On some well covered graphs

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Abstract. If all maximal independent sets of a graph G have the same cardinality, then G is called a well covered graph. In this paper, we study the well covered property for several classes of graphs. We show that the line graph of a complete graphs $L(K_n)$ is a well covered graph. Furthermore, we show that (m, n)-lollipop graph $L_{m,n}$ is a well covered graph only if $n = 2$ and the line graph of (m, n)-lollipop graph $L(L_{m,n})$ is a well covered graph only if m is even and $n = 1$ or $n = 3$ or if m is odd and $n = 2$.

Keywords: maximal independent sets, well covered graphs, line graphs, complete graphs, lollipop graphs.

1. Introduction

All graphs in this paper are finite undirected simple graphs. For a graph G, we denote by $V(G)$ the vertex set of G, and $E(G)$ the edge set of G. The order of a graph G is equal to the cardinality of $V(G)$ and is denoted by $|G|$. The line graph $L(G)$ of a graph G, is a simple graph whose vertex set is the set of all edges of G, and two vertices of $L(G)$ are adjacent if the corresponding edges of G are adjacent. We follow Bondy and Murty [1] for any undefined terms.

For a graph G, a set $M \subseteq V(G)$ is said to be an independent set of G if any two vertices in M are non adjacent. An independent set M is said to be

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maximal if M cannot be included into another independent set. The set M is called a maximum independent set in G if it has the maximum cardinality among all maximal independent sets in G. It is not easy to determine the maximum and maximal independent sets of a graph G. This topic has been studied by different authors (see, [2] and [7]). A graph G is called well covered if all maximal independent sets in a graph G has the same cardinality. Equivalently, a graph G is well covered if every maximal independent set in G is a maximum independent set.

The study of the well covered property of a graph was introduced by Plummer [5]. Several studies on the subject have been published since then (see, [3] and [4]). Also, one of the most effective studies in this subject is Plummer and Michael (see, [6]).

However, there are certain families of well covered graphs can be discovered. Complete graphs K_n are well covered graphs, since each maximal independent set has cardinality one. Also, for a complete bipartite graphs $K_{m,n}$ with $m \neq n$, is not well covered because it has two maximal independent sets with different cardinality.

In this paper, we focus on the well covered property for several classes of graphs. This paper is structured as follows: In section 2, we show that the line graph of a complete graph $L(K_n)$ is well covered. In section 3, we discuss the well covered property of (m, n)-lollipop graph $L_{m,n}$, and its line graph $L(L_{m,n})$.

2. The graphs $L(K_n)$

In this section, we show that the line graph of a complete graph $L(K_n)$ is well covered.

Assume that the vertex set of K_n is $V(K_n) = \{u_1, u_2, ..., u_n\}$ and let the edge between u_i and u_j be denoted by $u_{i,j}$. Thus, $V(L(K_n)) = \{u_{i,j} : 1 \leq i < j \leq n\}$. Then $u_{i,j}$ and $u_{k,l}$ are adjacent if $i = k$ or $j = l$.

Now, for $n \leq 3$, $L(K_n)$ is a well covered graph, since each maximal independent set has cardinality 1. In general, we show that the line graph of a complete graph $L(K_n)$ is well covered for each n.

Theorem 2.1. *For $n \geq 4$, the graph $L(K_n)$ is well covered.*

Proof. In $L(K_n)$ the vertices $u_{i,j}$, $u_{k,l}$ are non adjacent iff $\{i, j\} \cap \{k, l\} = \phi$. So, all independent set of $L(K_n)$ have the form $A = \{u_{x,f(x)} : f : X \to Y\}$ where $f(x)$ is a one to one function, where $X = \{i_1, ..., i_k\}, Y = \{j_1, ..., j_h\}$ such that $X \cap Y = \phi$ and $X \cup Y \subseteq \{1, ..., n\}$.

Now, we will show that A is maximal if $|A| = \lfloor \frac{n}{2} \rfloor$. Indeed, we have two cases:

Case 1. n is even, we have $X = \{i_1, ..., i_{\frac{n}{2}}\}$ and $Y = \{j_1, ..., j_{\frac{n}{2}}\}$ such that $X \cap Y = \phi$ and $X \cup Y = \{1, ..., n\}$.

Let $u_{a,f(a)}, u_{b,f(b)} \in A$, with $a \neq b$. Since $f(x)$ is a one to one function, we have $f(a) \neq f(b)$. Therefore, $u_{a,f(a)}, u_{b,f(b)}$ are independent vertices. Now, let

$u_{i,j} \notin A$, then $i, j \in X$ or $i, j \in Y$. Since $f(x)$ is a one to one and $|X| = |Y| = \frac{n}{2}$, we have $u_{i,j}$ is adjacent to one vertex in A which is $u_{i,f(i)}$ or $u_{f^{-1}(j),j}$ respectively. Hence A is maximal and $|A| = |X| = \frac{n}{2}$.

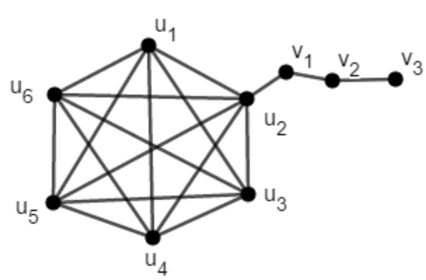
Case 2. n is odd, we have $X = \{i_1, ..., i_{\lfloor \frac{n}{2} \rfloor}\}, Y = \{j_1, ..., j_{\lceil \frac{n}{2} \rceil}\}$ such that $X \cap Y = \phi$ and $X \cup Y = \{1, ..., n\}$. Let $u_{a,f(a)}, u_{b,f(b)} \in A$, with $a \neq b$. Since $f(x)$ is a one to one function, we have $f(a) \neq f(b)$. Therefore, $u_{a,f(a)}, u_{b,f(b)}$ are independent vertices. Now, let $u_{i,j} \notin A$. Then $i, j \in X$ or $i \in X, j \in Y - f(X)$ or $i, j \in Y$. If i or j belongs to X, we have $u_{i,j}$ is adjacent to one vertex in A which is $u_{i,f(i)}$ or $u_{j,f(j)}$. Otherwise, if $i \in Y$ and $j \in Y$, we have $i \in f(X)$ or $j \in f(X)$ since $|Y| = |X| + 1$. Hence $u_{i,j}$ is adjacent to one vertex in A which is $u_{f^{-1}(i),i}$. Hence A is a maximal independent set in $L(K_n)$ and $|A| = |X| = \lfloor \frac{n}{2} \rfloor$. Since all maximal independent sets in $L(K_n)$ have the same cardinality; $|A| = \lfloor \frac{n}{2} \rfloor$, $L(K_n)$ is a well covered graph. □

3. The graph $L_{m,n}$ and its line graph $L(L_{m,n})$

In this section, we look at (m, n)-lollipop graph $L_{m,n}$ and their line graphs $L(L_{m,n})$ in terms of having the well covered property. We will show that the graph $L_{m,n}$ for all $n \neq 2$ are not well covered, while $L_{m,2}$ is a well covered graph for all m. Moreover, we will prove that the line graph of $L_{m,1}; L(L_{m,1})$ is well covered only if m is even, $L(L_{m,2})$ is well covered only if m is odd, and $L(L_{m,3})$ is well covered only if m is even. In the other cases the graph $L(L_{n,m})$ is not well covered.

An (m, n)-lollipop graph, denoted by $L_{m,n}$ is a graph on $m + n$ vertices obtained by connecting a complete graph K_m by a path graph P_n with an edge. Let $V(K_m) = \{u_1, u_2, ..., u_m\}$ be the set of vertex of K_m and $V(P_{n+1}) = \{u_i, v_1, ..., v_n\}$ be the set of vertex of P_{n+1} attached with a fixed vertex u_i for some $i : 1 \leq i \leq m$, of the complete graph K_m.

The following Figure shows the $(6, 3)$-lollipop graph where P_3 connected with the vertex u_2 of the complete graph K_6.



In the following theorem, we show that (m, n)-lollipop graph $L_{m,n}$ is well covered if $n = 2$.

Theorem 3.1. *The $(m, 2)$-lollipop graph $L_{m,2}$ is well covered.*

Proof. Let M be a maximal independent set of $(m,2)$-lollipop graph $L_{m,2}$. Let u_k represents the fixed vertex in $V(K_m)$ where P_2 has the vertex set $\{v_1, v_2\}$ and, P_2 is connected to a complete graph K_m. Then for any maximal independent set M, we have two cases.

Case 1. $u_k \in M$, then $v_1 \notin M$ since v_1 is adjacent to u_k and $u_i \notin M$, for all $i \neq k, 1 \leq i \leq m$. Therefore, $v_2 \in M$. Thus, $M = \{u_k, v_2\}$

Case 2. $u_k \notin M$, then $u_i \in M$ for some $i, 1 \leq i \leq m$ and $u_j \notin M$, for all $k \neq j \neq i, 1 \leq j \leq m$. Therefore, $v_1 \in M$ or $v_2 \in M$. That is $M = \{u_i, v_1\}$ or $M = \{u_i, v_2\}$. Thus, for any maximal independent set M of $L_{m,2}$, $|M| = 2$. This completes the proof. □

Now, for $n > 2$, the $(m,n)-$lollipop graph is not well covered. For, let u_k be the fixed vertex in $V(K_m)$ and $\{v_1, v_2, .., v_t : t > 2\}$, then $M_1 = \{u_k, v_i : i > 1, i$ is odd$\}$ and $M_2 = \{u_j, v_i : i$ is odd and $j \neq k\}$ are two maximal independent sets with cardinality $\lceil \frac{n}{2} \rceil$ and $\lceil \frac{n}{2} \rceil + 1$ respectively. Thus we have the following lemma.

Lemma 3.1. *For $m \geq 3$ and $n \neq 2$, the (m,n)-lollipop graph $L_{m,n}$ is not well covered.*

The line graph of a (m,n)-lollipop graph $L(L_{m,n})$ is a graph given by

$$V(L(L_{m,n})) = \{u_{i,j}, e_h : 1 \leq i < j \leq m, 1 \leq h \leq n\}.$$

Now, $L(L_{m,n})$ contains a line graph of complete graph $L(K_m)$ with $V(L(K_m)) = \{u_{i,j} : 1 \leq i < j \leq m\}$ and a path P_n with $V(P_n) = \{e_1, e_2, ..., e_n\}$. Suppose that in $L_{m,n}$, the vertex u_k is the fixed vertex in $V(K_m)$ where P_n is connected with the complete graph K_m. Then, e_1 is adjacent with $u_{1,k}, u_{2,k}, ..., u_{m,k}$ in $L(L_{m,n})$.

In the following theorem we show that if m is even then $L(L_{m,1})$ is a well covered graph.

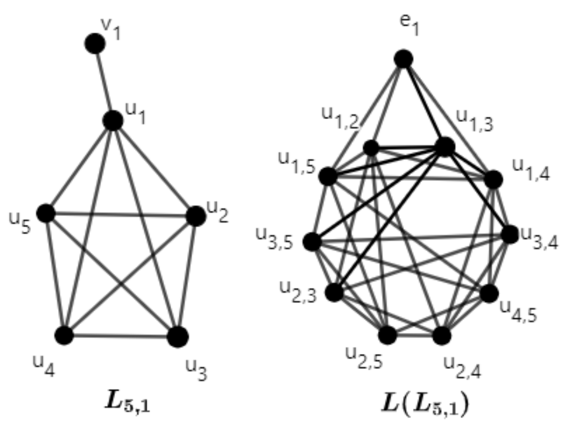
Theorem 3.2. *For any even number $m \geq 4$, the line graph of $(m,1)$-lollipop graph $L(L_{m,1})$ is well covered.*

Proof. Let u_k be the vertex of K_m that is adjacent with v_1 in P_1 by an edge e_1. Then, for any maximal independent set M in $L(L_{m,1})$, we have two cases; either $e_1 \in M$ or $e_1 \notin M$.

If $e_1 \notin M$, then M is a maximal independent set in $L(K_m)$, and hence by Theorem 2.1, $|M| = \frac{m}{2}$. If $e_1 \in M$, let $B = \{u_{k,j} : 1 \leq j \leq m, j \neq k\}$ be the set of all vertices with k is fixed. Since e_1 is incident with the vertices u_k and v_1, M and B have no common vertex. Thus, $M = \{e_1\} \cup M_k$, where $M_k = \{u_{r,s} : 1 \leq r < s \leq m, k \neq r, k \neq s\} \subset V(L(K_m))$. Now, let $D = \{r, s : u_{r,s} \in M_k\}$. Since m is even, k is excluded from D with exactly one j for some $j \neq k$ and $1 \leq j \leq m$. Thus $|D| = m-2$, i.e. $|M_k| = \frac{m-2}{2}$ and $|M| = 1 + \frac{m-2}{2} = \frac{m}{2}$. Hence any maximal independent set of $L(L_{m,1})$ has a cardinality $\frac{m}{2}$. That is $L(L_{m,1})$ is a well covered graph. □

In general, if m is odd, the line graph of $(m, 1)-$lollipop graph is not well covered graph.

For example, consider the line graph of the $(5, 1)-$lollipop graph. Then $M_1 = \{e_1, u_{3,4}, u_{2,5}\}$ and $M_2 = \{u_{1,2}, u_{3,4}\}$ are two maximal independent sets with different cardinality (see, the figure below). Indeed, if m is odd, we can always able to find two maximal independent sets with different cardinality.



The following theorem shows that if m is odd, then $L(L_{m,2})$ is a well covered graph.

Theorem 3.3. *For any odd number $m \geq 3$, the line graph of $(m, 2)$-lollipop graph $L(L_{m,2})$ is well covered.*

Proof. Let u_k be the vertex of K_m that adjacent with v_1 in P_2 by an edge e_1. Consider $L(L_{m,2})$ to be the line graph of $L_{m,2}$. Then, for any maximal independent set K in $L(L_{m,2})$, we have two cases;
Case 1. $e_1 \notin K$, then $K = \{e_2\} \cup M$, where M is a maximal independent set in $L(K_m)$ and hence by Theorem 2.1 $|M| = \lfloor \frac{m}{2} \rfloor + 1$.
Case 2. $e_1 \in K$, then $K = \{e_1\} \cup M_1$, where $M_1 = \{u_{r,s} : 1 \leq r < s \leq m\}$ and r, s are different from k. By Theorem 2.1 Since m is odd, M_1 is a maximal independent set in $L(K_m)$. So, we have $|M_1| = \lfloor \frac{m}{2} \rfloor$. Thus, for any maximal independent set K of $L(L_{m,2}), |K| = \lfloor \frac{m}{2} \rfloor + 1$. This completes the proof. □

In general, if m is even then $L(L_{m,2})$ is not a well covered graph. Let u_1 be the vertex of K_m that adjacent with v_1 in P_2 by an edge e_1. Then the sets $M_1 = \{e_1, u_{3,4}, u_{5,6}, ..., u_{m-1,m}\}$ and $M_2 = \{e_2, u_{1,2}, u_{3,4}, u_{5,6}, ..., u_{m-1,m}\}$ are two maximal independent sets with cardinality $\frac{m}{2}$ and $\frac{m}{2} + 1$ respectively.

The following theorem shows that if m is even, then $L(L_{m,3})$ is a well covered graph.

Theorem 3.4. *For any even number $m \geq 4$, the line graph of $(m, 3)$-lollipop graph $L(L_{m,3})$ is well covered.*

Proof. Let u_k be the vertex of K_m that adjacent with v_1 in P_3 by an edge e_1. Consider $L(L_{m,3})$ to be the line graph of $L_{m,3}$. Then, for any maximal independent set H in $L(L_{m,3})$, we have two cases;
Case 1. $e_1 \notin H$, then $H = \{e_2\} \cup M$, or $H = \{e_3\} \cup M$ where M is a maximal independent set in $L(K_m)$ and hence by Theorem 2.1 $|M| = \frac{m}{2} + 1$.
Case 2. $e_1 \in H$, then $H = \{e_3, e_1\} \cup M_k$. Then we can complete the proof by the similar technique of Theorem 3.2. Hence, any maximal independent set of $L(L_{m,3})$ has a cardinality $\frac{m}{2} + 1$. Thus we have $L(L_{m,3})$ is a well covered graph. □

In general, if m is odd then $L(L_{m,3})$ is not well covered. Let u_1 be the vertex of K_m that adjacent with v_1 in P_3 by an edge e_1. Then the sets $M_1 = \{e_3, e_1, u_{3,4}, u_{5,6}, ..., u_{m-2,m-1}, u_{2,m}\}$ and $M_2 = \{e_3, u_{1,2}, u_{3,4}, u_{5,6}, ..., u_{m-2,m-1}\}$ are two maximal independent sets with different cardinality.

Now, if $n \geq 4$, then $L(L_{m,n})$ is not a well covered graph for all m.

Theorem 3.5. *For all* $m \geq n \geq 4$*, the line graph of* (m, n)*-lollipop graph* $L(L_{m,n})$ *is not well covered.*

Proof. Without loss of generality, suppose u_1 be the vertex of K_m such that P_n is connected to the complete graph K_m. We have two cases;
Case 1. m is even, then $L(L_{m,n})$ have two maximal independent sets with different cardinality $M_1 = \{u_{1,2}, u_{3,4}, u_{5,6}, ..., u_{m-1,m}\} \cup \{e_i : i \text{ is even } 2 \leq i \leq n-1\}$ and $M_2 = \{u_{3,4}, u_{5,6}, ..., u_{m-1,m}\} \cup \{e_1, e_i : i \text{ is even } 4 \leq i \leq n-1\}$.
Case 2. m is odd, then $L(L_{m,n})$ have two maximal independent sets with different cardinality $M_1 = \{u_{1,2}, u_{3,4}, u_{5,6}, ..., u_{m-2,m-1}\} \cup \{e_i : i \text{ is odd } 3 \leq i \leq n-1\}$ and $M_2 = \{u_{3,4}, u_{5,6}, ..., u_{m-2,m-1}, u_{2,m}\} \cup \{e_i : i \text{ is odd } 1 \leq i \leq n-1\}$.
Thus, $L(L_{m,n})$ is not well covered for all $m \geq n \geq 4$. □

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