})$, $p(x_{k+1},x_{k+1})\le p(x_{k+1},x_{k+2})$ and $\frac{p(x_k,x_{k+1})}{1+p(x_k,x_{k+1})}<1$, we have

$$\begin{aligned}p(x_{k+1},x_{k+2})&\le\alpha p(x_k,x_{k+1})+\beta p(x_{k+1},x_{k+2})+\gamma p(x_k,x_{k+2})\\&+\delta p(x_k,x_{k+2})+\delta p(x_{k+1},x_{k+2})\\&=\alpha p(x_k,x_{k+1})+(\beta+\delta)p(x_{k+1},x_{k+2})+(\gamma+\delta)p(x_k,x_{k+2})\\&\le\alpha p(x_k,x_{k+1})+(\beta+\delta)p(x_{k+1},x_{k+2})\\&+(\gamma+\delta)(p(x_k,x_{k+1})+p(x_k,x_{k+1})-p(x_{k+1},x_{k+1}))\end{aligned}$$

implies

$$p(x_{k+1},x_{k+2})\le\Big(\frac{\alpha+\gamma+\delta}{1-\beta-\gamma-2\delta}\Big)p(x_k,x_{k+1})=rp(x_k,x_{k+1}),\tag{5}$$

where $r=\frac{\alpha+\gamma+\delta}{1-\beta-\gamma-2\delta}<1$.

Regarding (4) and (5), we conclude that $\{p(x_k,x_{k+1})\}$ is a nonnegative non-increasing sequence of real numbers and we obtain that

$$p(x_k,x_{k+1})\le r^n p(x_0,x_1),\tag{6}$$

for all $r=0,1,2,...$

Consider

$$\begin{aligned}d_p(x_{k+1},x_{k+2})&=2p(x_{k+1},x_{k+2})-p(x_{k+1},x_{k+1})-p(x_{k+2},x_{k+2})\\&\le 2p(x_{k+1},x_{k+2})\le 2r^{k+1}p(x_0,x_1).\end{aligned}\tag{7}$$

We conclude that

$$\lim_{k\to\infty}d_p(x_{k+1},x_{k+2})=0.\tag{8}$$

Moreover,

$$\begin{aligned}d_p(x_{k+1},x_{k+s})&\le d_p(x_{k+s},x_{k+s-1})+...+d_p(x_{k+2},x_{k+1})\\&\le 2(r^{k+s-1}+...+r^{k+1})p(x_0,x_1).\end{aligned}\tag{9}$$

After simple calculation we obtain $\{x_k\}$ a Cauchy sequence in (X, d_p). Since (X, p) is complete, by Lemma 1.1 (X, d_p) is also complete,hence the sequence $\{x_k\}$ converges to a point say $z \in X$. Again, by Lemma 1.1, we have

$$p(z,z) = \lim_{k\to\infty} p(x_k, z) = \lim_{k,m\to\infty} p(x_k, x_m). \tag{10}$$

Since $\{x_k\}$ a Cauchy sequence in (X, d_p), we have

$$\lim_{k,m\to\infty} d_p(x_k, x_m) = 0$$

implies that

$$\lim_{k,m\to\infty} p(x_k, x_m) = 0. \tag{11}$$

Without loss of generality, we may assume that $n > m$, we have

$$\begin{aligned} p(x_m, x_{m+2}) &\leq p(x_m, x_{m+1}) + p(x_{m+1}, x_{m+2}) - p(x_{m+1}, x_{m+1}) \\ &\leq p(x_n, x_{m+1}) + p(x_{m+1}, x_{m+2}) \end{aligned}$$

Similarly,

$$\begin{aligned} p(x_m, x_{m+3}) &\leq p(x_m, x_{m+1}) + p(x_{m+1}, x_{m+2}) + p(x_{m+2}, x_{m+3}) \\ &\quad - p(x_{m+1}, x_{m+1}) - p(x_{m+2}, x_{m+2}) \\ &\leq p(x_m, x_{m+1}) + p(x_{m+1}, x_{m+2}) + p(x_{m+2}, x_{m+3}). \end{aligned}$$

Inductively, we obtain

$$p(x_n, x_m) \leq p(x_m, x_{m+1}) + ... + p(x_{n-1}, x_n) \leq (r^m + ... + r^{n-1})p(x_0, x_1).$$

As $r < 1$, we can observe that $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. Regarding (10), we have

$$p(z,z) = \lim_{k\to\infty} p(x_k, z) = \lim_{k,m\to\infty} p(x_k, x_m) = 0. \tag{12}$$

Now, we shall show that z is a fixed point of f. On contrary, assume that $fz \neq z$, then $p(z, fz) > 0$. Let $\{x_{2k_i}\}$ be a subsequence of $\{x_{2k}\}$, so of $\{x_k\}$, we have

$$\begin{aligned} p(Sx_{2k_i}, fz) \leq \alpha p(x_{2k_i}, z) &+ \beta \frac{p(x_{2k_i}, gx_{2k_i}).p(z, fz)}{1 + p(x_{2k_i}, z)} + \gamma \frac{p(x_{2k_i}, fz).p(z, gx_2k_i)}{1 + p(x_{2k_i}, z)} \\ &+ \delta \frac{p(x_{2k_i}, gx_2k_i).p(x_{2k_i}, fz) + p(z, gx_2k_i).p(z, fz)}{1 + p(x_{2k_i}, z)}. \end{aligned}$$

Taking $i \to \infty$, we have $p(z, fz) = 0$ implies that $fz = z$. Similarly, we can obtain $gz = z$. Hence, $fz = gz = z$. Now, we can easily show that z is a unique fixed point of $f\&g$. □

Example 2.2. Let $X = [0, 1]$ and $p(x, y) = \max\{x, y\}$.Then (X, p) is a complete partial metric space. Clearly p is not metric on X. Let $f, g : X \to \infty$ defined by $fx = gx = \frac{x}{6}$, for all $x \in X$. Without loss of generality, we may assume $x > y$, then

$$\begin{aligned} p(fx, gy) &= \max\{\frac{x}{6}, \frac{y}{6}\} = \frac{x}{6} \leq \frac{1}{3}\max\{x, y\} \\ &\leq \alpha p(x, y) + \beta\frac{p(x, fx).p(y, gy)}{1 + p(x, y)} + \gamma\frac{p(x, gy).p(y, fx)}{1 + p(1 + p(x, y))} \\ &+ \delta\frac{p(x, fx).p(x, gy) + p(y, fx).p(y, gy)}{1 + p(x, y)}, \end{aligned}$$

with $\alpha = \frac{1}{3}$. Thus, all the conditions of the Theorem 2.2 are satisfied and 0 is a unique common fixed point of f and g.

Corollary 2.3. *Let (X, p) be a complete partial metric space. Let f be a self map on X. If there exists α, β, γ and δ in $(0, 1)$ with $\alpha + \beta + 2\gamma + 3\delta < 1$ such that*

$$\begin{aligned} p(fx, fy) &\leq \alpha p(x, y) + \beta\frac{p(x, fx).p(y, fy)}{1 + p(x, y)} + \gamma\frac{p(x, fy).p(y, fx)}{1 + p(x, y)} \\ &+ \delta\frac{p(x, fx).p(x, fy) + p(y, fy).p(y, fx)}{1 + p(x, y)}, \end{aligned}$$

for all $x, y \in X$. Then, there exists $z \in X$ such that $fz = z$.

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