Geometry of warped product pseudo slant submanifolds in a nearly quasi-Sasakian manifold

Shamsur Rahman
Department of Mathematics
Maulana Azad National Urdu University
Polytechnic, Setallite Campus
Darbhanga, Bihar 846002
India
shamsur@rediffmail.com

Abdul Haseeb*
Department of Mathematics
Faculty of Science
Jazan University
Jazan-2097
Kingdom of Saudi Arabia
malikhaseeb80@gmail.com
haseeb@jazanu.edu.sa

Hassan Almusawa
Department of Mathematics
Faculty of Science
Jazan University, Jazan-2097
Kingdom of Saudi Arabia
haalmusawa@jazanu.edu.sa
almusawah@mymail.vcu.edu

Abstract. In this paper we study distributions of pseudo slant submanifolds of a nearly quasi-Sasakian manifold and obtain some results related to integrability of the distributions and totally geodesic foliations determined by the distributions of pseudo-slant submanifold. Moreover, warped product pseudo slant submanifolds of a nearly quasi-Sasakian manifold are also studied and some characterization theorems for the existence of warped product isometrically immersed into nearly quasi-Sasakian manifolds have been proved.

Keywords: quasi-Sasakian manifold, pseudo slant submanifold, warped products.

1. Introduction

The geometry of warped product CR-submanifolds in a Kaehler manifold was studied by B. Y. Chen [8]. Further, within the different geometric aspects he tried to find the warping function in the sort of some partial differential equations. After that many research articles have been appeared on various kinds

*. Corresponding author

of warped product submanifolds of different class of structures [1-4, 9, 11-13, 15-17]. In [11], the author established general sharp inequality for the second fundamental form in terms of the warping function y in warped product submanifolds of a nearly Lorentzian para-Sasakian manifold. In 2012, S. Uddin et al. [14] obtained some characterization theorems on the existence or nonexistence of warped product pseudo-slant submanifolds of a nearly cosymplectic manifold in terms of endomorphisms. In the present work we investigate the properties of nontrivial warped product pseudo slant submanifold of the shape $M_\perp \times_y M_\theta$ which are the natural extension of CR-warped product submanifolds. It is clear that each CR-warped product submanifold is a non-trivial warped product pseudo slant submanifold of the shape $M_\perp \times_y M_\theta$ and $M_\theta \times_y M_\perp$ with slant angle $\theta = 0$. But the warped product pseudo slant submanifold never induces the CR-warped product submanifolds. Throughout the paper we consider M $=M_\perp \times_y M_\theta$, where M_θ and $M_\perp$ are proper slant and anti invariant submanifolds, respectively. Finally, we established some necessary and sufficient conditions involving some geometric conditions and properties of nearly quasi-Sasakian manifolds endowed with the warped product pseudo slant submanifolds.

2. Preliminaries

Let $\tilde{M}$ be an odd-dimensional almost contact manifold which carries a tensor field ϕ, a vector field ξ, called characteristic or Reeb vector field and a $1-$form υ satisfying

$$\phi^2 = -I + \upsilon \otimes \xi, \quad \upsilon(\xi) = 1, \tag{2.1}$$

where $I : T\tilde{M} \to T\tilde{M}$ is the identity map, $\phi\xi = 0$, $\upsilon o \phi = 0$ and the $(1, 1)$-tensor field ϕ has constant rank $n-1$. An almost contact manifold $\tilde{M}(\phi, \xi, \upsilon)$ is called normal if $N_\phi(Y, Z) + 2d\upsilon(Y, Z)\xi = 0$ or $(\bar{\nabla}_{\phi Y}\phi)Z = \phi(\bar{\nabla}_Y \phi)Z - < \bar{\nabla}_Y \xi, Z > \xi$, where N is the Nijenhuis tensor of ϕ. An almost contact manifold $\tilde{M}(\phi, \xi, \upsilon)$ admits a Riemannian metric $<,>$ such that

$$< \phi Y, \phi Z > = < Y, Z > -\upsilon(Y)\upsilon(Z)\xi, \ \forall\, Y, Z \in \Gamma(T\tilde{M}). \tag{2.2}$$

The induced metric $<,>$ on $\tilde{M}$ with the structure $(\phi, \xi, \upsilon, <,>)$ is said to be an almost contact metric manifold. As an instant significance of (2.2), we have $\upsilon(X) = < X, \xi >$ and $< \phi X, Y > = - < X, \phi Y > = \psi(X, Y)$. Let ξ be a Killing vector field with respect to $<,>$, then the contact metric structure is called a K- contact structure. An almost contact metric manifold $(\tilde{M}, \phi, \xi, \upsilon, <,>)$ is called quasi-Sasakian manifold if

$$(\bar{\nabla}_Y \phi)Z = \upsilon(Z)AY - < AY, Z > \xi, \quad \phi AY = A\phi Y, \tag{2.3}$$

for all $Y, Z \in \Gamma(T\tilde{M})$, where A is the symmetric linear transformation field. From the formula (2.3), it follows that $\tilde{\nabla}_X \xi = \phi AX$. An almost contact metric structure $(\phi, \xi, \upsilon, <,>)$ on $\tilde{M}$ is called a nearly quasi-Sasakian manifold if

$$(\tilde{\nabla}_Y \phi)Z + (\tilde{\nabla}_Z \phi)Y = \upsilon(Z)AY + \upsilon(Y)AZ - 2 < AY, Z > \xi. \tag{2.4}$$

Let M be a Riemannian manifold isometrically immersed into an almost contact metric manifold $\tilde{M}$ and let $<,>$ denotes the Riemannian metric induced on M. Let $\Gamma(TM)$ $(resp., \Gamma(T^{\perp}M))$ be the Lie algebra of the vector fields tangent to M (resp., normal to M) and $\nabla^{\perp}$ be the induced connection on $(T^{\perp}M)$. If ∇ is the Levi-Civita connection of M, then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_Y Z = \nabla_Y Z + h(Y, Z), \tag{2.5}$$

$$\tilde{\nabla}_Y N = -A_N Y + \nabla^{\perp}_Y N, \tag{2.6}$$

for all $Y, Z \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, where h is the second fundamental form and A_N is the shape operator (corresponding to the normal vector field N), respectively and are related by

$$< h(Y, Z), N >=< A_N Y, Z > . \tag{2.7}$$

Now, for any $Y \in \Gamma(TM)$, we have

$$\phi Y = PY + FY, \tag{2.8}$$

where PY and FY are the tangential and normal components of ϕY, respectively. Similarly for any $N \in \Gamma(T^{\perp}M)$, we have

$$\phi N = tN + fN, \tag{2.9}$$

where tN and fN are tangential and normal components of ϕN.

$$< Y, PZ >= - < PY, Z > . \tag{2.10}$$

If we put $Q = P^2$, then we have

$$(\tilde{\nabla}_Y Q)Z = \nabla_Y QZ - Q\nabla_Y Z, \tag{2.11}$$

$$(\tilde{\nabla}_Y P)Z = \nabla_Y PZ - P\nabla_Y Z, \tag{2.12}$$

$$(\tilde{\nabla}_Y F)Z = \nabla^{\perp}_Y FZ - F\nabla_Y Z, \tag{2.13}$$

for any $Y, Z \in \Gamma(TM)$. From (2.5), (2.8) and $\tilde{\nabla}_X \xi = \phi AX$, it follows that

$$\tilde{\nabla}_Y \xi = PAY, \tag{2.14}$$

$$h(Y, \xi) = FAY. \tag{2.15}$$

A submanifold M is said to be totally geodesic if $h(Y, Z) = 0$ and it is totally umbilical if $h(Y, Z) =<, > H$, where H is the mean curvature vector defined by $H = \frac{1}{n} trace(h) = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i)$. Now we defined a class of submanifolds called the slant submanifold.

Definition 2.1. *For each non zero vector Y tangent to M at P, such that Y is not proportional to ξ_p, we denote by $0 \leq \theta(Y) \leq \frac{\pi}{2}$, the angle between ϕY and T_pM is called the Wirtinger angle. If the angle $\theta(Y)$ is constant for all $Y \in T_pM - \{\xi\}$ and $p \in M$, then M is said to be a slant submanifold* [10] *and the angle θ is called slant angle of M. Obviously if $\theta = 0$ ($resp., \theta = \frac{\pi}{2}$), then M is called invariant $(resp., anti - invariant)$ submanifold. A slant submanifold is said to be proper slant if it is neither invariant nor anti-invariant.*

Proposition 2.1. *Let M be a submanifold of an almost contact metric manifold $\tilde{M}$ such that $\xi \in TM$, then M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that* [7]

$$P^2 = \lambda(-I + \upsilon \otimes \xi). \tag{2.16}$$

Furthermore, in such a case, if θ is a slant angle, then it satisfies that $\lambda = \cos^2\theta$.

Thus, for a slant submanifold M of an almost contact metric manifold $\tilde{M}$, the following relations are the consequences of the Proposition 2.1.

$$< PY, PZ >= \cos^2\theta(< Y, Z > -\upsilon(Y)\upsilon(Z)), \tag{2.17}$$

$$< FY, FZ >= \sin^2\theta(< Y, Z > -\upsilon(Y)\upsilon(Z)), \tag{2.18}$$

for any $Y, Z \in \Gamma(TM)$. Also we proceed to give an another characterization which is directly related to the consequence of the Proposition 2.1.

Proposition 2.2. *Let M be a slant submanifold of an almost contact metric manifold $\tilde{M}$ such that $\xi \in TM$, then*

$$(i)\ tFY = \sin^2\theta(-Y + \upsilon(Y)\xi) \qquad and \qquad (ii)\ fFY = -FPY, \tag{2.19}$$

for any $Y \in \Gamma(TM)$.

3. Pseudo slant submanifolds of a nearly quasi-Sasakian manifold

In this section we define pseudo slant submanifolds of an almost contact metric manifold by using slant distribution given in [6]. We find the geometry of leaves of the distributions involved in the definition of pseudo slant submanifolds of a nearly quasi-Sasakian manifold. We also obtain some necessary and sufficient conditions for such sub-immersions to be totally geodesic foliations which are used later in characterization theorem.

Definition 3.1. *A submanifold M of an almost contact metric manifold $\tilde{M}$ is called pseudo slant submanifold, if there exist a pair of orthogonal distributions $D^\perp$ and D^θ such that*

(i) The tangent bundle TM admits the orthogonal direct decomposition $TM = D^\perp \oplus D^\theta \oplus \{\xi\}$, where $\{\xi\}$ is 1-dimensional distribution spanned by ξ,

(ii) *The distribution* $D^\perp$ *is anti-invariant distribution under* ϕ, *i.e.,* $\phi D^\perp \subseteq T^\perp M$,

(iii) *The distribution* D^θ *is slant distribution with slant angle* $\theta \neq 0, \frac{\pi}{2}$.

Let m_1 and m_2 be the dimensions of the distributions $D^\perp$ and D^θ, respectively. If $m_2 = 0$, then M is anti invariant submanifold. If $m_1 = 0$ and $\theta = 0$, then M is invariant submanifold. If $m_1 = 0$ and $\theta \neq 0, \frac{\pi}{2}$, then M is proper slant submanifold, or if $\theta = \frac{\pi}{2}$, then M is anti invariant submanifold and if $\theta = 0$, then M is semi-invariant submanifold. If μ is an invariant subspace of normal bundle $T^\perp M$, then $T^\perp M$ can be decomposed as follows:

$$T^\perp M = \phi D^\perp \oplus F D^\theta \oplus \mu, \tag{3.1}$$

where μ is the even dimensional invariant sub bundle of $T^\perp M$. Now we establish the following theorem:

Theorem 3.1. *Let* M *be a pseudo slant submanifold of a nearly quasi-Sasakian manifold* $\tilde{M}$, *then the distribution* $D^\perp \oplus \xi$ *defines as totally geodesic foliation of* M *if and only if*

$$\begin{aligned} &< A_{\phi W} Z, PX > + < A_{\phi Z} W, PX > \\ &= -\upsilon(Z) < AW, PX > -\upsilon(W) < AZ, PX > . \end{aligned} \tag{3.2}$$

Proof. By using (2.5) and (2.9) in the relation

$$< \phi \tilde{\nabla}_Z W, \phi X > = < \tilde{\nabla}_Z W, X > -\upsilon(\tilde{\nabla}_Z W)\upsilon(X) = < \tilde{\nabla}_Z W, X >,$$

we have

$$< \tilde{\nabla}_Z W, X > = < \tilde{\nabla}_Z \phi W, PX > - < (\tilde{\nabla}_Z \phi) W, PX > - < \tilde{\nabla}_Z W, \phi FX > .$$

Now, using (2.5) and (2.7) in the last equation, we get

$$\begin{aligned} &< \tilde{\nabla}_Z W, X > = - < A_{\phi W} Z, PX > + < (\bar{\nabla}_W \phi) Z, PX > -\upsilon(W) < AZ, PX > \\ &- \upsilon(Z) < AW, PX > +2 < AZ, W > < \xi, PX > - < \bar{\nabla}_Z W, \phi FX > \end{aligned}$$

in which applying covariant derivative of endomorphism ϕ and using proposition 2.1 we easily obtain

$$\begin{aligned} &< \tilde{\nabla}_Z W, X > = - < A_{\phi W} Z, PX > + < (\bar{\nabla}_W \phi) Z, PX > -\upsilon(W) < AZ, PX > \\ &- \upsilon(Z) < AW, PX > - < \bar{\nabla}_Z W, tFX > - < \bar{\nabla}_Z W, nFX > . \end{aligned}$$

This implies

$$\begin{aligned} < \tilde{\nabla}_Z W, X > = &- < A_{\phi W} Z, PX > + < \bar{\nabla}_W \phi Z, PX > - < \phi \bar{\nabla}_W Z, PX > \\ &- \upsilon(W) < AZ, PX > -\upsilon(Z) < AW, PX > + \sin^2\theta < \tilde{\nabla}_Z W, X > \\ &+ < \tilde{\nabla}_Z W, FPX > . \end{aligned}$$

Using (2.6), (2.8), (2.11) and (2.14) we finally arrive at

$$\cos^2\theta < \tilde{\nabla}_Z W, X >= -< A_{\phi W}Z, PX > - < A_{\phi Z}W, PX > - < \bar{\nabla}_W Z, \phi PX >$$
$$- \upsilon(W) < AZ, PX > -\upsilon(Z) < AW, PX > + < \bar{\nabla}_Z W, FPX > .$$

Now, if this is totally geodesic foliation, then we have

$$0 = -< A_{\phi W}Z, PX > - < A_{\phi Z}W, PX > -\upsilon(W) < AZ, PX >$$
$$- \upsilon(Z) < AW, PX >,$$
$$< A_{\phi W}Z, PX > + < A_{\phi Z}W, PX >= -\upsilon(Z) < AW, PX >$$
$$- \upsilon(W) < AZ, PX >$$

the result follows immediately after applying it in the last expression. □

Theorem 3.2. *On a pseudo slant submanifold M of a nearly quasi-Sasakian manifold $\tilde{M}$, the distribution D^θ is integrable if and only if*

$$2\cos^2\theta < \nabla_X Y, Z > =< h(X, PY) + h(PX, Y), \phi Z > - < h(X, Z), FPY >$$
$$- < FPX, h(Y, Z) > -\upsilon(Z) < \bar{\nabla}_X \xi, Y >,$$

for every $Z \in \Gamma(D^\perp \otimes \xi)$ and $X, Y \in \Gamma(D^\theta)$.

Proof. By using (2.2), (2.5), (2.9) and the values $\upsilon(X) = 0, \upsilon(Y) = 0$, we have

$$< [X, Y], Z > =< \bar{\nabla}_X PY, \phi Z > + < \bar{\nabla}_X FY, \phi Z >$$
$$+ < (\bar{\nabla}_Y \phi)X, \phi Z > - < \bar{\nabla}_Y X, Z > -\upsilon(Z) < \bar{\nabla}_X \xi, Y > .$$

In view of (2.6), the last equation takes the form

$$< [X, Y], Z > =< h(X, PY), \phi Z > - < FY, (\tilde{\nabla}_X \phi)Z > + < \phi FY, \tilde{\nabla}_X Z >$$
$$+ < \tilde{\nabla}_Y PX, \phi Z > + < \tilde{\nabla}_Y FX, \phi Z > -2 < \tilde{\nabla}_Y X, Z >$$
$$- \upsilon(Z) < \tilde{\nabla}_X \xi, Y >$$

which immediately gives

$$\begin{aligned} < [X, Y], Z > &=< h(X, PY), \phi Z > + < h(PX, Y), \phi Z > \\ &- < FY, Q_X Z > + < tFY, \tilde{\nabla}_X Z > + < fFY, \bar{\nabla}_X Z > \\ &- < FX, \bar{\nabla}_Y \phi Z > -2 < \bar{\nabla}_Y X, Z > -\upsilon(Z) < \tilde{\nabla}_X \xi, Y > . \end{aligned} \tag{3.3}$$

Now, since $< \phi Y, \mathcal{Q}_X Z >=< Y, \phi\mathcal{Q}_X Z >= 0$, which by virtue of (2.9) turns to $< FY, \mathcal{Q}_X Z >=< \phi Y, \mathcal{Q}_X Z >$.

Considering $(\tilde{\nabla}_X \phi)Y = \mathcal{P}_X Y + \mathcal{Q}_X Y$, where $\mathcal{P}_X Y$ and $\mathcal{Q}_X Y$ are the tangential and normal parts of $(\tilde{\nabla}_X \phi)Y$, also taking account of Proposition 2.2, (2.6), (2.10) and (2.14), (3.3) leads to

$$< [X, Y], Z >=< h(X, PY), \phi Z > + < h(PX, Y), \phi Z > -\sin^2\theta < Y, \tilde{\nabla}_X Z >$$

$$- < FPY, h(X,Z) > + < \phi FX, \tilde{\nabla}_Y Z > -2 < \tilde{\nabla}_Y X, Z > -\upsilon(Z) < \tilde{\nabla}_X \xi, Y > .$$

Also, taking an account that X, Y are orthogonal to Z, the above equation settle to

$$\begin{aligned} < [X,Y], Z > =& < h(X,PY) + h(PX,Y), \phi Z > - \cos^2\theta < [X,Y], Z > \\ &+ \cos^2\theta < [X,Y], Z > - < h(X,Z), FPY > \\ &+ < -\sin^2\theta(X - \upsilon(X)\xi), \tilde{\nabla}_Y Z > \\ &- \sin^2\theta < Y, \tilde{\nabla}_X Z > - < FPX, \tilde{\nabla}_Y Z > -2 < \tilde{\nabla}_Y X, Z > \\ &- \upsilon(Z) < \tilde{\nabla}_X \xi, Y > . \end{aligned}$$

Now, calculating for a while we arrive

$$\begin{aligned} \sin^2\theta < [X,Y], Z > =& < h(X,PY) + h(PX,Y), \phi Z > -2\cos^2\theta < Z, \tilde{\nabla}_X Y > \\ &- < h(X,Z), FPY > - < h(Y,Z), FPX > \\ &- \upsilon(Z) < \tilde{\nabla}_X \xi, Y > . \end{aligned}$$

If D^θ is integrable, then finally we find

$$\begin{aligned} 2\cos^2\theta < \nabla_X Y, Z > =& < h(X,PY) + h(PX,Y), \phi Z > - < h(X,Z), FPY > \\ &- < FPX, h(Y,Z) > -\upsilon(Z) < \bar{\nabla}_X \xi, Y > \end{aligned}$$

which proves our assumption. Converse part of the theorem can be proved easily. □

4. Warped product submanifold of the form $M_\perp \times_y M_\theta$

One of the most important part of Riemannian product is warped product with warping function y, and it was introduced by R.L. Bishop and B. O'Neill [5]. They described these manifolds as: let y be a positive differentiable function which always be defined on leaves and (M^*, g_*) and $(M', g_{\prime})$ are two Riemannian manifolds. Then the warped product of M^* and M' are the Riemannian manifolds $M^* \times_y M' = (M^* \times_y M', g)$, where $g = g_* + y^2 g_{\prime}$. For a warped product, we have

$$\nabla_X Z = \nabla_Z X = X \ln y Z \tag{4.1}$$

for any vector fields X, Z and are tangents to g_* and $g_{\prime}$, respectively, where ∇ denotes the Levi-Civita connection on M [5]. On the other hand, $\nabla \ln y$ is the gradient of $\ln y$ which is defined as $g(\nabla \ln y, X) = X \ln y$. If the warping function y is constant, then a warped product manifold $M^* \times_y M'$ is called simply Riemannian product or trivial warped product manifold. For a warped product manifold $M = M^* \times_y M'$, M^* is said to be totally geodesic and M' is totally umbilical submanifolds of M, respectively. Now, we obtain some basic useful propositions.

Proposition 4.1. *If $M = M_\perp \times_y M_\theta$ is a warped product pseudo slant submanifold of a nearly quasi-Sasakian manifold $\tilde{M}$ such that the structure vector field ξ is tangent to $M_\perp$, then for any $X \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$, we have*

$$2 < h(X,Z), FPX > = \cos^2\theta (Z \ln y) \parallel X \parallel^2 \\ - < h(X,PX), \phi Z > - < h(Z,PX), FX > .$$

Proof. Let $M = M_\perp \times_y M_\theta$ be a warped product pseudo slant submanifold of a nearly quasi-Sasakian manifold $\tilde{M}$. Then from (2.6), we have

$$< \tilde{\nabla}_Z X, FPX > = < h(X,Z), FPX > .$$

Since ξ is tangent to $M_\perp$, then by using (2.9), (2.16) and (4.1), we have

$$< h(X,Z), FPX > = < (\bar{\nabla}_Z \phi)X, PX > - < \bar{\nabla}_Z \phi X, PX > \\ + \cos^2\theta < \bar{\nabla}_Z X, X >$$

which leads to

$$< h(X,Z), FPX > = < (\bar{\nabla}_Z \phi)X, PX > - < \bar{\nabla}_Z PX, PX > \\ - < \nabla_Z FX, PX > + \cos^2\theta (X \ln y) < Z, Z > .$$

Now, using (2.1), (2.5)-(2.7), (2.9), (2.14) and (4.1), we arrive at

$$< h(X,Z), FPX > = - < h(X,PX), \phi Z > + \cos^2\theta (Z \ln y) \parallel X \parallel^2 \\ - < h(X,Z), FPX > + < h(Z,PX), FX >, \\ 2 < h(X,Z), FPX > = \cos^2\theta (Z \ln y) \parallel X \parallel^2 \\ - < h(X,PX), \phi Z > - < h(Z,PX), FX >,$$

which completes the proof. □

Proposition 4.2. *Assume that $M_\perp \times_y M_\theta$ is a warped product pseudo slant submanifold of a nearly quasi-Sasakian manifold $\tilde{M}$, then*

$$< h(X,Z), FPX > = < h(Z,PX), FX >,$$

for any $X \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$.

Proof. By using (2.9) and (2.12), we get

$$< h(Z,PX), FX > = < \tilde{\nabla}_Z PX, \phi X > - < \tilde{\nabla}_Z PX, PX > .$$

Now, using (2.11), (2.12) and (4.1), after performing few steps we have

$$< h(Z,PX), FX > = < h(X,Z), FPX > -2\cos^2\theta (Z \ln y) \parallel X \parallel^2 \\ - < \tilde{\nabla}_{PX} Z, FX > - < \tilde{\nabla}_{PX} Z, PX > + < A_{\phi Z} PX, X > .$$

Now, using (2.6)-(2.9) and (4.1), after some simple calculating steps, we get

$$2 < h(Z,PX), FX > = -3\cos^2\theta(Z\ln y) \parallel X \parallel^2 + < h(X,Z), FPX > - < h(PX,X), \phi Z > . \tag{4.2}$$

Now, replacing X by PX in (4.2) and taking an account of (2.14), we get

$$2 < h(X,Z), FPX > = -3\cos^2\theta(Z\ln y) \parallel X \parallel^2 + < h(Z,PX), FX > - < h(PX,X), \phi Z > . \tag{4.3}$$

Now, from (4.2) and (4.3), we get finally

$$< h(X,Z), FPX >=< h(Z,PX), FX >,$$

which completes the proof. □

Proposition 4.3. *In a warped product pseudo slant submanifold $M = M_\perp \times_y M_\theta$ of a nearly quasi-Sasakian manifold $\tilde{M}$, we have*

$$< h(X,PX), \phi Z >= \cos^2\theta(Z\ln y) \parallel U \parallel^2 -3 < h(Z,PX), FX >,$$

for any $X \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$.

Proof. From Propositions 4.1 and 4.2, Propositions 4.3 easily follows. □

Theorem 4.1. *Let $\tilde{M}$ be a nearly quasi-Sasakian manifold and M be a proper pseudo-slant submanifold of $\tilde{M}$ such that the slant distribution is integrable. Then $M = M_\perp \times_y M_\theta$ is a locally warped product of proper slant and anti-invariant submanifolds if and only if*

$$3A_{FPX}Z + A_{\phi Z}PX = \cos^2\theta(\lambda Z)X, \tag{4.4}$$

for any $Z \in \Gamma(D^\perp \oplus \xi)$ and $X \in \Gamma(D^\theta)$.

Proof. From Proposition 4.3, we have

$$< h(X,PX), \phi Z >= \cos^2\theta(Z\lambda) \parallel X \parallel^2 -3 < h(Z,X), FPX >$$

and hence we get

$$< A_{\phi Z}PX, X >= -3 < A_{FPX}Z, X > + \cos^2\theta(Z\lambda) < X, X > .$$

This immediately gives

$$A_{\phi Z}PX = -3A_{FPX}Z + \cos^2\theta(Z\lambda)X, \tag{4.5}$$

which proves the necessary part. Now conversely, if M is a proper pseudo slant submanifold of a nearly quasi-Sasakian manifold $\tilde{M}$ holding (4.5), then taking the inner product in (4.5) with W and using the fact that X and W are orthogonal, we have

$$3 < h(Z,W), FPX >= - < h(PX,W), \phi Z > . \tag{4.6}$$

Interchanging Z and W, we have

$$3 < h(Z,W), FPX >= - < h(PX,Z), \phi W > . \tag{4.7}$$

Now, adding (4.6) and (4.7), we have

$$6 < h(Z,W), FPX >= - < h(PX,W), \phi Z > - < h(PX,Z), \phi W > . \tag{4.8}$$

Now, from

$$< h^\theta(X,Y), Z >=< \tilde{\nabla}_X Y, Z >=< \phi\tilde{\nabla}_X Y, \phi Z > + \upsilon(Z) < \xi, \tilde{\nabla}_X Y > .$$

Using (2.6), (2.8), (2.9), (2.11), (2.14) and supposing that X, Y are orthogonal to ξ, we arrive

$$\begin{aligned} < h^\theta(X,Y), Z > &= - < A_{\phi Y} PZ, X > + \sin^2\theta < \tilde{\nabla}_X Z, Y > \\ &\quad - \cos^2\theta \upsilon(Z) < \bar{\nabla}_X \xi, Y > + < A_{FPX} X, Y > \end{aligned} \tag{4.9}$$

which leads to

$$\begin{aligned} \cos^2\theta < h^\theta(X,Y), Z > &= - < A_{\phi Y} PZ, X > - \cos^2\theta \upsilon(Z) < \tilde{\nabla}_X \xi, Y > \\ &\quad + < A_{FPZ} X, Y > . \end{aligned} \tag{4.10}$$

Hence, we have

$$\begin{aligned} < h^\theta(X,Y), Z > &= (Z\lambda) < X, Y > - \upsilon(Z) < \tilde{\nabla}_X \xi, Y > \\ &=< X, Y >< \nabla\lambda, Z > - < \tilde{\nabla}_X \xi, Y >< \xi, Z > . \end{aligned} \tag{4.11}$$

This implies

$$h^\theta(X,Y) =< X, Y > \nabla\lambda - < \bar{\nabla}_X \xi, Y > \xi. \tag{4.12}$$

Interchanging X and Y in (4.11), we have

$$< h^\theta(X,Y), Z >=< X, Y >< \nabla\lambda, Z > - \upsilon(Z) < \bar{\nabla}_Y \xi, X > . \tag{4.13}$$

Now, by adding (4.11) and (4.13), we get

$$2 < h^\theta(X,Y), Z >= 2 < X, Y >< \nabla\lambda, Z > - \upsilon(Z)[< \bar{\nabla}_X \xi, Y > + < \bar{\nabla}_Y \xi, X >]$$

which gives

$$2 < h^\theta(X,Y), Z >= 2 < X, Y >< \nabla\lambda, Z > .$$

Thus, we get

$$< h^{\theta}(X,Y), Z >=< X,Y >< \nabla\lambda, Z > . \tag{4.14}$$

This gives

$$h^{\theta}(X,Y) =< X,Y > \nabla\lambda. \tag{4.15}$$

Thus, we have

$$H^{\theta} = \nabla\lambda \tag{4.16}$$

which is the mean curvature vector of M. □

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