Fuzzy (left)topological β-algebras

V.V.S. Ramachandram
Dadi Institute of Engineeting and Technology
Anakapalle - 531002
Visakapattinam (Andra Pradesh)
India
vvsrch@gmail.com

S. Nivetha
Sri Ramanas College Of Arts And Science For Women
Aruppukottai-626134,
India
nivethasoundar9127@gmail.com

M. Chandramouleeswaran*
Saiva Bhanu Kshatryia College
Aruppukottai-626101
India
moulee59@gmail.com

P. Muralikrishna
Muthurangam Government Arts College (Autonomous)
Vellore-632002, TamilNadu
India
pmkrishna@rocketmail.com

Abstract. In this paper we introduce the notions of fuzzy topological spaces on a β-algebra and (left) fuzzy topological β-algebras. We illustrate them with some examples and prove simple but elegant properties.
Keywords: fuzzy topological spaces, fuzzy continuity, β-algebras, (left)fuzzy topological β-algebras.

1. Introduction

In 1966, Imai and Isaki introduced two new classes of algebras namely, BCK algebras and BCI algebras [8]. As a generalization of these algebras sevaral other algebras were introduced by many authors. One such algebras are β algebras introduced by Neggeres and Kim [11]. In the year 1965, Zadeh [17] introduced the notion of fuzzy sets as a generalization of crisp sets, so as to include the uncertainities of the real world physical problems. This notion of fuzzy sets was studied by several authors in the areas of algebra, topology, etc.,. Three years after Zadeh's paper had appeared, Chang [5] made the first 'grafting' of the

∗. Corresponding author

notion of a fuzzy set on general topology. He introduced the notion, what we call now a Chang fuzzy space and made an attempt to develop basic topological notions for such spaces. This paper was followed by others in which Chang fuzzy spaces and other topological type structures for fuzzy set systems were considered. It was further studied by Chang [5], Wong [14], and Lowen [10]. D.H.Foster [7] combined the structure of a fuzzy topological space with that of a fuzzy group, introduced by A.Rosenfeld [13] and defined the term fuzzy topological groups.

The notion of fuzzy topology on fuzzy sets was introduced by Chakrabarty and Ahsaunllah [3]. Chanduraty and Das [4] studied several fundamental properties of such fuzzy topologies. In 1993, Jun [9], combined the structure of a fuzzy topological spaces with that of fuzzy BCK-algebras to formulate the elements of the theory of fuzzy topological BCK-algebras. In [2] Akram and Dar introduced the notions of fuzzy topological subalgebras and ideals in K-algebras. In [1], the authors studied the notion of fuzzy subalgebras of a β-algebra. This paper deals with the notion of (left)fuzzy topological β algebras and investigate some simple properties.

2. Preliminaries

In this section we recall some basic definitions and results that are needed for our work.

Definition 2.1 ([8]). *A BCK-algebra $(X, *, e)$ is a non-empty set X with a constant e and a binary operation $'*'$ satisfying the following axioms: For all $x_1, x_2, x_3 \in X$,*

1. $((x_1 * x_2) * (x_1 * x_3)) * (x_3 * x_2) = e$.
2. $(x_1 * (x_1 * x_2)) * x_2 = e$.
3. $x_1 * x_1 = e$
4. $x_1 * x_2 = e$ *and* $x_2 * x_1 = e \Rightarrow x_1 = x_2$
5. $e * x_1 = e$

Definition 2.2 ([8]). *A BCI-algebra $(X, *, e)$ is a non-empty set X with a constant e and a binary operation $'*'$ satisfying the following axioms: For all $x_1, x_2, x_3 \in X$,*

1. $((x_1 * x_2) * (x_1 * x_3)) * (x_3 * x_2) = e$.
2. $(x_1 * (x_1 * x_2)) * x_2 = e$.
3. $x_1 * x_1 = e$
4. $x_1 * x_2 = e$ *and* $x_2 * x_1 = 0 \Rightarrow x_1 = x_2$

Definition 2.3 ([11]). *A β-algebra is a non-empty set X with a constant 0 and two binary operations $'+'$ and $'-'$ satisfying the following axioms: For all $x, y, z \in X$,*

1. $x - 0 = x$.
2. $(0 - x) + x = 0$.
3. $(x - y) - z = x - (z + y)$

Example 2.1. A β-algebra is the set $X = (\{0, 1, 2, 3\}, +, -, 0)$ with the following Cayley table

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

-	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Example 2.2. Consider the set of natural numbers $\mathbb{N}$. Define the binary operation $'*'$ as $a * b = \frac{a}{(a,b)}$ $\forall\ a, b \in \mathbb{N}$ where (a, b) is the greatest common diviser of a and b. Then, $(\mathbb{N}, +, -, 1)$ where $x + y = x * y'$ with $y' = 0 * y$ and $x - y = x * y$ $\forall\ x, y \in \mathbb{N}$, is an infinite β-algebra.

Definition 2.4 ([1]). *Let μ be a fuzzy set in a β-algebra X. Then μ is called a fuzzy β-subalgebra of X if foralll $x, y \in X$, we have*

1. $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$;
2. $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$.

Example 2.3. Consider the β-algebra $(X, +, -, 0)$ in Example 2.1.

Define $\mu : X \to [0, 1]$ such that

$$\mu(x) = \begin{cases} 0, & x = 0, \\ 0.5, & x = 1, \\ 1, & x = 2, 3 \end{cases}$$

then μ is a fuzzy β-subalgebra in X.

Definition 2.5 ([10]). *The constant fuzzy set, denoted by k_c is defined by the membership function, $\mu_{k_c}(x) = c$, for all $x \in X$ and $c \in [0, 1]$. The fuzzy set k_1 corresponds to the set X and the fuzzy set k_0 denotes the empty set Φ.*

Definition 2.6 ([10]). *A fuzzy topology is a family $\mathscr{T}$ of fuzzy sets in X which satisfies the following conditions*

1. *for all $c \in I$, $k_c \in \mathscr{T}$;*

2. $\bigcap_{i=1}^{n} A_i \in \mathscr{T}$;

3. $\bigcup_{i\in I} A_i \in \mathscr{T}$;

where $A_i \in \mathscr{T}$ *for each* $i \in I$, I *an indexing set.*

If X *is a set with a fuzzy topology* $\mathscr{T}$, *then* $(X, \mathscr{T})$ *is called a fuzzy topological space and any element in* $\mathscr{T}$ *is called a* $\mathscr{T}$*-open fuzzy set in* X.

Remark 2.1. Throughout this paper, by an open set in a fuzzy topological space $(X, \mathscr{T})$ we mean only a $\mathscr{T}$-open set.

3. Fuzzy topological spaces on β algebras

In this section, we introduce the notion of fuzzy topological spaces on a β algebra and fuzzy continuity on the fuzzy topological spaces on β algebras. Further, we shall prove some simple properties

Definition 3.1. *A fuzzy topology is a family* $\mathscr{T}$ *of fuzzy sets in the* β*-algebra* $(X, +, -, 0)$ *which satisfies the following conditions:*

1. for all $c \in I, k_c \in \mathscr{T}$;

2. $\bigcap_{i=1}^{n} A_i \in \mathscr{T}$;

3. $\bigcup_{i\in I} A_i \in \mathscr{T}$;

where $A_i \in \mathscr{T}$ *for each* $i \in I, I$ *an indexing set.*

If X *is a* β*-algebra with a fuzzy topology* $\mathscr{T}$, *then* $(X, \mathscr{T})$ *is called a fuzzy topological space on* β*-algebra and any element in* $\mathscr{T}$ *is called a* $\mathscr{T}$*-open fuzzy set in* X.

Example 3.1. 1. Let $X = \{0, x_1, x_2, x_3\}$ be a β-algebra with the following Cayley tables.

$+$	0	x_1	x_2	x_3
0	0	x_1	x_2	x_3
x_1	x_1	0	x_3	x_2
x_2	x_2	x_3	x_1	0
x_3	x_3	x_2	0	x_1

-	0	x_1	x_2	x_3
0	0	x_1	x_3	x_2
x_1	x_1	0	x_2	x_3
x_2	x_2	x_3	0	x_1
x_3	x_3	x_2	x_1	0

Let A be a fuzzy set on X defined by $A(0) = A(x_3) = 0$, $A(x_1) = 0.5$, $A(x_2) = 0.4$. Then $\mathscr{T} = \{0, A, 1\}$ is a fuzzy topology on X and $(X, \mathscr{T})$ is a fuzzy topological space on the β-algebra X.

2. For any β-algebra $X, \mathscr{T} = \{0, 1\}$ is called an indiscrete fuzzy topology on X.

3. For any β-algebra X the discrete fuzzy topology $\mathscr{T}$ on X contains all the fuzzy sets in X.

Definition 3.2. *Let $(X, \mathscr{T})$ be a fuzzy topological space on a β-algebra X. A fuzzy topology $\mathscr{U}$ on X is said to be coarser than $\mathscr{T}$ if and only if $\mathscr{U} \subseteq \mathscr{T}$.*

In ordinary topological space, we consider the neighbourhood of a point. However, in the fuzzy topological space $(X, \mathscr{T})$, on a β-algebra X, we consider the neighbourhood of a fuzzy set in X.

Definition 3.3. *Let $(X, \mathscr{T})$ be a fuzzy topological space on the β-algebra $(X, +, -, 0)$. Let A be a fuzzy set in X. A fuzzy set $U \in \mathscr{T}$ is said to a neighbourhood of A if there exists a $\mathscr{T}$-open fuzzy set O such that $A \subset O \subset U$. That is, $A(x) \leq O(x) \leq U(x)$, for all $x \in X$.*

From the definition the following result follows: Let $(X, \mathscr{T})$ be a fuzzy topological space on the β-algebra $(X, +, -, e)$. Let A be a fuzzy set in X. A is open if and only if for each fuzzy set B such that $B \subseteq A, A$ is a neighbourhood of B.

Definition 3.4. *The neighbourhood system $\mathcal{N}$ of a fuzzy set A in a fuzzy topological space $(X, \mathscr{T})$ on the β-algebra $(X, +, -, e)$ is the family of all neighbourhoods of A.*

Theorem 3.1. *Let $(X, \mathscr{T})$ be a fuzzy topological space on the β-algebra and let A be a given fuzzy set in X. If $\mathcal{N}$ is the neighbourhood system of A, then finite intersections of members of $\mathcal{N}$ also belong to $\mathcal{N}$. Further, each fuzzy set of X that contains a member of $\mathcal{N}$ belongs to $\mathcal{N}$.*

Proof. Given that $(X, \mathscr{T})$ is a fuzzy topological space on the β-algebra X and A a fuzzy subset in X.

Since $\mathcal{N}$ is the neighbourhood system of A, it contains all the neighbourhoods of A.

Choose $R, S \in \mathcal{N}$. Then, both R and S are neighbourhoods of the fuzzy set A. Hence, there exists open sets $R_0 \subseteq R$ and $S_0 \subseteq S$ such that

$$A \subseteq R_0 \subseteq R \text{ and } A \subseteq S_0 \subseteq S$$

respectively, and hence $R_0 \cap S_0 \subseteq R \cap S$. That is, $R_0 \cap S_0$ is also a neighbourhood of A, thus proving our result. Further, if a fuzzy set R in X contains a neighbourhood of A, it contains an open neighbourhood of A. This proves that R itself is a neighbourhood of A. That is, $R \in \mathcal{N}$. □

Definition 3.5 ([5])**.** *Let A and B be fuzzy sets in a fuzzy topological space $(X, \mathscr{T})$ on the β-algebraX. Let $A \supset B$. Then B is called an interior of A if A is a neighbourhood of B. The union of all interior fuzzy sets of A is again an interior fuzzy set of A and is denoted by A^0*

Theorem 3.2. *Let $(X, \mathscr{T})$ be a fuzzy topological space on the β-algebra X and let A be a fuzzy set in X. Then, A^0 is open and is the largest open set contained in A. Moreover, the fuzzy set A is open if and only if $A^0 = A$.*

Proof. By Definition 3.5, A^0 itself is an interior fuzzy set of A. Therefore, there exists an open set O in X such that $A^0 \subseteq O \subseteq A$.

Since O is an interior fuzzy set of A, $O \subseteq A^0$, proving that $A^0 = O$. This proves that A^0 is an open set and is the largest open set contained in A. Further, if A is open then $A \subset A^0$ as A itself is an interior fuzzy set of A, forcing $A = A^0$.

On the other hand, if $A^0 = A$, then A^0 is the largest open set contained in A and hence A is open.

From the above theorem we define the $\mathscr{T}$-open fuzzy set and the $\mathscr{T}$-closed fuzzy set. □

Definition 3.6. *Let A be fuzzy a set in a fuzzy topological space $(X, \mathscr{T})$ on the β-algebraX. Any fuzzy subset $A \in X$ is called a $\mathscr{T}$-open fuzzy set, if $A^0 = A$. Any fuzzy subset $A \in X$ is called a $\mathscr{T}$-closed fuzzy set, if $A' \in \mathscr{T}$.*

Definition 3.7. *Let $(X, \mathscr{T})$ be a given fuzzy topological space on the β-algebra X. Let $\{A_i / i \in \mathbb{I}\}$ be a sequence of fuzzy sets in X where, $\mathbb{I}$ is an indexing set and let A be any fuzzy set in X. Then the sequence*

1. *$\{A_i\}$ is said to be eventually contained in A if and only if there exists an integer N such that $A_i \subset A$, $\forall\, i \geq N$.*

2. *$\{A_i\}$ is said to be frequently contained in A if and only if for each i there is an integer N such that $A_i \subset A$, $\forall\, i \geq N$.*

3. *$\{A_i\}$ is said to be fuzzy convergent to A if and only if $\{A_i\}$ is eventually contained in each neighbourhood of A.*

4. *A fuzzy set A in X is said to be a cluster fuzzy set of $\{A_i\}$ if and only if $\{A_i\}$ is frequently contained in every neighbourhood of A.*

From the definition one can prove the following.

Theorem 3.3. *Let $(X, \mathscr{T})$ be a fuzzy topological space on the β-algebra X. If the neighbourhood system of each fuzzy set A in X is countable, then*

1. *A is open if and only if each sequence of fuzzy sets $\{A_i\}$ converge to $B(\subset A)$ is eventually contained in A.*

2. *whenever A is a cluster fuzzy set of $\{A_i\}$ then there is a subsequence of $\{A_i\}$ converging to A.*

Definition 3.8. *Let $(X, \mathscr{T})$ and $(Y, \mathscr{U})$ be two fuzzy topological spaces on the β-algebras $(X, +, -, 0)$ and $(Y, +, -, 0)$ respectively. A function $f : X \to Y$ is called a fuzzy continuous (F-continuous) function if the inverse image of each $\mathscr{U}$ - fuzzy open set is a $\mathscr{T}$ - fuzzy open set.*

Example 3.2. Consider the β-algebra $(X, +, -, 0)$ where $X = \{0, 1, 2, 3, 4, 5\}$ and + and-are defined by the following Cayley tables:

+	0	1	2	3	4	5
1	1	2	0	4	5	3
2	2	0	1	5	3	4
3	3	5	4	0	2	1
4	4	3	5	1	0	2
5	5	4	3	2	1	0

-	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Let the fuzzy sets $\mu_i : X \to [0, 1], i = 1, 2, 3, 4, 5, 6, 7, 8, \ldots$ be given by

$$\mu_1(x) = \begin{cases} .7, & \text{if } x = 0 \\ .5, & \text{if } x = 1, 3 \\ .4, & \text{if } x = 2, 4, 5 \end{cases}, \quad \mu_2(x) = \begin{cases} .3, & \text{if } x = 0 \\ .2, & \text{if } x = 1, 3 \\ 0, & \text{if } x = 2, 4, 5 \end{cases}$$

$$\mu_3(x) = \begin{cases} .6, & \text{if } x = 0 \\ .4, & \text{if } x = 1, 3 \\ .2, & \text{if } x = 2, 4, 5 \end{cases}, \quad \mu_4(x) = \begin{cases} .5, & \text{if } x = 0 \\ .4, & \text{if } x = 1, 3 \\ .1, & \text{if } x = 2, 4, 5 \end{cases}$$

$$\mu_5(x) = \begin{cases} .8, & \text{if } x = 0 \\ .7, & \text{if } x = 1, 3 \\ .5, & \text{if } x = 2, 4, 5 \end{cases}, \quad \mu_6(x) = \begin{cases} .7, & \text{if } x = 0 \\ .4, & \text{if } x = 1, 3 \\ .3, & \text{if } x = 2, 4, 5 \end{cases}$$

$$\mu_7(x) = \begin{cases} .8, & \text{if } x = 0 \\ .6, & \text{if } x = 1, 3 \\ .5, & \text{if } x = 2, 4, 5 \end{cases}, \quad \mu_8(x) = \begin{cases} .8, & \text{if } x = 0 \\ .7, & \text{if } x = 1, 3 \\ .4, & \text{if } x = 2, 4, 5 \end{cases} \quad \ldots$$

Then, the collection $\mathscr{T} = \{0, 1, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \mu_8, \ldots\}$ is a fuzzy topology on X. Hence, $(X, \mathscr{T})$ is a fuzzy topological space on β-algebra, on X.

Consider the β-algebra $(Y, +, -, 0)$ where $Y = \{0, a, b, c\}$ and the binary operations + and − are difined by the following Cayley tables:

+	0	a	b	c
0	0	a	b	c
a	a	b	c	0
b	b	c	0	a
c	c	0	a	b

-	0	a	b	c
0	0	c	b	a
a	a	0	c	b
b	b	a	0	c
c	c	b	a	0

Let the fuzzy sets $\sigma_j : Y \to [0, 1], j = 1, 2, 3, 4, \ldots$ be given by

$$\sigma_1(y) = \begin{cases} .8, & \text{if } y = 0 \\ .5, & \text{if } y = a, b \\ .7, & \text{if } y = c \end{cases}, \quad \sigma_2(y) = \begin{cases} .5, & \text{if } y = 0 \\ .1, & \text{if } y = a, b \\ .4, & \text{if } y = c \end{cases}$$

$$\sigma_3(y) = \begin{cases} .6, & \text{if } \ y=0 \\ .2, & \text{if } \ y=a,b \\ .4, & \text{if } \ y=c \end{cases}, \quad \sigma_4(y) = \begin{cases} .7, & \text{if } \ y=0 \\ .3, & \text{if } \ y=a,b \\ .4, & \text{if } \ y=c \end{cases} \quad \ldots$$

Then, the collection $\mathscr{U} = \{0, 1, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \ldots\}$ is a fuzzy topology on Y, and hence $(Y, \mathscr{U})$ is a fuzzy topological space on β-algebra Y.

Let$f : X \to Y$ be the function given by, $f(0) = 0, f(1) = c, f(2) = a, f(3) = c, f(4) = a, f(5) = b$. Then, f is F-continuous.

Definition 3.9. *Let $(X, \mathscr{T})$, $(Y, \mathscr{U})$ be two fuzzy topological spaces on β-algebras X and Y respectively. A mapping $f : X \to Y$ is said to be fuzzy open if and only if the image of each $\mathscr{T}$- open fuzzy set is a $\mathscr{U}$- open fuzzy set.*

Theorem 3.4. *Let $(X, \mathscr{T})$ and $(Y, \mathscr{U})$ be the fuzzy topological spaces on β-algebras X and Y respectively. Let $f : X \to Y$ be a function from X to Y. Then the function f is F- continuous if and only if the inverse image of every fuzzy closed set in Y is a fuzzy closed set in X.*

Proof. Suppose that the function f is F- continuous. That is, the inverse image of each fuzzy $\mathscr{U}$- open set is $\mathscr{T}$- open.

Let U' be the set of fuzzy closed sets in Y. Then $\mu_{f^{-1}(U')}(x) = \mu'_U(f(x)) = \mu_{U'}(f(x)) = 1 - \mu_U(f(x)) = 1 - \mu_{f^{-1}(U)}(x) = \mu'_{f^{-1}(U)}(x) \Rightarrow f^{-1}(U') = \{f^{-1}(U)\}'$, for all x in X.

Since f is F- continuous, the inverse image of every fuzzy closed set in Y is a fuzzy closed set in X.

Conversely, assume that the inverse image of every fuzzy closed set of Y is fuzzy closed in X.

Let U be a fuzzy open set in Y. Then, U' is a fuzzy closed set in Y. By hypothesis, $f^{-1}(U')$ is a fuzzy closed set in X. That is, $(f^{-1}(U))'$ is fuzzy closed in X. Hence, $(f^{-1}(U))$ is a fuzzy open set in X. Since the inverse image of every fuzzy open set is open, f is F- continuous. □

We conclude this section with the definitions of restrictrd mapping and relatively fuzzy continuous mapping.

Definition 3.10. *Consider the fuzzy topological spaces $(X, \mathscr{T})$ and $(Y, \mathscr{U})$ on β-algebras X and Y respectively. Let $(A, \mathscr{T}_A)$ and $(B, \mathscr{U}_B)$ be fuzzy subspaces of $(X, \mathscr{T})$ and $(Y, \mathscr{U})$ respectively. The mapping $f : X \to Y$ is said to be a mapping of $(A, \mathscr{T}_A)$ into $(B, \mathscr{U}_B)$ if $f[A] \subset B$.*

Definition 3.11. *Let $(A, \mathscr{T}_A)$ and $(B, \mathscr{U}_B)$ be fuzzy subspaces of fuzzy topological spaces on β-algebras $(X, \mathscr{T})$ and $(Y, \mathscr{U})$ respectively. $f : (A, \mathscr{T}_A) \to (B, \mathscr{U})$ is said to be relatively F-continuous if and only if for each open fuzzy set V in $\mathscr{U}_B$ the intersection $f^{-1}[V] \cap A$ is in $\mathscr{T}_A$. f is said to be relatively fuzzy open if and only if for each fuzzy open set U in $\mathscr{T}_A$ the image $f[U]$ is in $\mathscr{U}_B$.*

4. Fuzzy (left) topological β-algebras

In this section, we introduce the notion of fuzzy left topological β-algebras and prove some simple properties. We recall that : A function $f : X \to Y$ of fuzzy topological spaces is called a fuzzy homeomorphism if and only if f is bijective, fuzzy continuous (F-continuous) and f^{-1} is fuzzy continuous (F-continuous).

Definition 4.1. *Let $(X, +, -, 0)$ and $(Y, +, -, 0)$ be two β- algebras. Let $(X, \mathscr{T})$, $(Y, \mathscr{U})$ be two fuzzy topological spaces on X and Y respectively. A function $f : X \times X \to Y$ is said to be fuzzy left continuous if f is fuzzy continuous with respect to the fuzzy topology on the product $X \times X$ generated by the collection $\{U \times V : U, V \in \mathscr{T}\}$ where*

$$(U \times V)(s,t) = \begin{cases} V(t), & \text{if } U(s) > 0 \\ 0, & \text{otherwise} \end{cases}.$$

Definition 4.2. *Let $(X, +, -, 0)$ be a β-algebra and $\mathscr{T}$ be a fuzzy topology on X such that, for all $x, y \in X$,*

1. *$(x, y) \mapsto x + y$ is fuzzy left continuous;*
2. *$(x, y) \mapsto x - y$ is fuzzy left continuous.*

The pair $(X, \mathscr{T})$ is called a fuzzy left topological β-algebra.

We define $U + V, U - V$ for non zero fuzzy sets U, V on X as follows:

$$(U + V)(x) = \sup\{V(x + s) : U(s) > 0\},$$

$$(U - V)(x) = \sup\{V(x - s) : U(s) > 0\}.$$

With this definition of $U + V$ and $U - V$ we modify the definition of fuzzy topological β-algebra as a fuzzy topological space $(X, \mathscr{T})$ on a β-algebra $(X, +, -, 0)$ that satisfies the following condition:

1. For each $x, y \in X$ and each fuzzy neighbourhood W of $x + y$ there exist fuzzy neighbourhoods U and V of x and y respectively such that $U + V \subseteq W$.
2. For each $x, y \in X$ and each fuzzy neighbourhood W of $x - y$ there exist fuzzy neighbourhoods U and V of x and y respectively such that $U - V \subseteq W$.

Example 4.1. Consider the β- algebra $(X, +, -, 0)$ where $X = \{0, 1, 2, 3\}$ $+$ and $-$ are defined by the following Cayley Tables:

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

-	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

Define the fuzzy sets, $\mu_i : X \to [0,1]$ $i = 1, 2, \dots$ by

$$\mu_1(x) = \begin{cases} 0.7, & \text{if } x = 0 \\ 0.4, & \text{if } x = 1,2 \\ 0.5, & \text{if } x = 3 \end{cases}, \quad \mu_2(x) = \begin{cases} 0.3, & \text{if } x = 0 \\ 0, & \text{if } x = 1,2 \\ 0.2, & \text{if } x = 3 \end{cases}$$

$$\mu_3(x) = \begin{cases} 0.6, & \text{if } x = 0 \\ 0.2, & \text{if } x = 1,2 \\ 0.4, & \text{if } x = 3 \end{cases}, \quad \mu_4(x) = \begin{cases} 0.5, & \text{if } x = 0 \\ 0.1, & \text{if } x = 1,2 \\ 0.4, & \text{if } x = 3 \end{cases},$$

$$\mu_5(x) = \begin{cases} 0.8, & \text{if } x = 0 \\ 0.5, & \text{if } x = 1 \\ 0.72, & \text{if } x = 3 \end{cases}, \quad \mu_6(x) = \begin{cases} 0.7, & \text{if } x = 0.3 \\ 0.4, & \text{if } x = 1 \\ 0.4, & \text{if } x = 2 \\ 0.6 \text{ if } x = 3 \end{cases} \dots$$

$0 : X \to [0,1]$ defined by $0(x) = 0$, for all $x \in X$. Similarly, $1 : X \to [0,1]$ defined by $1(x) = 1$, for all $x \in X$. $\Rightarrow 0 = \phi$ and $1 = X$. Then $(X, \mathscr{T})$ is a fuzzy topological system β-algebra, where $\mathscr{T}_X = \{0, 1, \mu_1, \mu_2, \mu_3, \dots\}$.

Consider the set $Y = X \times X$, then Y is a β-algebra under componentwise operations. Y can be made into a topological space under the topology,

$$\mathscr{T}_Y = \{\mu_i \times \mu_j / \mu_i, \mu_j \in \mathscr{T}_X, i, j = 1, 2, 3, \dots\}$$

by defining $(\mu_i, \mu_j)(x_1, x_2) = \begin{cases} \mu_j(x_2), & \mu_i(x_1) > 0 \\ 0, & \text{otherwise} \end{cases}$. Thus $(Y, \mathscr{T}_Y)$ becomes a topological space.

$+ : Y \to X$ is given by $+(x_1, x_2) = x_1 + x_2$ and $- : Y \to X$ is given by $-(x_1, x_2) = x_1 - x_2$. Now,

$$+^{-1}(\mu_1 \times \mu_2)(x_1, x_2) = (\mu_1 \times \mu_2)(+(x_1, x_2)) = (\mu_1 \times \mu_2)(x_1 + x_2) = \mu_2(x_1 + x_2).$$

This proves that $+$ defined on the β-algebra X is fuzzy left continuous on X.

Similarly, we can show that $-$ is fuzzy left continuous on X. Thus $(X, \mathscr{T})$ given above becomes a fuzzy left topological β-algebra.

Theorem 4.1. *Consider the fuzzy left topological β-algebra $(X, \mathscr{T})$. For any two fuzzy points x_α, y_α on X with $0 < \alpha < 1$, the following results hold:*

1. $x_\alpha + y_\alpha = (x + y)_\alpha$
2. $x_\alpha - y_\alpha = (x - y)_\alpha$

Proof. Consider the fuzzy points x_α and y_α respectively. Then, μ_{x_α} and μ_{y_α} are two fuzzy sets defined by

$$\mu_{x_\alpha}(z) = \begin{cases} \alpha, & \text{if } x = z \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \mu_{y_\alpha}(z) = \begin{cases} \alpha, & \text{if } y = z \\ 0, & \text{otherwise} \end{cases}$$

$(\mu_{x_\alpha} + \mu_{y_\alpha})(z) = \sup\{\mu_{y_\alpha}(y_1 + y_2)$, whenever $\mu_{x_\alpha}(z) > 0$ such that $z = y_1 + y_2, y_1, y_2 \in X\} = \sup\{\min\{\mu_{y_\alpha}(y_1), \mu_{y_\alpha}(y_2)\}\} = \alpha = \sup\{\mu_{y_\alpha}(y_2)\} = \sup\{(\mu_{x_\alpha} \times \mu_{y_\alpha})(y_1, y_2)/z = y_1+y_2\} = \mu_{(x+y)_\alpha}(z)$. Hence, $(\mu_{x_\alpha}+\mu_{y_\alpha})(z) = \mu_{(x+y)_\alpha}(z)$. Since it is true, for all $z \in X$, $\mu_{x_\alpha} + \mu_{y_\alpha} = \mu_{(x+y)_\alpha}$.

In other words, $x_\alpha + y_\alpha = (x + y)_\alpha$. Analogously, we can prove, $x_\alpha - y_\alpha = (x - y)_\alpha$. □

Theorem 4.2. *In a fuzzy left topological β-algebra $(X, \mathscr{T})$, for any fuzzy sets, S_1, S_2, T_1, T_2 with $S_1 \leq S_2$ and $T_1 \leq T_2$, the following results hold.*

1. $S_1 + T_1 \leq S_2 + T_2$;
2. $S_1 - T_1 \leq S_2 - T_2$.

Proof. For all $(x, y) \in X \times X$, $(S_1 + T_1)(x, y) = \sup\{T_1(y)/S_1(x) > 0\} \leq \sup\{T_2(y)/0 < S_1(x) \leq S_2(x)\} = \sup\{T_2(y)/S_2(x) > 0\}$
$= (S_2 + T_2)(x, y)$. Hence, in general, $S_1 + T_1 \leq S_2 + T_2$. Analogously, we can prove $S_1 - T_1 \leq S_2 - T_2$. □

Corollary 4.1. *In a fuzzy left topological β-algebra $(X, \mathscr{T})$, for any fuzzy sets, S_1, S_2, T_1, T_2 with $S_1 \leq S_2$ and $T_1 \leq T_2$ and, for any $\alpha \in (0, 1]$,*

1. $x_\alpha + S_1 \leq x_\alpha + S_2$;
2. $S_1 + x_\alpha \leq S_2 + x_\alpha$;
3. $x_\alpha - S_1 \leq x_\alpha - S_2$;
4. $S_1 - x_\alpha \leq S_2 - x_\alpha$.

Proof. By taking $S_1 = S_2 = x_\alpha$ and $T_1 = S_1,\ T_2 = S_2$ in 4.2(1) and 4.2(2), respectively we obtain the results (1) and (3). Similarly, by taking, $T_1 = T_2 = x_\alpha$ in 4.2(1) and 4.2(2) respectively, we obtain the results (2) and (4). □

Theorem 4.3. *Let $(X, \mathscr{T})$ be a fuzzy left topological β-algebra then for each $a \in X$, $f_a : X \to X$, given by $f_a(x) = a + x$ is a fuzzy homeomorphism.*

Proof. Let $(X, \mathscr{T})$ be a fuzzy topological space on the β-algebra X and define a map $f : X \to X$ by $f_a(x) = a + x$ for every $a \in X$.

Let W be an open set such that, $(x+a)_\alpha \subseteq W$. Since, X is a fuzzy topological space, we have $(x + a)_\alpha = x_\alpha + a_\alpha$. Thus there exists open sets U and V of x_α and a_α respectively such that,

$$W(z) = (U + V)(z) = \sup_{z=x_\alpha+a_\alpha} \{\min\{U(x_\alpha), V(a_\alpha)\}\}.$$

By our choice $0 < \alpha \leq U(a)$, from we observe that $W(z) = V(y + a)$. Now,

$$f_a(V(z)) = \sup\{V(t)/f_a(t) = z\} = \sup\{V(t)/t + a = z\} = V(z).$$

Thus, $(U+V)(z) \geq f_a(V)(z)$, for all $z \in X$. Hence, $f_a(V) \subseteq U+V \subseteq W$ $\Rightarrow V \subseteq f_a^{-1}(W)$. Hence, f_a is fuzzy continuous, since this is true for any element of X, we conclude that f_a is fuzzy continuous for all $a \in X$.

Now, define $g_a : X \to X$ by $g_a(x) = x - a$. Then, it easily follows that

$$f_a(g_a(x)) = x = 1(x) \text{ and } g_a(f_a(x)) = x = 1(x), \quad \forall\, x \in X.$$

That is, $f_a^{-1} = g_a \Rightarrow g_a$ is bijective. Further $f_a^{-1} = f_{-a}$. Thus $g_a = f_{-a}$. Hence g_a is also fuzzy continuous, forcing us to conclude that f_a is a fuzzy homeomorphism. $\square$

Since f_{-a} is a fuzzy homeomorphism, for each $a \in X$,

$$f_{-a}^{-1}(V(x)) = V(f_{-a}(x)) = V(-a+x) = a_\alpha + V(x),$$

for all $x \in X$. Hence V is fuzzy open if and only if $x_\alpha + V$ is fuzzy open for any fuzzy set V in a left fuzzy topological β-algebra $(X, \mathscr{T})$, for some α, $0 < \alpha \leq 1$, and $x \in X$. V is fuzzy closed if and only if V' is fuzzy open. Thus we obtain the following theorem.

Theorem 4.4. *Let $(X, \mathscr{T})$ be a left fuzzy topological β-algebra. Let V be a fuzzy subset of X for each α with $0 < \alpha \leq 1$ and $x \in X$, the following results hold.*

1. *V is fuzzy open if and only if $x_\alpha + V$ is fuzzy open.*
2. *V is fuzzy closed if and only if $x_\alpha - V$ is fuzzy closed.*

5. Conclusion

In this paper, we have introduced the notions of fuzzy topological spaces on a β-algebra and the fuzzy(left) topological β-algebras. One can extend this work to study different topologies on this structure, fuzzy compactness and fuzzy connectedness on fuzzy left topological β-algebra. Further one can identify some real life applications in signal processing, pattern classification and networks problem. In our future work we will introduce the concept of supra topology on the space of β-algebras that lead to the study of supra topological β-algebras, the concepts of compactness and separation axioms on supra topological spaces [6].

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