Separation axioms in intuitionistic topological spaces

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Abstract. In this paper we study that every classical topological space is also an intuitionistic topological space but the converse is not true in general. This notion opens up a new conception of generalization of classical topological space. Besides this, by using the notions of separation axioms (T_0, T_1, T_2) under intuitionistic set we define the relations among them. The hereditary and topological properties of intuitionistic topological spaces have been also investigated. Finally, it is showed that under some conditions the images and homeomorphic images preserve in intuitionistic topological spaces.
Keywords: intuitionistic set, intuitionistic topological space, hereditary, separation axioms.

1. Introduction

Ever since the invention of topological spaces, many researchers have been paying remarkable contribution in this field. By investigating different properties on classical topological spaces, they also added new notions for its generalization. After the introduction of the fuzzy set by Zadeh [23] the classical topological space is directed to a new dimension namely "Fuzzy Topological Spaces" which is defined by Chang [8]. After that, Atanassov [5, 6] introduced the notion of intuitionistic fuzzy set which is a generalization of fuzzy set. Later, Coker et al. [9, 10, 11, 12] defined intuitionistic fuzzy topological spaces, intuitionistic sets and intuitionistic topological spaces. From the onwards, Coker and S. Bayhan [9, 10, 11, 12], Singh and Srivastava [22], S. J. Lee and E. P Lee [16], Saadati and Park [21], Estiaq Ahmed et al. [1, 2, 3] initiated different studies on intuitionistic fuzzy topological spaces by using intuitionistic fuzzy sets. In this paper, we

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investigate various properties of separation axioms in intuitionistic topological spaces.

2. Notation and preliminaries

In this paper X is noted as an non-empty set, $\mathcal{T}$ is topology, $(X, \mathcal{T})$ is topological space, $\check{\mathcal{T}}$ is an intuitionistic topology, $(X, \check{\mathcal{T}})$ is intuitionistic topological space, $G = (G_1, G_2)$ and $H = (H_1, H_2)$ are intuitionistic sets, f is a function, $\check{\mathcal{T}}_A$ is an intuitionistic relative topology on A where $A \subseteq X$, $\emptyset = (\emptyset, X)$ and $X = (X, \emptyset)$ are also noted as intuitionistic sets.

Definition 2.1 ([12]). *Suppose X is a non-empty set. An intuitionistic set A on X is an object having the form $A = (X, A_1, A_2)$ where A_1 and A_2 are subsets of X satisfying $A_1 \cap A_2 = \emptyset$. The set A_1 is called the set of member of A while A_2 is called the set of non-member of A. In this paper, we use the simpler notation $A = (A_1, A_2)$ instead of $A = (X, A_1, A_2)$ for an intuitionistic set.*

Remark 2.1 Every subset A of a nonempty set X may obviously be regarded as an intuitionistic set having the form $A = (A, A^c)$ where $A^c = X/A$

Definition 2.2 ([12]). *Let the intuitionistic sets A and B in X be of the forms $A = (A_1, A_2)$ and $B = (B_1, B_2)$ respectively. Furthermore, let $\{A_j, j \in J\}$ be an arbitrary family of intuitionistic sets in X, where $A_j = (A_j^{(1)}, A_j^{(2)})$. Then:*

a. $A \subseteq B$ if and only if $A_1 \subseteq B_1$ and $A_2 \supseteq B_2$;
b. $A = B$ if and only if $A \subseteq B$ and $B \supseteq A$;
c. $\bar{A} = (A_1, A_2)$ denotes the complement of A;
d. $\cap A_j = (\cap A_j^{(1)}, \cup A_j^{(2)})$;
e. $\cup A_j = (\cup A_j^{(1)}, \cap A_j^{(2)})$;
f. $\emptyset_\sim = (\emptyset, X)$ and $X_\sim = (X, \emptyset)$.

Definition 2.3 ([6]). *Let X be a non-empty set. A family $\check{\mathcal{T}}$ of intuitionistic sets in X is called an intuitionistic topology on X if the following conditions hold:*

a. $\emptyset_\sim, X_\sim \in \check{\mathcal{T}}$;
b. $A \cap B \in \check{\mathcal{T}}$ for all $A, B \in \check{\mathcal{T}}$;
c. $\cup A_j \in \check{\mathcal{T}}$ for any arbitrary family $\{A_j \in \check{\mathcal{T}}, j \in J\}$.

The pair $(X, \check{\mathcal{T}})$ is called an intuitionistic topological space (ITS, in short), member of $\check{\mathcal{T}}$ are called intuitionistic open sets (IOS, in short) in X and their complements are called intuitionistic closed sets (ICS, in short) in X.

Definition 2.4 ([3]). *let $(X, \check{\mathcal{T}})$ and (Y, δ) be IFTSs. A function $f : X \to Y$ is called continuous if $f^{-1}(B) \in \mathcal{T}$ for all $B \in \delta$ and f is called open if $f(A) \in \delta$ for all $A \in \check{\mathcal{T}}$.*

Definition 2.5. *Let $(X, \check{\mathcal{T}})$ be an intuitionistic topological space and $A \subseteq X$. If $\check{\mathcal{T}}_A$ is a topology generated by $\{G \cap (A, \emptyset); G \in \check{\mathcal{T}}\}$ then $\check{\mathcal{T}}_A$ is also an intuitionistic topology on A and the space $(A, \check{\mathcal{T}}_A)$ is a subspace of $(X, \check{\mathcal{T}})$.*

Definition 2.6 ([3]). *An intuitionistic topological space* $(X, \check{\mathcal{T}})$ *is called* T_0 *if for all* $x, y \in X$ *with* $x \neq y$ *there exists intuitionistic sets* $A = (A_1, A_2) \in \check{\mathcal{T}}$ *such that* $x \in A_1, y \in A_2$ *and* $y \in A_1, x \in A_2$.

Definition 2.7 ([6]). *An intuitioinistic topological space* $(X, \check{\mathcal{T}})$ *is called* T_1 *if for all* $x, y \in X$ *with* $x \neq y$ *there exists intuitionistic sets* $A = (A_1, A_2), B = (B_1, B_2) \in \check{\mathcal{T}}$ *such that* $x \in A_1, y \notin A_1$ *and* $y \in B_1, x \notin B_2$.

Definition 2.8 ([6]). *An intuitionistic topological space* $(X, \check{\mathcal{T}})$ *is called* T_1 *if for all* $x, y \in X$ *with* $x \neq y$ *there exists intuitionistic sets* $A = (A_1, A_2), B = (B_1, B_2) \in \check{\mathcal{T}}$ *such that* $x \in A_1, y \notin A_1$ *and* $y \in B_1, x \notin B_1$ *and* $A \cap B = \emptyset$.

Definition 2.9. *Let* X *and* Y *be two nonempty sets and* $f : X \to Y$ *be a function. If* $A = (A_1, A_2) \subseteq X$ *then* $f(A)$ *is defined as* $f(A) = (f(A_1), f(A_2))$ *and if* $B = (B_1, B_2) \subseteq Y$ *then* $f^{-1}(B)$ *is defined as* $f^{-1}(B) = (f^{-1}(B_1), f^{-1}(B_2))$.

3. Separation axioms in intuitionistic topological space

In this section, we will prove different properties of separation axioms in intuitionistic topological space. Beside this, by using some examples we will show that intuitionistic topological space is a generalization of classical topological space.

Theorem 3.1. *Every classical topological space can be represented as an intuitionistic topological space but the converse is not true.*

Proof of Theorem 3.1. We know that, if $(X, \mathcal{T})$ is a topological space then it follows the following axioms:

i. $X, \emptyset \in \mathcal{T}$;

ii. $A, B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T}$;

iii. $A_j \in \mathcal{T} \Rightarrow \cup A_j \in \mathcal{T}$.

Now, we shall show that $(X, \mathcal{T})$ can be represented as an intuitionistic topological space $(X, \check{\mathcal{T}})$.

i. X can be represented as $(X, \emptyset) \in \check{\mathcal{T}}$, $\emptyset$ can be represented as $(\emptyset, X) \in \check{\mathcal{T}}$.

ii. $A, B \in \mathcal{T} \Rightarrow (A, A^c) \in \check{\mathcal{T}}$, $(B, B^c) \in \check{\mathcal{T}}$ and $A \cap B = (A \cap B, A^c \cup B^c) \in \check{\mathcal{T}}$.

iii. If $A_i \in \mathcal{T}$ then $(A_i, A_i^c) \in \check{\mathcal{T}}$ and $(\cup A_i, \cap A_i^c) \in \check{\mathcal{T}}$. Thus, $(X, \mathcal{T})$ is represented as an intuitionistic topological space $(X, \check{\mathcal{T}})$. But the converse is not true. For example, let $X = \{a, b, c\}$, $\check{\mathcal{T}} = \{A = (\{a\}, \{c\}), B = (\{b\}, \{c\}), C = (\{a, b\}, \{c\}), D = (\emptyset, \{c\}, \emptyset_\sim, X_\sim)\}$. Therefore, the intuitionistic topological space $(X, \check{\mathcal{T}})$ cannot be represented as a classical topological space.

Hence the proof.

Theorem 3.2. *Every classical* T_0 *topological space is an intuitionistic* T_0 *topological space but the converse is not true.*

Proof of Theorem 3.2. Let $(X, \mathcal{T})$ be a T_0 space. So according to the axioms of T_0 space we have $a, b \in X$ with $a \neq b$ $\exists$ a $G \in \mathcal{T}$ such that $a \in G, b \notin G$.

The above axioms imply that $\forall a, b \in X$ with $a \neq b \; \exists \; A = (G, G^c)$ and $A \in \check{\mathcal{T}}$ such that $a \in G, b \neq G^c$. Thus $(X, \check{\mathcal{T}})$ is an intuitionistic T_1 space.

But the converse is not true. For example, consider $X = \{a, b, c\}$ and $A = (\{a\}, \{b, c\})$, $B = (\{b\}, \{c\})$, $C = (\{c\}, \{a, b\})$. Therefore, $(X, \check{\mathcal{T}})$ is an intuitionistic T_0 space generated by $\{A, B, C\}$. But this topological space cannot be represented as a classical T_0 space. Because the element c in X bellongs to $B \in \check{\mathcal{T}}$ and $C \in \check{\mathcal{T}}$ as well.

Hence the proof.

Theorem 3.3. *Every classical T_1 space is an intuitionistic T_1 space but the converse is not true.*

Proof of Theorem 3.3. Let $(X, \mathcal{T})$ be a T_1 space. So according to the axioms of T_1 space we have $a, b \in X$ with $a \neq b \; \exists \; G, H \in T$ such that $a \in G, b \notin G$ and $b \in H, a \notin H$, G and H are not necessarily disjoint.

The above axioms imply that $\forall a, b \in X$ with $a \neq b \; \exists \; A = (G, G^c)$ and $B = (H, H^c) \in \check{\mathcal{T}}$ such that $a \in G, b \notin G$ and $b \in H, a \notin H$. Thus $(X, \check{\mathcal{T}})$ is an intuitionistic T_1 space.

But the converse is not true. For example, consider $X = \{a, b, c\}$ and $A = (\{a\}, \{b\})$, $B = (\{b\}, \{c\})$, $C = (\{c\}, \{a, b\})$. Therefore, $(X, \check{\mathcal{T}})$ is an intuitionistic T_1 space generated by $\{A, B, C\}$. But this cannot be represented as a classical T_1 space. Because the element b in X belongs to $A \in \check{\mathcal{T}}$ and $B \in \check{\mathcal{T}}$ as well.

Hence the proof.

Theorem 3.4. *Every classical T_2 space is an intuitionistic T_2 space but the converse is not true.*

Proof of Theorem 3.4. Let $(X, \mathcal{T})$ be a T_2 space. So according to the axioms of T_2 space we have $a, b \in X$ with $a \neq b \; \exists \; G, H \in \mathcal{T}$ such that $a \in G, b \notin G$ and $b \in H, a \notin H$, $G \cap H = \emptyset$.

The above axioms imply that $\forall a, b \in X$ with $a \neq b \; \exists \; A = (G, G^c)$ and $B = (H, H^c) \in \check{\mathcal{T}}$ such that $a \in G, b \notin G$ and $b \in H, a \notin H$ and $A \cap B = (G, G^c) \cap (H, H^c) = (G \cap H, G^c \cup H^c) = (\emptyset, D)$ where $G^c \cup H^c = D \subseteq X$. Thus, $(X, \check{\mathcal{T}})$ is an intuitionistic T_2 space. But, the converse is not true. For example, consider $X = \{p, q, r, s\}$ and $A = (\{p\}, \{q, s\})$, $B = (\{q\}, \{p, r, s\})$, $C = (\{s\}, \{p, r\})$, $D = (\{r\}, \{p, s\})$. Therefore, $(X, \check{\mathcal{T}})$ is an intuitionistic T_2 space generated by $\{A, B, C, D\}$. But this topological space cannot be represented as a classical T_2 space.

Hence the proof.

Theorem 3.5. *Let $(X, \check{\mathcal{T}})$ be an intuitionistic T_0 space then $(A, \check{\mathcal{T}}_A)$ is also T_0.*

Proof of Theorem 3.5. Let $a, b \in A$ with $a \neq b$. This implies $a, b \in X$ with $a \neq b$. Since $(X, \check{\mathcal{T}})$ is an intuitionistic T_0 space the $\exists \; G = (G_1, G_2) \in \check{\mathcal{T}}$ such that $a \in G_1, b \notin G_1$. Now we have $G \in \check{\mathcal{T}} \Rightarrow G \cap (A, \emptyset) \in \check{\mathcal{T}}_A$.

But $G \cap (A, \emptyset) = (G_1, G_2) \cap (A, \emptyset) = (G_1 \cap A, G_2 \cup \emptyset) = (G_1 \cap A, G_2)$. Furthermore, $a, b \in A$ and $a \in G_1, b \notin G_1 \Rightarrow a \in G_1 \cap A, b \notin G_1 \cap A$. Finally, we get $\forall a, b \in A$ with $a \neq b$ $\exists$ a $(G_1 \cap A, G_2) \in \check{\mathcal{T}}_A$ such that $a \in G_1 \cap A, b \notin G_1 \cap A$. Therefore, $(A, \check{\mathcal{T}}_A)$ is an intuitionistic T_0 space.

Hence T_0 property is hereditary. (Proved)

Theorem 3.6. *Let $(X, \check{\mathcal{T}})$ be an intuitionistic T_1 space then $(A, \check{\mathcal{T}}_A)$ is also T_1.*

Proof of Theorem 3.6. Let $a, b \in A$ with $a \neq b$. This implies $a, b \in X$ with $a \neq b$. Since $(X, \check{\mathcal{T}})$ is intuitionistic T_1 space then $\exists$ $G = (G_1, G_2) \in \check{\mathcal{T}}$. Therefore, $a \in G_1, b \notin G_1$ and $b \in H_1, a \notin H_1$. Now we have $G, H \in \check{\mathcal{T}} \Rightarrow G \cap (A, \emptyset), H \cap (A, \emptyset) \in \check{\mathcal{T}}_A$.

But $G \cap (A, \emptyset) = (G_1, G_1) \cap (A, \emptyset) = (G_1 \cap A, G_2 \cup \emptyset) = (G_1 \cap A, G_2)$ $H \cap (A, \emptyset) = (H_1, H_2) \cap (A, \emptyset) = (H_1 \cap A, H_2 \cup \emptyset) = (H_1 \cap A, H_2)$.

Further, we set $a, b \in A$ and $a \in G_1, b \notin G_1 \Rightarrow a \in G_1 \cap A, b \notin G_1 \cap A$. Again $b \in H_1, a \notin H_1 \Rightarrow b \in H_1 \cap A, a \notin H_1 \cap A$. Finally, we get $\forall a, b \in A$ with $a \neq b$ $\exists$ $(G_1 \cap A, G_2)$ and $(H_1 \cap A, H_2) \in \check{\mathcal{T}}_A$ such that $a \in G_1 \cap A, b \notin G_1 \cap A$ and $a \in H_1 \cap A, b \in H_1 \cap A$. Therefore, $(A, \check{\mathcal{T}}_A)$ is an intuitionistic T_1 space.

Hence, T_1 property is hereditary. (Showed)

Theorem 3.7. *Let $(X, \check{\mathcal{T}})$ be an intuitionistic T_2 space then $(A, \check{\mathcal{T}}_A)$ is also T_2.*

Proof of Theorem 3.7. Let $a, b \in A$ with $a \neq b$. This implies $a, b \in X$ with $a \neq b$. Since $(X, \check{\mathcal{T}})$ is intuitionistic T_1 space then $\exists$ $G = (G_1, G_2) \in \check{\mathcal{T}}$ and $H = (H_1, H_2) \in \check{\mathcal{T}}$ such that $a \in G_1, b \notin G_1$ and $b \in H_1, a \notin H_1$ and $G \cap H = \emptyset$. Now we have $G, H \in \check{\mathcal{T}} \Rightarrow G \cap (A, \emptyset), H \cap (A, \emptyset) \in \check{\mathcal{T}}_A$.

But $G \cap (A, \emptyset) = (G_1, G_2) \cap (A, \emptyset) = (G_1 \cap A, G_2 \cup \emptyset) = (G_1 \cap A, G_2)$ $H \cap (A, \emptyset) = (H_1, H_2) \cap (A, \emptyset) = (H_1 \cap A, H_2 \cup \emptyset) = (H_1 \cap A, H_2)$.

Furthermore, $a, b \in A$ and $a \in G_1, b \notin G_1 \Rightarrow a \in G_1 \cap A, b \notin G_1 \cap A$. Again $b \in H_1, a \notin H_1 \Rightarrow b \in H_1 \cap A, b \notin H_1 \cap A$. Consider $(G \cap A) \cap (H \cap A) \neq (\emptyset \cap A)$, where $A \subseteq X \Rightarrow (G_1 \cap A, G_2) \cap (H_1 \cap A, H_2) \neq (\emptyset \cap A) \Rightarrow ((G_1 \cap A) \cap (H_1 \cap A), G_2 \cup H_2) \neq (\emptyset \cap A)$. Therefore, $(G_1 \cap A) \cap (H_1 \cap A) \neq \emptyset$ i.e. $\exists$ at least one $y \in A \Rightarrow y \in X$ such that $y \in G_1 \cap A$ and $y \in H_1 \cap A \Rightarrow y \in G_1$ and $y \in A$ and $y \in H_1, y \in A$. Since $(X, \check{\mathcal{T}})$ is intuitionistic T_1 space so $\nexists$ a y which belongs to G_1 and H_1 simultaneously. So, this is a contradiction to the fact that $(G_1 \cap A) \cap (H_1 \cap A) \neq \emptyset$. Therefore, $(G_1 \cap A) \cap (H_1 \cap A) = (\emptyset, A)$.

Finally, we get $\forall a, b \in A$ with $a \neq b$ $\exists$ $(G_1 \cap A, G_2)$ and $(H_1 \cap A, H_2) \in \check{\mathcal{T}}_A$ such that $a \in G_1 \cap A, b \notin G_1 \cap A$ and $a \notin H_1 \cap A, b \in H_1 \cap A$ and $(G_1 \cap A) \cap (H_1 \cap A) = (\emptyset, A)$. Therefore, $(A, \check{\mathcal{T}}_A)$ is an intuitionistic T_2 space. Hence, T_2 property is hereditary. (Proved)

Theorem 3.8. *Let $(X, \check{\mathcal{T}})$ and $(Y, \check{\delta})$ be two intuitionistic topological spaces and $f : X \to Y$ be a one-one, onto and open map. If $(X, \check{\mathcal{T}})$ is T_0 space then $(Y, \check{\delta})$ is also T_0 space.*

Proof of Theorem 3.8. Let $(X, \check{\mathcal{T}})$ and $(Y, \check{\delta})$ be two intuitionistic topological spaces and $f : X \to Y$ be one-one, onto and open map. Let $(X, \check{\mathcal{T}})$ be intuitioistic T_0 space, we shall show that $(Y, \check{\delta})$ is also intuitionistic T_0 space.

Suppose $a, b \in Y$ with $a \neq b$. Since f is onto then $\exists\ p, q \in X$ such that $f(p) = a$ and $f(q) = b$.

Again, since $a \neq b \Rightarrow f(p) \neq f(q) \Rightarrow p \neq q$ as f is one-one. Further since $p, q \in X$, $p \neq q$ and $(X, \check{\mathcal{T}})$ is T_0 space then $\exists\ G = (G_1, G_2) \in \check{\mathcal{T}}$ such that $p \in G_1, q \notin G_1$.

Since $G, H \in \check{\mathcal{T}} \Rightarrow f(G), f(H) \in \check{\delta}$ as f is open.

We know, $f(G) = (f(G_1), f(G_1))$. Furthermore $a = f(p) \in f(G_1)$ and $b = f(q) \notin f(G_1)$. Finally, we get $a, b \in Y$ with $a \neq b\ \exists\ f(G) \in \check{\delta}$ such that $a \in f(G_1), b \notin f(G_1)$. Therefore, $(Y, \check{\delta})$ is T_0 space. $\therefore$ Every homeomorphic image of intuitionistic T_0 space is also T_0 space.

Hence intuitionistic T_0 is a topological property. (Proved)

Theorem 3.9. *Let $(X, \check{\mathcal{T}})$ and $(Y, \check{\delta})$ be two intuitionistic topological spaces and $f : X \to Y$ be a one-one, onto and open map. If $(X, \check{\mathcal{T}})$ is T_1 space then $(Y, \check{\delta})$ is also T_1 space.*

Proof of Theorem 3.9. Let $(X, \check{\mathcal{T}})$ and $(Y, \check{\delta})$ be two intuitionistic topological spaces and $f : X \to Y$ be a one-one, onto and open map. Let $(X, \check{\mathcal{T}})$ be intuitionistic T_1 space, we shall show that $(Y, \check{\delta})$ is also T_1 space.

Suppose $a, b \in Y$ with $a \neq b$. Since f is onto then $\exists\ p, q \in X$ such that $f(p) = a$ and $f(q) = b$.

Again since $a \neq b \Rightarrow f(p) \neq f(q) \Rightarrow p \neq q$ as f is one-one. Further since $p, q \in X, p \neq q$ and $(X, \check{\mathcal{T}})$ is T_1 space then $\exists\ G = (G_1, G_2) \in \check{\mathcal{T}}$ and $H = (H_1, H_2) \in \check{\mathcal{T}}$ such that $p \in G_1, q \notin G_1$ and $q \in H_1, p \notin H_1$.

Since, $G, H \in \check{\mathcal{T}} \Rightarrow f(G), f(H) \in \check{\delta}$ as f is open.

Now, we have, $f(G) = (f(G_1), f(G_2))$ and $f(H) = (f(H_1), f(H_2))$. Furthermore, $a = f(p) \in f(G_1)$ and $b = f(q) \in f(H_1)$. Again, since $q \notin G_1 \Rightarrow b = f(q) \notin f(G_1)$. Finally, we get $a, b \in Y$ with $a \neq b\ \exists\ f(G)$ and $f(H) \in \check{\delta}$ such that $a \in f(G_1), b \notin f(G_1)$ and $b \in f(H_1), a \notin f(H_1)$.

$\therefore$ Every homeomorphic image of intuitionistic T_1 space is also T_1 space.

Hence, intuitionistic T_1 is a topological property. (Proved)

Theorem 3.10. *Let $(X, \check{\mathcal{T}})$ and $(Y, \check{\delta})$ be two intuitionistic topological spaces and $f : X \to Y$ be one-one, onto and open map. If $(X, \check{\mathcal{T}})$ is T_2 space then $(Y, \check{\delta})$ is also T_2 space.*

Proof of Theorem 3.10. Let $(X, \check{\mathcal{T}})$ and $(Y, \check{\delta})$ be two intuitionistic topological spaces and $f : X \to Y$ be a one-one, onto and open map. Let $(X, \check{\mathcal{T}})$ be intuitionistic T_2 space, we shall show that $(Y, \check{\delta})$ is also T_2 space.

Suppose $a, b \in Y$ with $a \neq b$. Since f is onto then $\exists\ p, q \in X$ such that $f(p) = a$ and $f(q) = b$.

Again since $a \neq b \Rightarrow f(p) \neq f(q) \Rightarrow p \neq q$ as f is one-one. Further since $p, q \in X, p \neq q$ and $(X, \check{\mathcal{T}})$ is T_2 space then $\exists\ G = (G_1, G_2) \in \check{\mathcal{T}}$ and $H = (H_1, H_2) \in \check{\mathcal{T}}$ such that $p \in G_1, q \notin G_1$ and $q \in H_1, p \notin H_1$ with $G \cap H = (\emptyset, A)$.

Since, $G, H \in \check{\mathcal{T}} \Rightarrow f(G), f(H) \in \check{\delta}$ as f is open.

Now, we have, $f(G) = (f(G_1), f(G_2))$ and $f(H) = (f(H_1), f(H_2))$. Furthermore $a = f(p) \in f(G_1)$ and $b = f(q) \in f(H_1)$. Consider $f(G) \cap f(H) \neq (\emptyset, A)$ where $A \subseteq X \Rightarrow (f(G_1) \cap f(H_1), f(G_2) \cup f(H_2) \neq (\emptyset, A) \Rightarrow f(G_1) \cap f(H_1) \neq \emptyset$ i.e. there exists at least one $y \in Y$ for which $y \in f(G_1) \cap f(H_1) \Rightarrow y \in f(G_1)$ and $y \in f(H_1)$. Then, there exists $m \in G_1$ and $n \in H_1$. Such that $f(m) = f(n) = y$ $\Rightarrow m = n$ as f is one-one $\Rightarrow m = n \in G_1 \cap H_1$ which is a contradiction to the fact that $G \cap H = \emptyset$. Therefore, we get $f(G) \cap f(H) = \emptyset$. Finally, we get $a, b \in Y$ with $a \neq b\ \exists\ f(G)$ and $f(H) \in \check{\delta}$ such that $a \in f(G_1), b \notin f(H_1)$ and $b \in f(H_1), a \notin f(H_1)$ and, $f(G) \cap f(H) = \emptyset$. Therefore, $(Y, \check{\delta})$ is T_2 space. $\therefore$ Every homeomorphic image of intuitionistic T_2 space is also T_2 space.

Hence intuitionistic T_2 is a topological property. (Proved)

Theorem 3.11. *Let $(X, \check{\mathcal{T}})$ and $(Y, \check{\delta})$ be two intuitionistic topological spaces and $f : X \to Y$ be one-one and continuous. If $(Y, \check{\delta})$ is T_2 space then $(X, \check{\mathcal{T}})$ is also T_2 space.*

Proof of Theorem 3.11. Let $a, b \in X$ with $a \neq b \Rightarrow f(a), f(b) \in Y$ with $f(a) \neq f(b)$ as f is one-one.

Since $f(a), f(b) \in Y$ and $(Y, \check{\delta})$ is T_2 space then $\exists\ G = (G_1, G_2)$, $H = (H_1, H_2) \in \check{\delta}$ such that $f(a) \in G_1, f(b) \in H_1$ and $G \cap H = (\emptyset, A)$, where $A \subseteq X$. But we have $G \cap H = (G_1, G_2) \cap (H_1, H_2) = (G_1 \cap H_1, G_2 \cup H_2) = (\emptyset, A)$ i.e. $G_1 \cap H_1 = \emptyset, G_2 \cup H_2 = A$. We have, $f^{-1}(G) = (f^{-1}(G_1), f^{-1}(G_2))$ and $f^{-1}(H) = (f^{-1}(H_1), f^{-1}(H_2))$. Now, $f(a) \in G_1 \Rightarrow f^{-1}f(a) \in f^{-1}(G_1)$ $\Rightarrow a \in f^{-1}(G_1)$. And, $f(b) \in H_1 \Rightarrow f^{-1}f(b) \in f^{-1}(H_1) \Rightarrow b \in f^{-1}(H_1)$. Furthermore $f^{-1}(G) \cap f^{-1}(H) = (f^{-1}(G_1), f^{-1}(G_2)) \cap (f^{-1}(H_1), f^{-1}(H_2)) = (f^{-1}(G_1) \cap f^{-1}(H_1), f^{-1}(G_2) \cup f^{-1}(H_2))$. Suppose $f^{-1}(G_1) \cap f^{-1}(H_1) \neq \emptyset$ $\Rightarrow f(f^{-1}(G_1) \cap f^{-1}(H_1)) \neq f(\emptyset) \Rightarrow ff^{-1}(G_1) \cap ff^{-1}(H_1) \neq \emptyset \Rightarrow G_1 \cap H_1 \neq \emptyset$, which is a contradiction $\therefore f^{-1}(G_1) \cap f^{-1}(H_1) \neq \emptyset$. Again $f^{-1}(G_2) \subseteq X$, $f^{-1}(H_2) \subseteq X$. Thus $f^{-1}(G_2) \cup f^{-1}(H_2) \subseteq X$. Finally, $f^{-1}(G) \cap f^{-1}(H) \in \check{\mathcal{T}}$. Therefore, $f^{-1}(G) \cap f^{1}(H) = (\emptyset, A)$. Finally, we get $a, b \in X$ with $a \neq b\ \exists$ $f^{-1}(G)$ and $f^{-1}(H) \in \check{\mathcal{T}}$ such that $a \in f^{-1}(G), b \notin f^{-1}(G)$ and $b \in f^{-1}(H), a \notin f^{-1}(H)$ and $f^{-1}(G) \cap f^{-1}(H) = (\emptyset, A)$. Therefore, $(X, \check{\mathcal{T}})$ is T_2 space. (Proved)

Theorem 3.12. *Let $(X, \check{\mathcal{T}})$ and $(Y, \check{\delta})$ be two intuitionistic topological spaces and $f : X \to Y$ be one-one and continuous. If $(Y, \check{\delta})$ is T_1 space then $(X, \check{\mathcal{T}})$ is also T_1 space.*

Proof of Theorem 3.12. Let $a, b \in X$ with $a \neq b \Rightarrow f(a), f(b) \in Y$ with $f(a) \neq f(b)$ as f is one-one.

Since $f(a), f(b) \in Y$ and $(Y, \check{\delta})$ is T_1 space then $\exists\ G = (G_1, G_2)$, $H = (H_1, H_2) \in \check{\delta}$ such that $f(a) \in G_1, f(b) \in H_1$.

Since f is continuous and $G, H \in \check{\delta}$ then $f^{-1}(G), f^{-1}(H) \in \check{\mathcal{T}}$.

We have, $f^{-1}(G) = (f^{-1}(G_1), f^{-1}(G_2))$ and $f^{-1}(H) = (f^{-1}(H_1), f^{-1}(H_2))$.

Now, $f(a) \in G_1 \Rightarrow f^{-1}f(a) \in f^{-1}(G_1) \Rightarrow a \in f^{-1}(G_1)$ and $f(b) \in H_1 \Rightarrow f^{-1}f(b) \in f^{-1}(H_1) \Rightarrow b \in f^{-1}(H_1)$. Finally, we have $a, b \in X$ with $a \neq b$ then there exists $f^{-1}(G_1), f^{-1}(G_1) \in \check{\mathcal{T}}$ such that $a \in f^{-1}(G), b \notin f^{-1}(G)$ and $a \notin f^{-1}(H), b \in f^{-1}(H)$.

Furthermore, $f(b) \notin G \Rightarrow f^{-1}f(b) \notin f^{-1}(G) \Rightarrow b \notin f^{-1}(G)$ and $f(a) \notin H \Rightarrow f^{-1}f(a) \notin f^{-1}(H) \Rightarrow a \notin f^{-1}(H)$. Thus, $(X, \check{\mathcal{T}})$ is a T_1 topological space. (Proved)

Theorem 3.13. *Let $(X, \check{\mathcal{T}})$ and $(Y, \check{\delta})$ be two intuitionistic topological spaces and $f : X \to Y$ be one-one and continuous. If $(Y, \check{\delta})$ is T_0 space then $(X, \check{\mathcal{T}})$ is also T_0 space.*

Proof of Theorem 3.13. This proof is obvious.

Theorem 3.14. *Show that every intuitionistic T_1 space is also intuitionistic T_0 space. But the converse is not true.*

Proof of Theorem 3.14. Let $(X, \check{\mathcal{T}})$ be an intuitionistic T_1 space. Then $\forall a, b \in X$ with $a \neq b$ $\exists$ $G = (G_1, G_2)$ and $H = (H_1, H_2) \in \check{\mathcal{T}}$ such that $a \in G_1, b \notin G_1$ and $b \in H_1, a \notin H_1$.

So, for $G = (G_1, G_2) \in \check{\mathcal{T}}$ we have $\forall a, b \in X$ with $a \neq b$ we get $a \in G_1, b \notin G_1$. Or, for $H = (H_1, H_2) \in \check{\mathcal{T}}$ we have $\forall a, b \in X$ with $a \neq b$ we get $b \in H_1, a \notin H_1$. Thus, this topological space follows the axioms of T_0 space. Hence, every intuitionistic T_1 space is also intuitionistic T_0 space.

To prove the converse, we have an exampple:

Consider $X = \{a, b, c\}$ and $A = (\{a\}, \{b, c\}), B = (\{a, b\}, \{c\}), C = (\{c\}, \{b\})$. Suppose $\check{\mathcal{T}}$ be an intuitionistic topology generated by $\{A, B, C\}$. Therefore, $(X, \check{\mathcal{T}})$ is an intuitionistic T_0 space. But this is not an intuitionistic T_1 space because $a, b \in X$ with $a \neq b$. We have an intuitionistic set A where $a \in \{a\}, b \notin \{a\}$, but we cannot have any intuitionistic set $D = (D_1, D_2)$ such that $a \notin D_1, b \in D_1$.

Theorem 3.15. *Show that every intuitionistic T_2 space is also intuitionistic T_1 space. But the converse is not true.*

Proof of Theorem 3.15. Let $(X, \check{\mathcal{T}})$ be an intuitionistic T_2 space. Then $\forall a, b \in X$ with $a \neq b$ $\exists$ $G = (G_1, G_2)$ and $H = (H_1, H_2) \in \check{\mathcal{T}}$ such that $a \in G_1, b \notin G_1$ and $b \in H_1, a \notin H_1$ and $G \cap H = (\emptyset, A)$ where $A \subseteq X$.

So, for $G \in \check{\mathcal{T}}$ we get $a \in G_1, b \notin G_1$ and for $H \in \check{\mathcal{T}}$ we have $b \in H_1, a \notin H_1$. Thus, this topological space is also intuitionistic T_1 space. Hence, every intuitionistic T_2 space is also T_1 space. But, the converse is not true.

For example, consider $X = \{a, b, c\}$ and $A = (\{a, c\}, \{b\})$, $B = (\{b, c\}, \{a\})$, $C = (\{a, b\}, \{c\})$. Suppose $\check{\mathcal{T}}$ be an intuitionistic topology generated by $\{A, B, C\}$. Therefore, $(X, \check{\mathcal{T}})$ is an intuitionistic T_1 space. But this is not an intuitionistic T_2 space. Because we cannot have two nonempty intuitionistic open set D, E such that $D \cap E = (\emptyset, M)$, where $M \subseteq X$.

4. Conclusion

In this paper, we observe that intuitionistic topological space is more general that the classical topological space. We also see that separation axioms defined under intuitionistic set satisfy hereditary and topological property. Moreover, they are preserve under one-one and open mapping. Thus, it is certain that this study will add a new dimension of further investigation in intuitionistic topological space.

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