Applications of Fibonacci matrix to characterize some Banach spaces through weakly unconditional Cauchy series

Kuldip Raj*
Department of Mathematics
Shri Mata Vaishno Devi University
Katra-182320, J & K (India)
kuldipraj68@gmail.com

Anu Choudhary
Department of Mathematics
Shri Mata Vaishno Devi University
Katra-182320, J & K (India)
anuchoudhary407@gmail.com

M. Aiyub
Department of Mathematics
College of Sciences
University of Bahrain
Manam, Kingdom of Bahrain
maiyub2002@@gmail.com

Sonali Sharma
Department of Mathematics
Shri Mata Vaishno Devi University
Katra-182320, J & K (India)
sonalisharma8082@gmail.com

Abstract. In this article, we introduce new sequence spaces $U(\mathcal{T}(x), \mathcal{F}, p, u), U(\mathcal{T}_w(x), \mathcal{F}, p, u)$ and $U(\mathcal{T}_{w^*}(h), \mathcal{F}, p, u)$ by using Fibonacci matrix and sequence of modulus functions. We make an effort to study some characterizations of weakly unconditionally Cauchy series associated to newly formed sequence spaces. Finally, the barreledness of a normed space Y is characterized by means of *weakly** unconditionally Cauchy series in Y^*.

Keywords: modulus function, weakly unconditionally Cauchy series, Fibonacci sequence spaces, barreledness.

1. Introduction

Fibonacci number satisfy the recurrence relation $t_k = t_{k-1} + t_{k-2};\ k \geq 2$ with initial condition $t_0 = 1$ and $t_1 = 1$, where t_k is the kth term of the sequence. As a result the sequence $1, 2, 3, 5, 8, 13, 21, 34, 55, \cdots$ is obtained which is called as Fibonacci sequence. Fibonacci numbers has many applications in arts, sciences

*. Corresponding author

and architecture. These numbers has many interesting properties like as division of two successive sequence of Fibonacci numbers converges to the golden ratio. In [8] Kara introduced the Fibonacci band matrix $\mathcal{T} = (t_{kn})$ as

$$t_{kn} = \begin{cases} -\frac{t_{k+1}}{t_k}, & (n = k-1), \\ \frac{t_k}{t_{k+1}}, & (n = k), \\ 0, & (0 \leq n < k-1 \text{ or } n > k), \end{cases}$$

where $k, n \in \mathbb{N}$. He studied some results related to Fibonacci sequence spaces $\ell_p(\hat{F})$ for $1 \leq p < \infty$ and $\ell_\infty(\hat{F})$. By ω, c, c_0 and ℓ_∞ we denote the space of all real valued sequences, convergent, null and bounded sequences respectively. In [2] the sequence spaces $c_0(\hat{F})$ and $c(\hat{F})$ were introduced and studied by Başarir et al.. Boussayoud et al. [4] have introduced new operator to derive new symmetric properties of Fibonacci numbers. Khan et al.[9] introduced intuitionistic fuzzy I-convergent Fibonacci difference sequence spaces. In [10] Kılınc and Candan generalized Fibonacci difference sequence spaces by using sequence of modulus functions. Recently, Raj et al. [20] introduced the following Fibonacci difference sequence spaces for modulus functions:

$$\ell_\infty(\mathcal{T}, \mathcal{F}, p, u) = \left\{ x = (x_k) \in \omega : \sup_{k \in \mathbb{N}} \left[u_k f_k \left(\left| \frac{t_k}{t_{k+1}} x_k - \frac{t_{k+1}}{t_k} x_{k-1} \right| \right) \right]^{p_k} < \infty \right\}$$

and

$$\ell(\mathcal{T}, \mathcal{F}, p, u) = \left\{ x = (x_k) \in \omega : \sum_k \left[u_k f_k \left(\left| \frac{t_k}{t_{k+1}} x_k - \frac{t_{k+1}}{t_k} x_{k-1} \right| \right) \right]^{p_k} < \infty \right\}.$$

Define a sequence space

$$c_0(\mathcal{T}, \mathcal{F}, p, u) = \left\{ x = (x_k) \in \omega : \lim_{k \to \infty} \left[u_k f_k \left(\left| \frac{t_k}{t_{k+1}} x_k - \frac{t_{k+1}}{t_k} x_{k-1} \right| \right) \right]^{p_k} = 0 \right\}.$$

Let $wuCs(Y), wcs(Y), ucs(Y), \ell_1(Y)$ and $cs(Y)$ be the Y-valued sequence spaces of weakly unconditionally Cauchy series, weakly convergent series, unconditionally convergent series, absolutely convergent series and convergent series respectively.

Several authors have given different attribution of $wuCs$ ([1],[3],[6],[14]) as follows:

(i) A sequence $x = (x_k) \in wuCs(Y)$ iff $(s_k x_k) \in cs(Y)$ $\forall$ $s = (s_k) \in c_0$ and $x = (x_k) \in ucs(Y)$ iff $(s_k x_k) \in cs(Y)$ $\forall$ $s = (s_k) \in \ell_\infty$.

(ii) A sequence $x = (x_k) \in wuCs(Y)$ iff $B : c_0 \to Y$ is a bounded operator given by $B(s) = \sum_k s_k x_k$ with $Ze_m = x_m$, where e^m is the sequence with $e^m_m = 1$ and $e^m_k = 0$ for $k \neq m$ $(m \in \mathbb{N})$.

(iii) In a normed space Y, the series $\sum_k x_k$ is weakly unconditionally Cauchy series if and only if the set

$$G = \left\{ \sum_{k=1}^{m} s_k x_k : |s_k| \leq 1,\ k = 1, 2, \cdots, m; m \in \mathbb{N} \right\} \tag{1}$$

is bounded.
(iv) In a real Banach space Y a series $\sum_k x_k$ is called wuCs if $\sum_k |f(x_k)| < \infty$ for every $f \in Y^*$ and is called ucs if for every permutation ζ of $\mathbb{N}$ the series $\sum_k x_{\zeta(k)}$ is convergent.

A sequence space Y with a linear topology is called a K-space, provided each of the maps $q_n : Y \to \mathbb{R}$ defined by $q_n(x) = x_n$ is continuous for all $n \in \mathbb{N}$. A K-space Y is called an FK-space, if Y is a complete linear metric space. An FK-space Y has AD if the set c_{00} of all finitely non-zero sequences defined as $c_{00} = span\{e^m : m \in \mathbb{N}\}$ is dense in Y.

The idea of modulus was structured by Nakano [19]. A function $f : [0, \infty) \to [0, \infty)$ is said to be a modulus function if it satisfies the following conditions:-

1. $f(x) = 0$ if and only if $x = 0$,
2. $f(x + y) \leq f(x) + f(y)$, for all $x, y \geq 0$,
3. f is increasing,
4. f is continuous from the right at 0.

Ruckle [22] used the idea of modulus function to construct the sequence space

$$Y(f) = \{x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty\}.$$

Recently Mohiuddine et al.[16] introduced difference sequence spaces of Lucas band matrix and modulus function.

Let Y and Z be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, for $n, k \in \mathrm{N}$, where $\mathrm{N} = \mathbb{N} \cup \{0\}$. Then, we say that A defines a matrix transformation from Y into Z and we denote it by writing $A : Y \to Z$ if for every sequence $x = (x_k) \in Y$, the sequence $Ax = \{A_n(x)\}$ and the A-transform of x is in Z, where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k \quad (n \in \mathrm{N}).$$

The matrix domain Y_A of an infinite matrix A in a sequence space Y is defined by

$$Y_A = \{x = (x_k) \in \omega : Ax \in Y\}.$$

For a triangle infinite matrix many mathematicians have defined sequence spaces with the help of matrix domain see ([8], [11], [17], [18]). To know more about sequence spaces see ([5], [7], [12], [13], [15], [21], [23]) and references therein.
Let $\mathcal{T} = (t_{kn})$ denotes the Fibonacci band matrix, $\mathcal{T}(s_k)$ denotes a matrix transformation of a sequence (s_k), $\mathcal{F} = (f_k)$ be a sequence of modulus functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of positive real numbers. In normed spaces Y and Y^*(the dual spaces

of Y) consider two sequences $x=(x_k)$ and $h=(h_k)$ respectively. In this paper, we define the following sequence spaces:

$$U(\mathcal{T}(x),\mathcal{F},p,u)=\Big\{s=(s_k)\in\ell_\infty(\mathcal{T},\mathcal{F},p,u):\sum_k[u_kf_k(|\mathcal{T}(s_k)|)]^{p_k}x_k \text{ exists}\Big\},$$

$$U(\mathcal{T}_w(x),\mathcal{F},p,u)=\Big\{s=(s_k)\in\ell_\infty(\mathcal{T},\mathcal{F},p,u):w-\sum_k[u_kf_k(|\mathcal{T}(s_k)|)]^{p_k}x_k \text{ exists}\Big\}$$

and

$$U(\mathcal{T}_{w^*}(h),\mathcal{F},p,u)$$
$$=\Big\{s=(s_k)\in\ell_\infty(\mathcal{T},\mathcal{F},p,u):w^*-\sum_k[u_kf_k(|\mathcal{T}(s_k)|)]^{p_k}h_k \text{ exists}\Big\}.$$

These spaces are endowed with sup norm called the space of convergence, *weak* convergence and *weak** convergence of the series $\sum_k x_k$ respectively. Also, $w-\sum_k[u_kf_k(|\mathcal{T}(s_k)|)]^{p_k}x_k$ denotes the limit in weak topology and

$$w^*-\sum_k[u_kf_k(|\mathcal{T}(s_k)|)]^{p_k}h_k$$

denotes the limit in weak* topology.

2. Main results

Proposition 2.1. *If $U(\mathcal{T}(x),\mathcal{F},p,u)$ is complete, then $c_0(\mathcal{T},\mathcal{F},p,u)\subseteq U(\mathcal{T}(x),\mathcal{F},p,u)$.*

Proof. In order to show this let us consider $c_0(\mathcal{T},\mathcal{F},p,u)\nsubseteq U(\mathcal{T}(x),\mathcal{F},p,u)$. Then, there exist a sequence $s^0=(s_k^0)\in c_0(\mathcal{T},\mathcal{F},p,u)$ such that

$$\sum_k[u_kf_k(|\mathcal{T}(s_k^0)|)]^{p_k}x_k$$

is not convergent. Since $c_0(\mathcal{T},\mathcal{F},p,u)$ is a $AD-$space, there exist a Cauchy sequence $s=(s_k^n)\in c_{00}$ (also, in $U(\mathcal{T}(x),\mathcal{F},p,u)$) such that

$$\lim_{n\to\infty}s_k^n=s_k^0.$$

Then, $U(\mathcal{T}(x),\mathcal{F},p,u)$ is not complete. This implies that $c_0(\mathcal{T},\mathcal{F},p,u)\subseteq U(\mathcal{T}(x),\mathcal{F},p,u)$ only if $U(\mathcal{T}(x),\mathcal{F},p,u)$ is complete. □

Theorem 2.1. *Let $\mathcal{F}=(f_k)$ be a sequence of modulus functions, $u=(u_k)$ be a sequence of positive real numbers, $p=(p_k)$ be a bounded sequence of positive real numbers. Suppose Y is a Banach space with $x=(x_k)\in Y$. Then, $x\in wuCs(Y)$ iff $U(\mathcal{T}(x),\mathcal{F},p,u)$ is a Banach space.*

Proof. The set G defined in (1) is bounded, as $x \in wuCs$. So, there exist $N > 0$ such that $\|G\| \leq N$. Let (s^n) be a Cauchy sequence in $U(\mathcal{T}(x), \mathcal{F}, p, u)$. Since $\ell_\infty(\mathcal{T}, \mathcal{F}, p, u)$ is a Banach space, there exist $s = (s_k^0) \in \ell_\infty(\mathcal{T}, \mathcal{F}, p, u)$ such that $\lim_n s^n = s^0 \in \ell_\infty(\mathcal{T}, \mathcal{F}, p, u)$. We need to show that $s^0 \in U(\mathcal{T}(x), \mathcal{F}, p, u)$. For $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and $k \in \mathbb{N}$

$$\big|[u_k f_k\big(|\mathcal{T}(s_k^n)|\big)]^{p_k} - [u_k f_k\big(|\mathcal{T}(s_k^0)|\big)]^{p_k}\big| < \frac{\epsilon}{3N}.$$

This implies

$$\frac{3N}{\epsilon}\left|[u_k f_k\big(|\mathcal{T}(s^n)|\big)]^{p_k} - [u_k f_k\big(|\mathcal{T}(s^0)|\big)]^{p_k}\right| < 1.$$

Hence,

$$\frac{3N}{\epsilon}\sum_{k=1}^{m}\big([u_k f_k\big(|\mathcal{T}(s_k^n)|\big)]^{p_k} - [u_k f_k\big(|\mathcal{T}(s_k^0)|\big)]^{p_k}\big)x_j \in G.$$

Now, for $n > n_0$,

$$\left\|\sum_{k=1}^{m}\Big([u_k f_k\big(|\mathcal{T}(s_k^n)|\big)]^{p_k} - [u_k f_k\big(|\mathcal{T}(s_k^0)|\big)]^{p_k}\Big)x_k\right\| < \frac{\epsilon}{3}.$$

Since (s^n) is a Cauchy sequence in $U(\mathcal{T}(x), \mathcal{F}, p, u)$, there exists a sequence $(y_n) \subset Y$ such that for $m \geq m_0$

$$\left\|\sum_{k=1}^{m}[u_k f_k\big(|\mathcal{T}(s_k^n)|\big)]^{p_k}x_k - y_n\right\| < \frac{\epsilon}{3}.$$

Therefore, for $r > s > n_0$ and $m \in \mathbb{N}$, we have $\|y_r - y_s\| < \epsilon$. Hence (y_n) is a Cauchy sequence in Y. Then, for $\epsilon > 0$, there exists $y_0 \in Y$ such that for $n > n_1$

$$\|y_n - y_0\| < \frac{\epsilon}{3}.$$

Let us suppose $n_2 = \max\{n_0, n_1\}$. By using the above inequalities, we have

$$\begin{aligned}
&\left\|\sum_{k=1}^{m}[u_k f_k\big(|\mathcal{T}(s_k^0)|\big)]^{p_k}x_k - y_0\right\| \\
&\leq \left\|\sum_{k=1}^{m}\big([u_k f_k\big(|\mathcal{T}(s_k^0)|\big)]^{p_k} - [u_k f_k\big(|\mathcal{T}(s_k^n)|\big)]^{p_k}\big)x_k\right\| \\
&+ \left\|\sum_{k=1}^{m}[u_k f_k\big(|\mathcal{T}(s_k^n|\big)]^{p_k}x_k - y_n\right\| + \|y_n - y_0\| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

This implies $s^0 \in U(\mathcal{T}(x), \mathcal{F}, p, u)$. Now, we show that if $U(\mathcal{T}(x), \mathcal{F}, p, u)$ is a Banach space, then x must belongs to weakly unconditionally Cauchy series. Consider a sequence x that does not belongs to wuCs(Y). Then $\exists$ a $h \in Y^*$ such that $\sum_k |h(x_k)| = \infty$. Now, let us construct a sequence $s = (s_k) \in c_0(\mathcal{T}, \mathcal{F}, p, u) \setminus U(\mathcal{T}(x), \mathcal{F}, p, u)$. Take $n_1 \in \mathbb{N}$ such that $\sum_{k=1}^{n_1} |h(x_k)| > 2.2$. Define

$$s_k = \begin{cases} \dfrac{1}{2}\displaystyle\sum_{j=0}^{k}\left[u_k f_k\left(\frac{t_{k+1}^2}{t_j t_{j+1}}\right)\right]^{p_k}, & \text{if } h(x_k) \geq 0, \\ -\dfrac{1}{2}\displaystyle\sum_{j=0}^{k}\left[u_k f_k\left(\frac{t_{k+1}^2}{t_j t_{j+1}}\right)\right]^{p_k}, & \text{if } h(x_k) < 0, \end{cases}$$

for $k \in \{1, 2, \cdots, n_1\}$. Similarly, we can choose $n_2 > n_1$ such that $\sum_{k=n_1+1}^{n_2} |h(x_k)| > 3.3$. Define

$$s_k = \begin{cases} \dfrac{1}{3}\displaystyle\sum_{j=0}^{k}\left[u_k f_k\left(\frac{t_{k+1}^2}{t_j t_{j+1}}\right)\right]^{p_k}, & \text{if } h(x_k) \geq 0, \\ -\dfrac{1}{3}\displaystyle\sum_{j=0}^{k}\left[u_k f_k\left(\frac{t_{k+1}^2}{t_j t_{j+1}}\right)\right]^{p_k}, & \text{if } h(x_k) < 0, \end{cases}$$

for $k \in \{n_1 + 1, 2, \cdots, n_2\}$. Proceeding in this way, we obtain an increasing sequence $(n_k) \in \mathbb{N}$ and the sequence $s = (s_k) \in c_0(\mathcal{T}, \mathcal{F}, p, u)$ such that

$$\sum_{k=1}^{\infty} [u_k f_k(|\mathcal{T}(s_k)|)]^{p_k} h(x_k) = \infty.$$

Then, $s \notin U(\mathcal{T}(x), \mathcal{F}, p, u)$ and hence $c_0(\mathcal{T}, \mathcal{F}, p, u) \nsubseteq U(\mathcal{T}(x), \mathcal{F}, p, u)$. From Proposition 2.1, we can conclude that $U(\mathcal{T}(x), \mathcal{F}, p, u)$ is not a Banach space. $\square$

Remark 2.2. If Y is not a Banach space then $\exists$ a sequence $x = (x_k) \in \ell_1(Y) \setminus cs(Y)$ such that $\forall\, k \in \mathbb{N}$ and $x^{**} \in Y^{**} \setminus Y$. $\|x_k\| < \frac{1}{k2^k}$ and $\sum_k x_k = x^{**}$. Define the sequence $v = (v_k)$ by

$$v_k = \begin{cases} kx_k, & \text{if } k \text{ is odd}, \\ -kx_k, & \text{if } k \text{ is even}. \end{cases}$$

This implies $v = (v_k) \in wuCs(Y)$. Now, let us consider the sequence $s = (s_k) \in c_0(\mathcal{T}, \mathcal{F}, p, u)$ such that

$$s_k = \begin{cases} \dfrac{1}{2}\displaystyle\sum_{j=0}^{k}\frac{1}{k}\left[u_k f_k\left(\frac{t_{k+1}^2}{t_j t_{j+1}}\right)\right]^{p_k}, & \text{if } k \text{ is odd}, \\ -\dfrac{1}{2}\displaystyle\sum_{j=0}^{k}\frac{1}{k}\left[u_k f_k\left(\frac{t_{k+1}^2}{t_j t_{j+1}}\right)\right]^{p_k}, & \text{if } k \text{ is even}. \end{cases}$$

Then, $\sum_k [u_k f_k(|\mathcal{T}(s_k)|)]^{p_k} v_k = \frac{1}{2}x^{**} \in Y^{**} \setminus Y$. Thus, $s \notin U(\mathcal{T}(v), \mathcal{F}, p, u)$ and this implies $c_0(\mathcal{T}, \mathcal{F}, p, u) \nsubseteq U(\mathcal{T}(v), \mathcal{F}, p, u)$. Hence, $U(\mathcal{T}(v), \mathcal{F}, p, u)$ is not complete.

Theorem 2.3. *If Y be a normed space and $x = (x_k) \in Y$. Then, a linear operator $B : U(\mathcal{T}(x), \mathcal{F}, p, u) \to Y$, defined as $s \to B(s) = \sum_k [u_k f_k(|\mathcal{T}(s_k)|)]^{p_k} x_k$ is continuous iff $x = (x_k) \in wuCs(Y)$.*

Proof. Firstly, we show that if B is continuous, then $x = (x_k) \in wuCs$. Since B is continuous, there exists $L > 0$ such that $\|B(s_k)\| \le L\|s_k\|$ for $s = (s_k) \in U(\mathcal{T}(x), \mathcal{F}, p, u)$. Let $d = (d_k) \in B_{c_{00}}$. Then, there exist a sequence $s = (s_k) \in c_{00}(\mathcal{T}, \mathcal{F}, p, u)$ such that $[u_k f_k(|\mathcal{T}(s_k)|)]^{p_k} = d_k$ for every $k \in \mathbb{N}$. Since $c_{00} \subseteq U(\mathcal{T}(x), \mathcal{F}, p, u)$, so we have

$$\begin{aligned} \left\| \sum_{k=1}^{m} d_k x_k \right\| &= \left\| \sum_{k=1}^{m} [u_k f_k(|\mathcal{T}(s_k)|)]^{p_k} x_k \right\| \\ &\le L\|(s_k)\|. \end{aligned}$$

Hence, the set G defined in (1) is bounded and $x = (x_k) \in wuCs(Y)$.

Conversely, suppose $x \in wuCs(Y)$. As G is bounded, $\exists\ L > 0$ such that $\|g\| < L$ for every $g \in G$. Now, for $m \in \mathbb{N}$ and $(s_k) \in U(\mathcal{T}(x), \mathcal{F}, p, u)$, we have

$$\left\| \sum_{k=1}^{m} \frac{[u_k f_k(|\mathcal{T}(s_k)|)]^{p_k}}{\|[u_k f_k(|\mathcal{T}(s_k)|)]^{p_k}\|} x_k \right\| \le L.$$

Hence, B is continuous. □

Theorem 2.4. *If Y is a Banach space, then $x \in wuCs(Y)$, iff $U(\mathcal{T}_w(x), \mathcal{F}, p, u)$ is a Banach space.*

Proof. As x is in weakly unconditionally Cauchy series. Suppose $\|g\| \le N\ \forall\ g \in G$ and (s^n) be a Cauchy sequence in $U(\mathcal{T}_w(x), \mathcal{F}, p, u)$ such that $(s^n) \to (s^0) \in \ell_\infty(\mathcal{T}, \mathcal{F}, p, u)$ as $n \to \infty$. Let $\epsilon > 0$ and $n_0 \in \mathbb{N}$ such that $\forall\ n \ge n_0$

$$\left|[u_k f_k(|\mathcal{T}(s_k^n)|)]^{p_k} - [u_k f_k(|\mathcal{T}(s_k^0)|)]^{p_k}\right| < \frac{\epsilon}{3N}.$$

Because $\frac{3N}{\epsilon}\left|[u_k f_k(|\mathcal{T}(s^n)|)]^{p_k} - [u_k f_k(|\mathcal{T}(s^0)|)]^{p_k}\right| < 1$ and

$$\frac{3N}{\epsilon} \sum_{k=1}^{m} \left([u_k f_k(|\mathcal{T}(s_k^n)|)]^{p_k} - [u_k f_k(|\mathcal{T}(s_k^0)|)]^{p_k}\right) x_j \in G.$$

So, for every $n > n_0$

$$\left\| \sum_{k=1}^{m} \left([u_k f_k(|\mathcal{T}(s_k^n)|)]^{p_k} - [u_k f_k(|\mathcal{T}(s_k^0)|)]^{p_k}\right) x_k \right\| < \frac{\epsilon}{3}.$$

Also (s^n) is a Cauchy sequence in $U(\mathcal{T}_w(x), \mathcal{F}, p, u)$, $\exists$ a sequence $(z_n) \subset Y$ such that for $m \geq m_0$ and $\forall\ f \in Y^*$

$$\Big|\sum_{k=1}^{m}[u_k f_k\big(|\mathcal{T}(s_k^n)|\big)]^{p_k} f(x_k) - f(z_n)\Big| < \frac{\epsilon}{3}.$$

By using Hahn- Banach theorem, we have $f \in Y^*$ such that

$$\|z_r - z_s\| = |f(z_r - z_s)|.$$

Hence, we have

$$\|z_r - z_s\| < \epsilon,$$

for $r > s > n_0$ and $m \in \mathbb{N}$. Therefore (z_n) is a Cauchy sequence in Y. Consider that $z_0 \in Y$, such that for $n > n_1$

$$\|z_n - z_0\| < \frac{\epsilon}{3}.$$

Take $n_2 = \max\{n_0, n_1\}$, then we have $|\sum_{k=1}^{m}[u_k f_k\big(|\mathcal{T}(s_k^0)|\big)]^{p_k} f(x_k) - f(z_0)|$

$$\begin{aligned}
&\leq \Big|\sum_{k=1}^{m}\big([u_k f_k\big(|\mathcal{T}(s_k^0)|\big)]^{p_k} - [u_k f_k\big(|\mathcal{T}(s_k^n)|\big)]^{p_k}\big) f(x_k)\Big| \\
&+ \Big|\sum_{k=1}^{m}[u_k f_k\big(|\mathcal{T}(s_k^n)|\big)]^{p_k} f(x_k) - f(z_n)\Big| \\
&+ |f(z_n) - f(z_0)| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon.
\end{aligned}$$

Thus, $(s^0) \in U(\mathcal{T}_w(x), \mathcal{F}, p, u)$. Now, we need to show that if $U(\mathcal{T}_w(x), \mathcal{F}, p, u)$ is a Banach space, then $x \in wuCs(Y)$. Consider that $h \in Y^*$ such that $\sum_k |h(x_k)| = \infty$.

We can establish $s = (s_k) \in c_0(\mathcal{T}, \mathcal{F}, p, u)$ such that

$$\sum_{k=1}^{\infty}[u_k f_k\big(|\mathcal{T}(s_k)|\big)]^{p_k} h(x_k) = \infty.$$

Hence, $s = (s_k) \notin U(\mathcal{T}_w(x), \mathcal{F}, p, u)$. Thus, $U(\mathcal{T}_w(x), \mathcal{F}, p, u)$ is not complete. □

Theorem 2.5. *Consider $\mathcal{F} = (f_k)$ be a sequence of modulus functions and Y be a normed space with $x=(x_k) \in Y$. Then, a linear operator $B : U(\mathcal{T}_w(x), \mathcal{F}, p, u) \to Y$, defined as $s \to B(s) = w-\sum_k [u_k f_k\big(|\mathcal{T}(s_k)|\big)]^{p_k} x_k$ is continuous iff $x = (x_k) \in wuCs(Y)$.*

Proof. From Theorem 2.3, we get the result as desired. □

Theorem 2.6. *Consider a sequence $\mathcal{F} = (f_k)$ of modulus functions and a normed space Y with $h = (h_j) \in Y^*$. Then, we have:*

(*i*) $h \in wuCs(Y^*)$,

(*ii*) $U(\mathcal{T}_{w^*}(h), \mathcal{F}, p, u) = \ell_\infty(\mathcal{T}, \mathcal{F}, p, u)$ *and*

(*iii*) $h \in w^*ucs(Y^*)$

are equivalent iff Y is a barreled normed space.

Proof. Firstly, we prove $(i) \to (ii)$.

As we know $U(\mathcal{T}_{w^*}(h), \mathcal{F}, p, u) \subset \ell_\infty(\mathcal{T}, \mathcal{F}, p, u)$, now we only need to show that $\ell_\infty(\mathcal{T}, \mathcal{F}, p, u) \subset U(\mathcal{T}_{w^*}(h), \mathcal{F}, p, u)$.

Let $s = (s_k)$ be a sequence in $\ell_\infty(\mathcal{T}, \mathcal{F}, p, u)$, then $([u_k f_k(|\mathcal{T}(s_k)|)]^{p_k} h_k) \in wuCs(Y^*)$. Hence, $\sum_{k=1}^{\infty}[u_k f_k(|\mathcal{T}(s_k)|\ )]^{p_k} h_k$ is $weak^*$ convergent in Y^*. So, $s = (s_k) \in U(\mathcal{T}_{w^*}(h), \mathcal{F}, p, u)$.

Since $(ii) \to (iii)$ is clear. Now, to show $(iii) \to (i)$. For this define G' by

$$G' = \left\{ \sum_{k=1}^{m} s_k h_k : |s_k| \le 1,\ k = 1, 2, \cdots, m; m \in \mathbb{N} \right\}.$$

One can see that G' is pointwise bounded and we know that Y is barreled. Thus, $(h_k) \in wuCs(Y^*)$. Conversely, we need to show that Y is a barreled space if $(iii) \to (i)$. Consider that Y is not barreled space. Then $\exists$ a $weak^*$ bounded set $M \subseteq Y^*$ which is not bounded. Suppose $(h_k) \in M$ such that $\|h_k\| > 2^k.2^k$ for $k \in \mathbb{N}$. For $z_k = \frac{1}{2^k} h_k$ and $k \in \mathbb{N}, (z_k(x)) \in \ell_1\ \forall\ x \in Y$. The series $\sum_{k=1}^{\infty} \frac{1}{2^k} z_k$ does not converge because $\|z_k\| > 2^k$ for every $k \in \mathbb{N}$. Thus, we can conclude that $(z_k) \notin wuCs(Y^*)$. □

3. Conclusion

Not only in the field of Mathematics, but in other disciplines also the term "Modulus Function" attained many applications. In our day to day life, modulus function is used every now and then like if one travel anywhere the cost is positive. By geophysict to look at the total amount of the energy used, by scuba divers to discuss their location in regards to a sea level. Applied problems such as ranges of possible values can also be solved using the absolute value function. Similarly Fibonacci sequence serves many applications in real life, for example the number of petals in a flower consistently follows the Fibonacci sequences. Fibonacci numbers include computer algorithms such as the Fibonacci search technique and the Fibonacci heap data structure and graphs called Fibonacci cubes used for interconnecting parallel and distributed systems. In this paper we introduce some sequence spaces associated to sequence of modulus functions and Fibonacci sequence. We shall give certain characterizations of weakly unconditionally Cauchy series by means of completeness of these sequence spaces and through continuity of operator $B : U(\mathcal{T}(x), \mathcal{F}, p, u)(U(\mathcal{T}_w(x), \mathcal{F}, p, u)) \to Y$. Also, we characterize the barreledness of a normed space Y through the behavior of its *weakly** unconditionally Cauchy series in Y^*.

References

[1] F. Albiac, N.J. Kalton, *Topics in Banach spaces theory*, Springer, New York, 2006.

[2] M. Başarir, F. Başar, E. E. Kara, *On the spaces of Fibonacci difference null and convergent sequences*, arXiv:1309.0150v1 [math.FA], (2013).

[3] C. Bessaga, A. Pelczynski, *On bases and unconditional convergence of series in Banach spaces*, Stud. Math., 17 (1958), 151-164.

[4] A. Boussayoud, M. Kerada, S. Araci, M. Acikgoz and A. Esi, *Symmetric functions of binary products of Fibonacci and orthogonal polynomials*, Filomat, 33 (2019), 1495-1504.

[5] A. Choudhary, K. Raj, M. Mursaleen, *Compact operators on spaces of binomial fractional difference sequences*, Math. Sci., (2021), https://doi.org/10.1007/s40096-021-00396-3.

[6] J. Diestel, *Sequences and series in Banach spaces*, Springer, New York, 1984.

[7] B. Hazarika, A. Esi, *Lacunary ideal quasi Cauchy Sequences*, An. Univ. Craiova Ser. Mat. Inform., 45 (2018), 220-231.

[8] E. E. Kara, *On some topological and geometric properties of new Banach sequence spaces*, J. Inequal. Appl., 38 (2013).

[9] V. A. Khan, E. E. Kara, H. Altaf, N. Khan, M. Ahmad, *Intutionistic fuzzy I-convergent Fibonacci difference sequence spaces*, J. Inequal. Appl., 202 (2019).

[10] G. Kılınc, M. Candan, *Some generalized Fibonacci difference spaces defined by a sequence of modulus functions*, Facta Univ. Ser. Math. Inform., 32 (2017), 95-116.

[11] M. Kirişçi, F. Başar, *Some new sequence spaces derived by the domain of generalized difference matrix*, Comput. Math. Appl., 60 (2010), 1299-1309.

[12] P. Kòrus, *On Λ^2-strong convergence of numerical sequences revisited*, Acta Math. Hungar., 148 (2016), 222-227.

[13] P. Kòrus, *On the uniform convergence of double sine integrals with generated monotone coefficients*, Period. Math. Hungar., 63 (2011), 205-214.

[14] C. W. McArthur, *On relationships amongst certain spaces of sequences in an arbitrary Banach space*, Canad. J. Math., 8 (1956), 192-197.

[15] S. A. Mohiuddine, K. Raj, *Vector valued Orlicz-Lorentz sequence spaces and their operator ideals*, J. Nonlinear Sci. Appl., 10 (2017), 338-353.

[16] S. A. Mohiuddine, K. Raj, A. Choudhary, *Difference sequence spaces based on Lucas band matrix and modulus function*, $\tilde{S}$ao Paulo J. Math. Sci., (2021), Doi: https://doi.org/10.1007/s40863-020-00203-2.

[17] M. Mursaleen, A. K. Noman, *On the spaces of $\lambda-$convergent and bounded sequences*, Thai J. Math., 8 (2010), 311-329

[18] M. Mursaleen, A. K. Noman, *On some new sequence spaces of non-absolute type related to the spaces ℓ_p and ℓ_∞ I*, Filomat, 25 (2011), 33-51.

[19] H. Nakano, *Concave modulars*, J. Math. Soc. Japan, 5 (1953), 29-49.

[20] K. Raj, Suruchi Pandoh, Seema Jamwal, *Fibonacci difference sequence spaces for modulus functions*, Le Matematiche, 70 (2015), 137-156.

[21] K. Raj, A. Choudhary, C. Sharma, *Almost strongly Orlicz double sequence spaces of regular matrices and their applications to statistical convergence*, Asian-Eur. J. Math., 11 (2018), 1850073(14 pages).

[22] W. H. Ruckle, *FK spaces in which the sequence of coordinate vectors is bounded*, Canad. J. Math., 25 (1973), 973-978.

[23] M. Shafiq, A. Esi, *On some spaces of Lacunary I-convergent sequences of interval numbers defined by sequence of modulli*, Proyecciones, 36 (2017), 325-346.

Accepted: June 18, 2021