Numerical approximation of vorticity transport equation using implicit-Euler finite difference scheme

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Abstract. This paper is devoted to study the numerical solutions of the two-dimensional vorticity transport equation (VTE) with homogenous Dirichlet boundary conditions. For this problem, we derive the implicit-Euler difference equation using Tayler expansion and with approximating the spatial partial derivatives by finite difference operators. In addition, the matrix formulation for the discrete problems are shown. The proof of consistency and the order of accuracy for the proposed scheme is provided by deriving the local truncation errors of the difference equations. Moreover, the necessary and sufficient condition for the stability of the proposed scheme is considered. Furthermore, the stability and convergence of the proposed method are tested by solving a numerical experiment for different space-steps. Additionally, the discrete graphs for the vorticity and stream functions are curried out to support the numerical results. The results show that the obtained numerical solutions are stable, convergent and decreasing over time.

Keywords: implicit-Euler scheme, vorticity transport equation, truncation-error, stream function, Reynolds number.

1. Introduction

The numerical solutions of time-dependent differential equations, used to model many phenomena in various scientific fields such as physics, engineering, population and fluid dynamics, have been studied by many authors, see for instance [1, 2, 3, 4, 5, 6, 7, 8]. Different scheme have been proposed to solve these problems, such as finite difference and finite element schemes. It known that the implicit finite difference methods are more recommended to solve many types of problems than explicit methods due to their strong stability properties and high convergence-order. However, the main drawback about using implicit schemes is that, at each advance time-level, we need to deal with complicated linear (non-

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linear) systems arising from the numerical approximations of the considered differential equations. One of the common-used implicit schemes for solving linear (semilinear) diffusion equations is Euler scheme [6, 9]. The reason behind that is the implicit Euler scheme is a one-step method and it is easier to use with less complicated computational-steps compared with other implicit methods, such as Crank-Nicolson and ADI methods.

In the present work, we consider the two-dimensional vorticity transport equation (VTE), which is also known as the streamfunction-vorticity formulation [10, 11]. It is well known that the two dimensional vorticity transport equation is used to represent the two-dimensional unsteady flow problem. Moreover, it has several applications in fluid dynamics such as analyzing of laminar to turbulent flow transition, solving the two-dimensional viscous incompressible flow, modeling of turbulent flows, studying on free and mixed convection. For further details regarding the importance, the derivation and the applications of this problem, see [12, 13].

Due to the complexity of nonlinear terms appearing in the two-dimensional vorticity transport equation, it cannot be solved analytically. Therefore, different schemes have been proposed to solve it numerically, subject to different boundary-conditions, such as upwind-Petrov-Galerkin method, boundary-domain integral method, ADI method and Crank-Nicolson scheme, see[10, 11, 14, 15, 16, 17, 18, 19, 20].

In this work, we propose the implicit-Euler finite difference scheme to solve the two-dimensional vorticity transport equation with homogenous Dirichlet boundary conditions. The aim of this work is to prove the consistency and to show the order of accuracy of the proposed scheme. In addition, to test the stability and convergence of the proposed scheme by solving a numerical experiment.

This paper contains eight sections, in section two the mathematical problem is presented, whereas its numerical approximation is presented in section three. Section four is devoted to the matrix formulation of the discrete problem, also the algorithm-steps are stated in this section. The theorems of consistency and stability are provided in section five and six, respectively. For a numerical experiment, the numerical results are presented in form of table and figures and discussed in section seven. The last section is devoted to point out the observed conclusions based on the numerical results.

2. The mathematical problem

The two-dimensional vorticity transport equation with homogenous Dirichlet boundary conditions is given as follows [19, 20]:

$$(1)\quad \frac{\partial \omega}{\partial t} = \frac{1}{R}\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right) - \left(\frac{\partial \psi}{\partial y}\right)\left(\frac{\partial \omega}{\partial x}\right) + \left(\frac{\partial \psi}{\partial x}\right)\left(\frac{\partial \omega}{\partial y}\right), (x, y) \in \Omega,\ t > 0,$$

(2) $$\omega = -\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right), (x, y) \in \Omega,$$

(3) $$\omega(x, y, 0) = \omega_0(x, y), \quad \psi(x, y, 0) = \psi_0(x, y), (x, y) \in \Omega,$$

(4) $$\omega(x, y, t) = \psi(x, y, t) = 0, \quad (x, y) \in \partial\Omega, \ t > 0,$$

where $\Omega = \{(x, y) : a < x < b; a < y < b\}$; $\partial\Omega = \{(a, y), (b, y), (x, a), (x, b)\}$; ω and ψ are the vorticity function and the stream function, respectively; ω_0, ψ_0 are smooth, nonnegative functions satisfying (2); and the positive value R is the Reynolds number.

3. Numerical discretization

Throughout this work, we choose equal space-steps (h) in x, y-directions and let k refers to the time-steps:

$$x_0 = a, \quad x_i = x_0 + ih, \quad x_m = b,$$
$$y_0 = a, \quad y_i = y_0 + jh, \quad y_m = b,$$

for $i, j = 1, 2, 3, \ldots m-1$, $h = (b-a)/m$

$$t_n = nk, \quad n = 0, 1, 2, \ldots, k > 0.$$

The approximate values to the $\omega(x_i, y_j, t_n)$ and $\psi(x_i, y_j, t_n)$, is denoted by $\omega_{i,j}^n$ and $\psi_{i,j}^n$, respectively.

In addition, the discrete space $\Omega_{i,j}^n$, takes the form:

$$\Omega_{i,j}^n = \{ \ (x_i, y_j, t_n) \ ; \ i, j = 0, 1, 2 \ldots m \ ; n = 0, 1, 2, \ldots \ \}.$$

By using Taylor expansion for $\omega(x, y, (n+1)k)$ about $\omega(x, y, nk)$, it follows:

$$\omega(x, y, (n+1)k) = \left(1 + k\frac{\partial}{\partial t} + \frac{k^2}{2}\frac{\partial^2}{\partial t^2} + \ldots\right)\omega(x, y, nk).$$

Thus

$$\omega(x, y, (n+1)k) = \exp(k\frac{\partial}{\partial t})\omega(x, y, nk).$$

We can rewrite the last equation as follows:

(5) $$\exp\left(-k\frac{\partial}{\partial t}\right) \omega(x, y, (n+1)k) = \omega(x, y, nk).$$

By equations (1), (5), we obtain:

$$\exp\left(-k\left[\frac{1}{R}\left(\frac{\partial^2}{x^2} + \frac{\partial^2}{y^2}\right) - \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right]\right)\omega(x, y, (n+1)k) = \omega(x, y, nk).$$

Now, we replace (ω, ψ), by $(\omega_{i,j}^n , \psi_{i,j}^n)$, and approximate the partial derivatives ω_x , ψ_x ,ω_y, and ψ_y by using the central finite difference operators of first order, δ_x , δ_y , while, the partial derivatives, ω_{xx} , ω_{yy} is approximated by the central finite difference operators of second order, ${\delta^2}_x$, ${\delta^2}_y$, then the last equation becomes:

$$\exp\left(-k\left[\frac{1}{R}\left(\frac{{\delta^2}_x}{h^2}+\frac{{\delta^2}_y}{h^2}\right)-\left(\frac{\delta_y\psi^n_{i,j}}{2h}\right)\left(\frac{\delta_x}{2h}\right)+\left(\frac{\delta_x\psi^n_{i,j}}{2h}\right)\left(\frac{\delta_y}{2h}\right)\right]\right)\omega^{n+1}_{i,j}=\omega^n_{i,j}.$$

For simplicity, we rewrite the last equation as follows:

$$\exp\left(-\frac{r}{R}\left({\delta^2}_x+{\delta^2}_y\right)+\frac{r}{4}\left(\delta_y\psi^n_{i,j}\right)\left(\delta_x\right)-\frac{r}{4}\left(\delta_x\psi^n_{i,j}\right)\left(\delta_y\right)\right)\omega^{n+1}_{i,j}=\omega^n_{i,j},$$

where $r = k/h^2$.

By taking the Taylor expansion for the left hand side of the above equation and truncating after the second term, it follows that

$$\left(1-\frac{r}{R}\left({\delta^2}_x+{\delta^2}_y\right)+\frac{r}{4}\left(\delta_y\psi^n_{i,j}\right)\left(\delta_x\right)-\frac{r}{4}\left(\delta_x\psi^n_{i,j}\right)\left(\delta_y\right)\right)\omega^{n+1}_{i,j}=\omega^n_{i,j},$$

$$i,j=1,2,3,\ldots..m-1;\quad n\geq 0.$$

Now, approximating the space-derivatives in equation (2) by the central finite difference operator of second order, we obtain

$$\omega^n_{i,j}=-\left(\frac{{\delta^2}_x\psi^n_{i,j}}{h^2}+\frac{{\delta^2}_y\psi^n_{i,j}}{h^2}\right).$$

For simplify, the last two equations can be rewritten as follows:

$$\begin{aligned}&\left(1+\frac{4r}{R}\right)\omega^{n+1}_{i,j}-\frac{r}{R}\left(\omega^{n+1}_{i+1,j}+\omega^{n+1}_{i-1,j}+\omega^{n+1}_{i,j+1}+\omega^{n+1}_{i,j-1}\right)\\&+\frac{r}{4}\left(\psi^n_{i,j+1}-\psi^n_{i,j-1}\right)\left(\omega^{n+1}_{i+1,j}-\omega^{n+1}_{i-1,j}\right)\\&-\frac{r}{4}\left(\psi^n_{i+1,j}-\psi^n_{i-1,j}\right)\left(\omega^{n+1}_{i,j+1}-\omega^{n+1}_{i,j-1}\right)=\omega^n_{i,j},\end{aligned}\tag{6}$$

$$-h^2\omega^n_{i,j}=\psi^n_{i+1,j}+\psi^n_{i-1,j}+\psi^n_{i,j+1}+\psi^n_{i,j-1}-4\psi^n_{i,j}.\tag{7}$$

For $i,j=1,2,3,\ldots..m-1, n=0,1,2,\ldots.$

The initial-boundary conditions (3) and (4), in the discrete space, $\Omega^n_{i,j}$, are written as follows:

$$\omega^0_{i,j}=\omega_0\left(x_i,y_i\right),\psi^0_{i,j}=\psi_0\left(x_i,y_i\right),\quad i,j=0,1,2,\ldots m,\tag{8}$$

$$\omega^n_{0,j}=\omega^n_{m,j}=\omega^n_{i,0}=\omega^n_{i,m}=0,\tag{9}$$

$$\psi^n_{0,j}=\psi^n_{m,j}=\psi^n_{i,0}=\psi^n_{i,m}=0,\quad i,j=1,2,3,\ldots m-1, n=1,2,3,\ldots\tag{10}$$

4. Matrix formulations and algorithm steps

4.1 Matrix formulations

We can write the implicit-Euler finite difference equation (6) and (7) in matrix forms as follows:

$$\left(I - \frac{r}{R}C + \frac{r}{4}\ V_2^n A - \frac{r}{4}\ V_1^n B\right)\omega^{n+1} = \omega^n, \tag{11}$$

$$-\,h^2\omega^n = C\psi^n, n = 0, 1, 2, \ldots \tag{12}$$

$$V_1^n = A\psi^n\ ,\ V_2^n = B\psi^n, \tag{13}$$

where

$$\begin{aligned}\omega^n = (&\omega_{1,1}^n,\ \omega_{2,1}^n, \ldots, \omega_{m-1,1}^n;\ \omega_{1,2}^n,\ \omega_{2,2}^n, \ldots, \omega_{m-1,2}^n; \ldots;\\ &\omega_{1,m-1}^n,\ \omega_{2,m-1}^n, \ldots, \omega_{m-1,m-1}^n),\\ \psi^n = (&\psi_{1,1}^n,\ \psi_{2,1}^n, \ldots, \psi_{m-1,1}^n;\ \psi_{1,2}^n,\ w_{2,2}^n, \ldots, \psi_{m-1,2}^n; \ldots;\\ &\psi_{1,m-1}^n,\ \psi_{2,m-1}^n, \ldots, \psi_{m-1,m-1}^n),\end{aligned}$$

M_1, M_2, A and B take the following forms:

$$A = \begin{bmatrix} 0 & I_1 & & 0 \\ -I_1 & 0 & I_1 & \\ & & \ddots & \\ 0 & & -I_1 & 0 \end{bmatrix}_{(m-1)^2\times(m-1)^2},\quad B = \begin{bmatrix} B_1 & & & 0 \\ & B_1 & & \\ & & \ddots & \\ 0 & & & B_1 \end{bmatrix}_{(m-1)^2\times(m-1)^2},$$

$$B_1 = \begin{bmatrix} 0 & 1 & & 0 \\ -1 & 0 & 1 & \\ & & \ddots & \\ 0 & & -1 & 0 \end{bmatrix}_{(m-1)\times(m-1)},\quad C = \begin{bmatrix} C_1 & I_1 & & 0 \\ I_1 & C_1 & I_1 & \\ & & \ddots & \\ 0 & & I_1 & C_1 \end{bmatrix}_{(m-1)^2\times(m-1)^2},$$

$$C_1 = \begin{bmatrix} -4 & 1 & & 0 \\ 1 & -4 & 1 & \\ & & \ddots & \\ 0 & & 1 & -4 \end{bmatrix}_{(m-1)\times(m-1)}$$

and $I,\ I_1$ are the identity matrices of orders $(m-1)^2$, $(m-1)$, respectively.

4.2 The proposed algorithm steps

The three required computational steps regarding using the implicit-Euler finite difference matrix forms (11)-(13), for solving problem (1)-(4) at each time-level $(n+1)$, are as follows:

First step: Find the vector ψ^n, by computing the solution of the linear system (12).

Second step: Compute V_1^n , V_2^n using (13) and then substitute them in system (11).

Third step: Solve the linear system (11), to get the solution-vector ω^{n+1} .

5. Consistency of the proposed method

The next theorem shows the order of accuracy of the implicit-Euler method by estimating the truncation-errors (consistency-errors) for both finite difference equations (6), (7).

Theorem 5.1. *Let $T^n_{i,j}$ and $\hat{T}^n_{i,j}$ be the local-truncation-errors at (x_i, y_j, t_n) for equations (6) and (7), respectively. Then $|T^n_{i,j}| \leq C_1k^2 + C_2kh^2, |\widehat{T}^n_{i,j}| \leq Ch^2, C_1, C_2, C > 0$.*

Proof of Theorem 5.1. Set $\omega|^n_{i,j} = \omega(x_i, y_j, t_n)$, $\psi|^n_{i,j} = \psi(x_i, y_j, t_n)$. By the implicit-Euler discrete equation (6), we obtain

$$T^n_{i,j} = \left(1 - \frac{r}{R}\left(\delta^2{}_x + \delta^2{}_y\right) + \frac{r}{4}\left(\delta_y\psi|^n_{i,j}\right)(\delta_x) - \frac{r}{4}\left(\delta_x\psi|^n_{i,j}\right)(\delta_y)\right)\omega|^{n+1}_{i,j} - \omega|^n_{i,j}\ .$$

Thus

$$\begin{aligned} T^n_{i,j} &= \omega|^{n+1}_{i,j} - \omega|^n_{i,j} - \frac{r}{R}\left(\delta^2{}_x + \delta^2{}_y\right)\omega|^{n+1}_{i,j} \\ &+ \frac{r}{4}\left(\delta_y\psi|^n_{i,j}\right)\left(\delta_x\omega|^{n+1}_{i,j}\right) - \frac{r}{4}\left(\delta_x\psi|^n_{i,j}\right)\left(\delta_y\omega|^{n+1}_{i,j}\right). \end{aligned}$$

By truncating the Taylor expansion, the above equation of LTE becomes:

$$\begin{aligned} T^n_{i,j} &= k\left(\frac{\partial\omega}{\partial t}|^n_{i,j} + \frac{k}{2}\frac{\partial^2\omega}{\partial t^2} + O(k^2)\right) - \frac{k}{R}\left[\left(\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2}\right)|^n_{i,j} + O\left(h^2\right)\right] \\ &+ k\left[\frac{\partial\psi}{\partial y}|^n_{i,j} + O\left(h^2\right)\right]\left[\frac{\partial\omega}{\partial x}|^n_{i,j} + O\left(h^2\right)\right] - k\left[\frac{\partial\psi}{\partial x}|^n_{i,j} + O\left(h^2\right)\right]\left[\frac{\partial\omega}{\partial y}|^n_{i,j} + O\left(h^2\right)\right]. \end{aligned}$$

By equation (1), we have

$$\left[\frac{\partial\omega}{\partial t} - \frac{1}{R}\left(\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2}\right) + \left(\frac{\partial\psi}{\partial y}\right)\left(\frac{\partial\omega}{\partial x}\right) - \left(\frac{\partial\psi}{\partial x}\right)\left(\frac{\partial\omega}{\partial y}\right)\right]|^n_{i,j} = 0.$$

It follows that $T^n_{i,j} = O\left(k^2\right) + O\left(kh^2\right)$, or $T^n_{i,j} = O\left(k^2 + kh^2\right)$.

As the partial derivatives of ω, ψ are bounded in the domain, $D^n_{i,j}$, there are two positive constants, $C_1, C_2 \in R$, such that $|T^n_{i,j}| \leq C_1k^2 + C_2kh^2$.

Next, the local truncation error for the difference equation (7) at the mesh points (x_i, y_j, t_n), can be derived as follows:

$$\hat{T}^n_{i,j} = \omega|^n_{i,j} + \left(\frac{\delta^2{}_x\psi|^n_{i,j}}{h^2} + \frac{\delta^2{}_y\psi|^n_{i,j}}{h^2}\right).$$

By truncating the Taylor expansion, the above equation of LTE becomes $\hat{T}^n_{i,j} = \omega|^n_{i,j} + \left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right)|^n_{i,j} + O\left(h^2\right)$. By equation (2), we get $\omega|^n_{i,j} + (\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2})|^n_{i,j} = 0$. Thus $\hat{T}^n_{i,j} = O\left(h^2\right)$, and there exists $C > 0$ such that $|\widehat{T}^n_{i,j}| \leq Ch^2$.

Definition 5.1 ([9]). *The finite difference-equation $F(x_i, y_i, t_n)$ is said to be consistent if,* $\frac{LTE}{k} \to 0$, *as* $h, k \to 0$.

By Definition 1 and Theorem 1, we can prove the following theorem.

Theorem 5.2. *The difference equation of implicit-Euler scheme* (6) *and* (7) *is consistent.*

6. Stability of the proposed method

In this section, we discuss the stability for the matrix form (11) and (12).

The matrix form implicit-Euler scheme (11) and (12) can be rewritten as follows:

$$\omega^{n+1} = H_n\, \omega^n, \quad \forall\, n, \tag{14}$$

where $H_n = \left(I - \frac{r}{R}C + \frac{r}{4}\, V_2^n A - \frac{r}{4}\, V_1^n B\right)^{-1}$.

Theorem 6.1. *The necessary and sufficient condition, which guarantees that the implicit-Euler scheme* (11) *and* (12) *is stable, is given as follows:*

$$\|H_n\| \le 1, \quad \textit{for all} \quad n, \tag{15}$$

where

$$\|H_n\|_2 = \max_s |\lambda_s|, \tag{16}$$

$\lambda_s (s = 1, 2, \ldots, (m-1)^2)$ *are the eigenvalues of* H_n .

The proof of this theorem is similar to that in [9], so it is omitted here.

7. Numerical example

Next, we apply the implicit-Euler finite difference algorithm, using (11)-(13), to find the numerical solution of problem (1)-(4), where $R = 50$, and the initial function for vorticity is given as follows:

$$\omega_0\,(x, y) = \left(1 - x^2\right)\left(1 - y^2\right), \qquad x, y \in [-1, 1]. \tag{17}$$

The numerical solutions are computed for ($h = 0.4, 0.2,\ 0.1$), and ($k = 0.002$).

Based on the type the chosen initial function (17), we know that the solution of problem (1)-(4) is symmetric and positive. Therefore, at any time-level, it is sufficient to find only the first β components of the approximate solution-vectors ω^n, ψ^n, where

$$\beta = \begin{cases} \frac{(m-1)^2}{2}, & \text{if } m \text{ is even,} \\ \frac{(m-1)^2+1}{2}, & \text{if } m \text{ is odd} \end{cases} .$$

Moreover, for each of $h = 0.4$, $h = 0.2$, and at the time-level n, we shall compute the errors -bounds that show the variations between the two numerical solutions (ω_h^n, ψ_h^n) and $(\omega_{h/2}^n,\ \psi_{h/2}^n)$ computed with h and $h/2$, respectively, at the points of the set:

$$\mathrm{H} = \{(x, y)\,,\ s.t.\ x = -1 + ih;\ y = -1 + jh;\ i = 1, 2, 3, 4, j = 1, 2, h = 0.4\}\,,$$

by using the errors-formulas:

$$E_h^n(\omega) = \frac{\sum_{(x,y)\in \mathrm{H}} \left|\omega_h^n(x, y) - \omega_{h/2}^n(x, y)\ \right|}{N = 8}, \tag{18}$$

$$E_h^n(\psi) = \frac{\sum_{(x,y)\in \mathrm{H}} \left|\psi_h^n(x, y) - \psi_{h/2}^n(x, y)\ \right|}{N = 8}. \tag{19}$$

7.1 Numerical results and discussion

The next tables show the numerical solutions of problem (1)-(4) with (17), for ($k = 0.002$) and ($R = 50$), obtained by using the implicit-Euler scheme, where Matlab software is used in the computational processes. In Tables 1, 2 and 3, the numerical results are shown, for $h = 0.4, 0.2, 0.1$, with respect to the time-levels 100, 200 and 400, respectively. In Table 4, the errors bounds of numerical solution are shown with $h = 0.4, 0.2$, at time levels: $n = 100, 200, 400$, using the formulas (18),(19). In Table 5, we show the values of $\|H_n\|_2$ defined in (16), for $h = 0.4, 0.2, 0.1$, at time levels: $n = 100, 200, 400$.

Table 1: Numerical solutions, where $n = 100$ ($t = 0.2$)

h	**0.4**		**0.2**		**0.1**	
(i, j)	ψ	ω	ψ	ω	ψ	ω
(1,1)	0.0767	0.3989	0.0743	0.3974	0.0738	0.3975
(2,1)	0.1222	0.6068	0.1189	0.6067	0.1181	0.6069
(3,1)	0.1215	0.5937	0.1183	0.5936	0.1174	0.5935
(4,1)	0.0755	0.3773	0.0734	0.3765	0.0727	0.3763
(1,2)	0.1208	0.5795	0.1174	0.5769	0.1166	0.5763
(2,2)	0.1936	0.9057	0.1888	0.9055	0.1876	0.9055
(3,2)	0.1932	0.8990	0.1885	0.8990	0.1872	0.8987
(4,2)	0.1201	0.5689	0.1169	0.5671	0.1160	0.5662

From Tables 1-3, we see that the numerical results for the vorticity and stream are decreasing as time increases. Table 4 shows that at any fixed time-level, as we refine the space-grids, the corresponding errors-bounds are decreasing, which indicates that the resulting numerical solutions are convergent. On the other hand, at any fixed space-step, the corresponding errors are increasing as time increases. Table 5 shows that, at any time-level, the stability-condition (15) is held.

Table 2: Numerical solutions, where $n = 200$ ($t = 0.4$)

$\boldsymbol{h}$	**0.4**		**0.2**		**0.1**	
(i, j)	$\boldsymbol{\psi}$	$\boldsymbol{\omega}$	$\boldsymbol{\psi}$	$\boldsymbol{\omega}$	$\boldsymbol{\psi}$	$\boldsymbol{\omega}$
(1,1)	0.0742	0.3874	0.0715	0.3820	0.0709	0.3819
(2,1)	0.1188	0.5988	0.1151	0.5981	0.1143	0.5987
(3,1)	0.1174	0.5740	0.1141	0.5739	0.1131	0.5734
(4,1)	0.0720	0.3487	0.0699	0.3469	0.0691	0.3462
(1,2)	0.1161	0.5458	0.1123	0.5377	0.1114	0.5351
(2,2)	0.1878	0.8887	0.1825	0.8879	0.1813	0.8880
(3,2)	0.1870	0.8761	0.1820	0.8762	0.1806	0.8753
(4,2)	0.1149	0.5274	0.1116	0.5229	0.1104	0.5198

Table 3: Numerical solutions, where $n = 400$ ($t = 0.8$)

$\boldsymbol{h}$	**0.4**		**0.2**		**0.1**	
(i, j)	$\boldsymbol{\psi}$	$\boldsymbol{\omega}$	$\boldsymbol{\psi}$	$\boldsymbol{\omega}$	$\boldsymbol{\psi}$	$\boldsymbol{\omega}$
(1,1)	0.0693	0.3628	0.0659	0.3466	0.0652	0.3438
(2,1)	0.1122	0.5810	0.1077	0.5778	0.1067	0.5794
(3,1)	0.1098	0.5376	0.1061	0.5373	0.1049	0.5360
(4,1)	0.0658	0.3007	0.0636	0.2972	0.0627	0.2952
(1,2)	0.1071	0.4830	0.1026	0.4619	0.1015	0.4539
(2,2)	0.1766	0.8522	0.1703	0.8484	0.1687	0.8484
(3,2)	0.1753	0.8302	0.1695	0.8300	0.1677	0.8275
(4,2)	0.1053	0.4554	0.1016	0.4461	0.1003	0.4379

Table 4: Errors bounds

$\boldsymbol{h}$	**0.4**		**0.2**	
$\boldsymbol{n}$	$\boldsymbol{E_h^n(\psi)}$	$\boldsymbol{E_h^n(\omega)}$	$\boldsymbol{E_h^n(\psi)}$	$\boldsymbol{E_h^n(\omega)}$
100	0.0034	8.8750e-04	8.8750e-04	3.0000e-04
200	0.0037	0.0027	9.8750e-04	0.0011
400	0.0043	0.0072	0.0012	0.0033

Table 5: $\|\boldsymbol{H_n}\|_2 = \boldsymbol{max_s|\lambda_s|}$

n	**100**	**200**	**400**
h	$\|H_n\|_2$	$\|H_n\|_2$	$\|H_n\|_2$
0.4	0.999601	0.999606	0.999616
0.2	0.999570	0.999574	0.999584
0.1	0.999550	0.999555	0.999566

7.2 Numerical simulations

In Figures 1,2 and 3, we present the discrete graphs of vorticity and stream obtained by using the implicit-Euler scheme results with $R = 50$, $(k = 0.002)$ and $(h = 0.1)$, at time levels $n = 0,\ 200$ and 400 , respectively. It is observed that the shown discrete graphs of both vorticity and stream are decreasing as time increases.

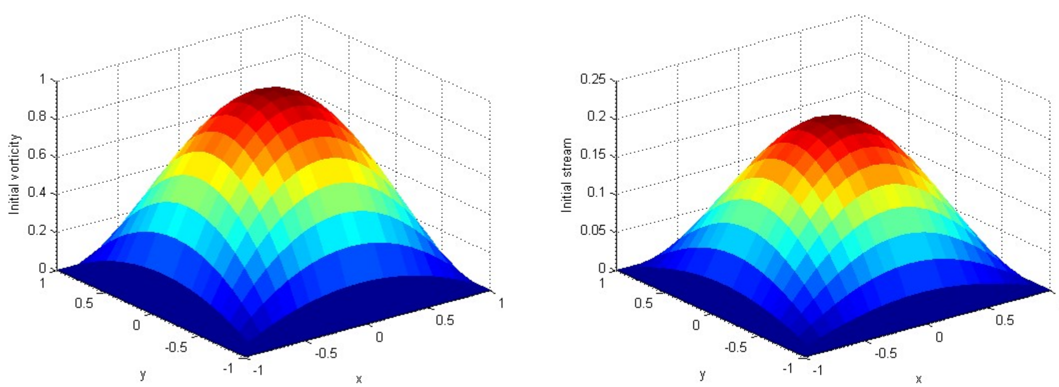

Figure 1: Numerical solutions at $\boldsymbol{t = 0}$

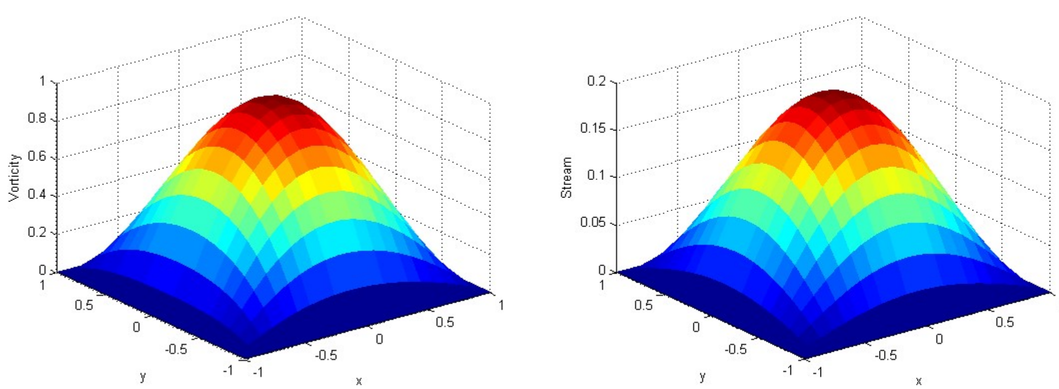

Figure 2: Numerical solutions at $\boldsymbol{t = 0.4}$

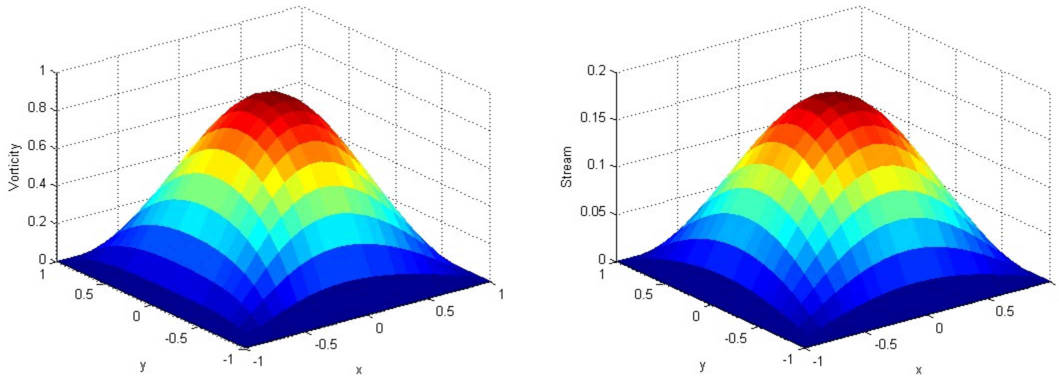

Figure 3: Numerical solutions at $\boldsymbol{t = 0.8}$

8. Conclusion

In this paper, the Implicit-Euler finite difference scheme is proposed to compute the numerical solution of the two-dimensional vorticity transport equation with homogenous Dirichlet boundary conditions. In addition, a numerical experiment is given, and the numerical results are presented and simulated in the form of tables and figures. From this work, we conclude that the Implicit-Euler finite difference scheme is consistent and stable, with any space-step and time level. Moreover, it is observed that the resulting numerical solutions are decreasing over time. Furthermore, at any fixed time-level, as we refine the space-grids, the corresponding errors-bounds are decreasing, which indicates that the resulting numerical solutions are convergent. On the other hand, at any fixed space-step, the corresponding errors are increasing as time increases. In order to overcome this problem, the space grids should be re-refined, as time goes farther.

Acknowledgments

The authors would like to thank Mustansiriyah University for supporting the present work.

References

[1] J. D. Anderson, J. Wendt, *Computational fluid dynamics*, 206, New York: McGraw-Hill, 1995.

[2] V.C. Loukopoulos, G.T. Messaris, G.C. Bourantas, *Numerical solution of the incompressible Navier–Stokes equations in primitive variables and velocity–vorticity formulation*, Applied Mathematics and Computation, 222 (2013), 575-588.

[3] M. A. Rasheed, S. Laverty, B. Bannish, *Numerical solutions of a linear age-structured population model*, AIP Conference Proceedings, 2096 (2019), 020002, doi.org/10.1063/1.5097799

[4] A. Kaushik, *Numerical study of 2D incompressible flow in a rectangular domain using chorin's projection method at high Reynolds number*, International Journal of Mathematical, Engineering and Management Sciences, 4 (2019), 157–169.

[5] M. A. Rasheed, F. N. Ghaffoori, *Numerical blow-up time and growth rate of a reaction-diffusion equation*, Italian journal of pure and applied mathematics, 44 (2020), 805-813.

[6] M. A. Rasheed, R. A. Hameed, S. K. Obeid, A. F. Jameel, *On numerical blow-up solutions of semilinear heat equations*, Iraqi Journal of Science, 61 (2020), 2077-2086.

[7] A. F. Jameel, S. A. J. Altaie, S. G. A. Aljabbari, A. AlZubaidi, N. H. Man, *Double parametric fuzzy numbers approximate scheme for solving one-dimensional fuzzy heat-like and wave-like equations*, Mathematics, 8 (2020), 1737, 1-26.

[8] R. H. Ibraheem, *Solving fuzzy differential equations by using power series*, Iraqi Journal of Science, special issue (2020), 92-107.

[9] A. R. Mitchell, *Computational methods in partial Differential equations*, Wiley, London, 1969.

[10] T. E. Tezduyar, J. Liou, D. K. Ganjoo, M. Behr, *Solution techniques for the vorticity-streamfunction formulation of two-dimensional unsteady incompressible flows*, Int. J. Numer. Methods Fluids, 11 (1990), 515–539.

[11] V. Ambethkar, M. Kumar, M. K. Srivastava, *Numerical solutions of 2-d unsteady incompressible flow in a driven square cavity using streamfunction-vorticity formulation*, Int. J. Appl. Math., 29 (2016), 727-757.

[12] C. G. Speziale, *On the advantages of the vorticity-velocity formulation of the equations of fluid dynamics*, Journal of Computational Physics, 73 (1987), 476-480.

[13] C. Pozrikidis, *Equation of motion and vorticity transport. In: Fluid Dynamics*, Springer, Boston, MA, 2017.

[14] S. C. R. Dennis ,*The numerical solution of the vorticity transport equation*, Proc. 3rd Int. Conf. on Numerical Methods in Fluid Mech., Lect. Notes Phys., (1973), 19-120, Springer.

[15] M. Joseph, *Finite difference representations of vorticity transport*, Computer Methods in Applied Mechanics and Engineering, 39 (1983).

[16] M. Napolitano and G. Pascazio, *A numerical method for the vorticity-velocity Navier-Stokes equations in two and three dimensions*, Computers and Fluids, 19 (1991).

[17] D. C. Lo, K. Murugesan, D. L. Young, *Numerical solution of three-dimensional velocity–vorticity Navier–Stokes equations by finite difference method*, Int. J. Numer. Methods Fluids, 47 (2005), 1469-1487.

[18] J. Ravnik , J. Tibaut, *Boundary-domain integral method for vorticity transport equation with variable viscosity*, International Journal of Computational Methods and Experimental Measurements, 6 (2018), 1087–1096.

[19] M. A. Rasheed, A. T. Balasim, and A.F. Jameel. , *Some results for the vorticity transport equation by using A.D.I scheme*, AIP Conference Proceedings, 2138 (2019), 030031, doi.org/10.1063/1.5121068

[20] M. A. Rasheed, S. N. Kadhim, *Numerical solutions of two-dimensional vorticity transport equation using Crank-Nicolson method*, Baghdad Science Journal, 19 (2022), 321-328.

Accepted: February 06, 2021