Topologizing a hyper BCI-algebra using its hyper-order

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Abstract. In this paper, we topologize a given hyper BCI-algebra using the hyper-order associated with the hyper-structure. Specifically, we consider a family of subsets of the hyper BCI-algebra that serves as a basis for some topology on this structure. Properties of the topological spaces generated in this way are subsequently investigated.
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1. Introduction

In 1934, F. Marty [3] introduced the hyperstructure theory (also called multi-valued algebras) at the 8th congress of Scandinavian Mathematicians. Xin [9] applied the concept of hyperstructure theory to a BCI-algebra, a class of abstract algebra introduced by Iséki [1] in 1966, and introduced the notion of a hyper BCI-algebra.

In [2], Y. B. Jun et al. initiated the study of topological BCI-algebras (briefly, TBCI-algebras) and some properties of this structure, and gave a characterization of a TBCI-algebra in terms of neighborhoods. They also gave a filter base Ω generating a BCI-topology, and made a BCI-algebra X into a TBCI-algebra for which Ω is a fundamental system of neighborhoods of 0. Moreover, they introduced the notions of topological subalgebras, topological ideals and topo-

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logical homomorphisms in topological BCI-algebras and studied some related properties.

In this study, we topologize a given hyper BCI-algebra by right application of the hyper-order associated with it. In this manner, we obtain a family of subsets of the hyper BCI-algebra that serves as a basis for some topology on this structure. Properties of the induced topological space generated in this way are investigated. This method of topologizing a hyperstructure had been previously utilized in [6] and [7].

2. Preliminaries

A *hyperoperation* on a nonempty set H is a map from $H \times H$ into $P^*(H) = P(H) \setminus \{\emptyset\}$. Let $\circledast$ be a hyperoperation on H and $(x, y) \in H \times H$. Then its image under $\circledast$, denoted by $x \circledast y$, is called the *hyperproduct* of x and y. If A and B are nonempty subsets of H, then $A * B$ is given by $A \circledast B = \bigcup_{a \in A, b \in B} a \circledast b$. We shall use $x \circledast y$ instead of $x \circledast \{y\}$, $\{x\} \circledast y$, or $\{x\} \circledast \{y\}$. When $A \subseteq H$ and $x \in H$, we agree to write $A \circledast x$ instead of $A \circledast \{x\}$. Similarly, we write $x \circledast A$ for $\{x\} \circledast A$. In effect, $A \circledast x = \bigcup_{a \in A} a \circledast x$ and $x \circledast A = \bigcup_{a \in A} x \circledast a$.

A *hyper BCI-algebra* $(H, \circledast, 0)$ is a nonempty set H endowed with a hyperoperation "$\circledast$" and a constant 0 satisfying the following axioms: for all $x, y, z \in H$,

(B_1) $((x \circledast z) \circledast (y \circledast z)) \ll x \circledast y$,

(B_2) $(x \circledast y) \circledast z = (x \circledast z) \circledast y$,

(B_3) $x \ll x$,

(B_4) $x \ll y$ and $y \ll x$ imply $x = y$,

(B_5) $0 \circledast (0 \circledast x) \ll x$, $x \neq 0$,

where for every $A, B \subseteq H$, $A \ll B$ if and only if for each $a \in A$, there exists $b \in B$ such that $0 \in a \circledast b$. In particular, for every $x, y \in H$, $x \ll y$ if and only if $0 \in x \circledast y$. In such case, we call "$\ll$" the *hyper order* in H (see [9]).

A hyper BCI-algebra $(H, \circledast, 0)$ is said to be *ordered* if for each x, y, $z \in H$, $x \ll y$ and $y \ll z$ implies $x \ll z$.

Example 2.1. [9] Let $H = \{0, 1, 2\}$. Define the hyperoperation "$\circledast$" by the Cayley table shown below.

$\circledast$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$

Then by routine calculations, $(H, \circledast, 0)$ is an ordered hyper BCI-algebra.

All throughout, we denote a hyper BCI-algebra $(H, \circledast, 0)$ by H, unless otherwise specified.

Let H be a hyper BCI-algebra and $A \subseteq H$. The sets $L_H(A)$ and $R_H(A)$ are given by $L_H(A) = \{x \in H \mid x \ll a, \forall\, a \in A\} = \{x \in H \mid 0 \in x \circledast a, \forall\, a \in A\}$ and $R_H(A) = \{x \in H \mid a \ll x, \forall\, a \in A\} = \{x \in H \mid 0 \in a \circledast x, \forall\, a \in A\}$. If $A = \{a\}$, we write $L_H(\{a\}) = L_H(a)$, and $R_H(\{a\}) = R_H(a)$.

An element a of a hyper BCI-algebra H is called a *hyperatom* if for each $x \in H$, $x \ll a$ implies $x = 0$ or $x = a$. Denote by $A(H)$ the set of all hyperatoms of H, and by $A^*(H)$ the set of all nonzero hyperatoms of H; that is, $A^*(H) = A(H) \setminus \{0\}$. A hyper BCI-algebra H is said to be *hyperatomic* if each element of H is a hyperatom, that is, $A(H) = H$ (see [5]).

Let X be a nonempty set and $p \in X$. The topology τ_p given by $\tau_p = \{\emptyset\} \cup \{A \subseteq X : p \in A\}$ is called the *particular point p topology* on X (see [8]).

Example 2.2. Consider the hyper BCI-algebra defined in Example 2.1. Then $R_H(0) = \{0, 1, 2\} = H$, $R_H(1) = \{1, 2\}$, $R_H(2) = \{2\}$, $R_H(\{0, 1\}) = \{1, 2\}$, $R_H(\{0, 2\}) = \{2\}$, $R_H(\{1, 2\}) = \{2\}$, and $R_H(H) = \{2\}$.

Example 2.3. Consider $H = \{0, a, b\}$ with the hyperoperation "$\circledast$" defined as follows:

$\circledast$	0	a	b
0	$\{0, a\}$	$\{0, a\}$	$\{b\}$
a	$\{a\}$	$\{0, a\}$	$\{b\}$
b	$\{b\}$	$\{b\}$	$\{0, a\}$

H is a hyper BCI-algebra [4]. By the definition of R_H, $R_H(0) = \{0, a\}$, $R_H(a) = \{a\}$, $R_H(b) = \{b\}$, $R_H(\{0, a\}) = \{a\}$, $R_H(\{0, b\}) = \emptyset$, $R_H(\{a, b\}) = \emptyset$, and $R_H(H) = \emptyset$.

3. Results

The following properties of a hyper BCI-algebra H are taken from [9].

Proposition 3.1 ([9])**.** *In any hyper BCI-algebra H, the following hold:*

(*i*) $x \ll 0$ *implies* $x = 0$.

(*ii*) $0 \in x \circledast (x \circledast 0)$.

(*iii*) $x \ll x \circledast 0$.

(*iv*) $0 \circledast (x \circledast y) \ll y \circledast x$.

(*v*) $A \ll A$.

(*vi*) $A \subseteq B$ *implies* $A \ll B$.

(*vii*) $A \ll \{0\}$ *implies* $A = \{0\}$.

(*viii*) *$x \circledast 0 \ll \{y\}$ implies $x \ll y$.*

(*ix*) *$y \ll z$ implies $x \circledast z \ll x \circledast y$.*

(*x*) *$x \circledast y = \{0\}$ implies $(x \circledast z) \circledast (y \circledast z) = \{0\}$ and $x \circledast z \ll y \circledast z$.*

(*xi*) *$A \circledast A = \{0\}$ implies A is a singleton.*

(*xii*) *$A \circledast \{0\} = \{0\}$ implies $A = \{0\}$,*

for all $x, y, z \in H$ and for all non-empty subsets A and B of H.

The next result gives some properties of the operator L_H.

Proposition 3.2 ([5])**.** *Let A and B be subsets of H. Then the following hold:*

(*i*) *$L_H(\emptyset) = H$.*

(*ii*) *$L_H(\{0\}) = \{0\}$.*

(*iii*) *If $A \subseteq B$, then $L_H(B) \subseteq L_H(A)$.*

(*iv*) *$L_H(A) = \bigcap_{a \in A} L_H(\{a\})$.*

(*v*) *If $x \in H$, then $x \in L_H(\{x\})$. Furthermore, $L_H(\{x\}) = \{0\}$ if and only if $x = 0$.*

Theorem 3.1. *Let A and B be subsets of a hyper BCI-algebra H. Then the following hold:*

(*i*) *$R_H(\emptyset) = H$.*

(*ii*) *If $A \subseteq B$, then $R_H(B) \subseteq R_H(A)$.*

(*iii*) *If H is ordered and $R_H(A) \neq \emptyset$, then $R_H(R_H(A)) \subseteq R_H(A)$.*

Proof of Theorem 3.1. Suppose $R_H(\emptyset) \neq H$. Then there exists $h \in H$ such that $h \notin R_H(\emptyset)$. Thus, there exists $a \in \emptyset$ such that $a \not\ll h$, which is absurd. Therefore, $R_H(\emptyset) = H$. This shows (*i*).

For (*ii*), let $x \in R_H(B)$. Then $b \ll x$ for all $b \in B$. Since $A \subseteq B$, $a \ll x$ for all $a \in A$. Thus, $x \in R_H(A)$. Hence, $R_H(B) \subseteq R_H(A)$.

Finally, let $x \in R_H(R_H(A))$. Then $b \ll x$ for all $b \in R_H(A)$. Since $a \ll b$ for all $a \in A$ and H is ordered, it follows that $a \ll x$ for all $a \in A$. Thus, $x \in R_H(A)$. Hence, $R_H(R_H(A)) \subseteq R_H(A)$. This proves (*iii*). □

Theorem 3.2. *Let H be a hyper BCI-algebra and $A \subseteq H$. Then*

(*i*) *$R_H(A) = \bigcap_{a \in A} R_H(a)$.*

(*ii*) *For any $x \in H$, $R_H(x) = \{z \in H : x \in L_H(z)\}$.*

(*iii*) *$R_H(x) \neq \emptyset$ for all $x \in H$. In particular, $x \in R_H(x)$.*

Proof of Theorem 3.2. Note that $R_H = \{x \in H | a \ll x, \forall\, a \in A\} = \{x \in H \mid x \in R_H(a), \forall\, a \in A\} = \bigcap_{a\in A} R_H(a)$. Hence, (i) holds.

For (ii), let $x \in H$ and let $D = \{z \in H : x \in L_H(z)\}$. If $z \in R_H(x)$, then $x \ll z$, that is, $x \in L_H(z)$. Thus, $z \in D$. On the other hand, let $z \in D$. Then $x \in L_H(z)$. Hence, $x \ll z$. That is, $z \in R_H(x)$.

By (B_3), $x \ll x$ for all $x \in H$. It follows that $x \in R_H(x)$. That is, (iii) holds. □

Corollary 3.1. *Let H be a hyper BCI-algebra and $A \subseteq H$. Then*

(i) *For any $x \in H \setminus \{0\}$, $0 \notin R_H(x)$. Consequently, for any $\emptyset \neq A \subseteq H$ such that $A \neq \{0\}$, $0 \notin R_H(A)$.*

(ii) *$R_H(0) = H$ if and only if $0 \in L_H(x)$ for all $x \in H$.*

Proof of Corollary 3.1. For (i), let $x \in H \setminus \{0\}$. By Theorem 3.2(ii), $0 \in R_H(x) = \{z \in H : x \in L_H(z)\}$ means that $x \in L_H(0) = \{0\}$ by Proposition 3.2(ii), a contradiction. Consequently, by Theorem 3.2(i), for any $\emptyset \neq A \subseteq H$ such that $A \neq \{0\}$, $0 \notin R_H(A)$.

For (ii), by Theorem 3.2(ii), $R_H(0) = \{z \in H : 0 \in L_H(z)\}$. Hence, $R_H(0) = H$ if and only if $0 \in L_H(x)$ for all $x \in H$. □

Lemma 3.1. *Let H be a hyper BCI-algebra and A a subset of H. Then $A \subseteq L_H(R_H(A))$. In particular, $L_H(R_H(H)) = H$.*

Proof of Lemma 3.1. Let $A \subseteq H$. If $R_H(A) = \emptyset$, then $A \subseteq H = L_H(\emptyset) = L_H(R_H(A))$ by Proposition 3.2(i). Suppose $R_H(A) \neq \emptyset$ and let $x \in R_H(A)$. Then $a \ll x$ for all $a \in A$. Let $b \in A$. Then $b \ll x$ for every $x \in R_H(A)$. This means that $b \in L_H(x)$ for every $x \in R_H(A)$. Hence, by Proposition 3.2(iv), $b \in \bigcap_{x\in R_H(A)} L_H(x) = L_H(R_H(A))$. Therefore, $A \subseteq L_H(R_H(A))$. In particular, since $L_H(R_H(H)) \subseteq H$, it follows that $L_H(R_H(H)) = H$. □

Theorem 3.3. *Let H be a hyper BCI-algebra with $|H| \geq 2$. Then $R_H(H)$ is a singleton if and only if there exists $a \in H \setminus \{0\}$ such that $L_H(a) = H$. In particular, $R_H(H) = \{a\}$.*

Proof of Theorem 3.3. Suppose that $R_H(H)$ is a singleton. By Corollary 3.1(i), $R_H(H) = \{a\}$ for some $a \in H \setminus \{0\}$. Then by Lemma 3.1, $H = L_H(R_H(H)) = L_H(a)$.

For the converse, suppose that there exists $a \in H\setminus\{0\}$ such that $L_H(a) = H$. Then $x \ll a$ for all $x \in H$. Hence, $a \in R_H(x)$ for all $x \in H$. This implies that $a \in \bigcap_{x\in H} R_H(x) = R_H(H)$ by Theorem 3.2(i). Thus, $\{a\} \subseteq R_H(H)$. Suppose further that there exists $b \in R_H(H) \setminus \{a\}$. Then $x \ll b$ for all $x \in H$. Since a, $b \in H$, it follows that $a \ll b$ and $b \ll a$. By (B_4), $a = b$, contrary to the assumption that $a \neq b$. Thus, $R_H(H) = \{a\}$. □

Theorem 3.4. *Let H be a hyper BCI-algebra. Then either $|R_H(H)| = 0$ or $|R_H(H)| = 1$.*

Proof of Theorem 3.4. *If $R_H(H)$ is a singleton, then we are done. Suppose that $|R_H(H)| \neq 1$. Let $a \in H$. Then by assumption and Theorem 3.3, $L_H(a) \neq H$. Hence, there exists $y \in H$ such that $y \not\ll a$, that is, there exists $y \in H$ such that $a \notin R_H(y)$. Thus, $a \notin R_H(H) = \bigcap_{z\in H} R_H(z)$. Since a is arbitrary, $R_H(H) = \emptyset$.* □

Lemma 3.2. *Let $\{A_\alpha : \alpha \in I\}$ be a collection of subsets of a hyper BCI-algebra H. Then $\bigcap_{\alpha\in I} R_H(A_\alpha) = R_H\left(\bigcup_{\alpha\in I} A_\alpha\right)$.*

Proof of Lemma 3.2. Suppose $\bigcap_{\alpha\in I} R_H(A_\alpha) = \emptyset$. Since $A_\alpha \subseteq \bigcup_{\alpha\in I} A_\alpha$, by Theorem 3.1, $R_H\left(\bigcup_{\alpha\in I} A_\alpha\right) \subseteq \bigcap_{\alpha\in I} R_H(A_\alpha) = \emptyset$. Thus, $R_H\left(\bigcup_{\alpha\in I} A_\alpha\right) = \emptyset$. Suppose $\bigcap_{\alpha\in I} R_H(A_\alpha) \neq \emptyset$. Then

$$\begin{aligned} x \in \bigcap_{\alpha\in I} R_H(A_\alpha) &\Leftrightarrow x \in R_H(A_\alpha) \text{ for all } \alpha \in I \\ &\Leftrightarrow a \ll x \text{ for all } a \in A_\alpha \text{ and for all } \alpha \in I \\ &\Leftrightarrow a \ll x \text{ for all } a \in \bigcup_{\alpha\in I} A_\alpha \\ &\Leftrightarrow x \in R_H\left(\bigcup_{\alpha\in I} A_\alpha\right). \end{aligned}$$

This proves the assertion. □

Theorem 3.5. *Let H be a hyper BCI-algebra. Then the family $\mathcal{B}_R(H) = \{R_H(A) : \emptyset \neq A \subseteq H\}$ is a basis for some topology on H.*

Proof of Theorem 3.5. Clearly, $H = \bigcup_{a\in H} R_H(a)$. Let A and B be nonempty subsets of H. Then by Lemma 3.2, $R_H(A) \cap R_H(B) = R_H(A \cup B) \in \mathcal{B}_R(H)$. Therefore, $\mathcal{B}_R(H)$ is a basis for some topology on H. □

Denote by $\tau_R(H)$ the topology generated by $\mathcal{B}_R(H)$.

Example 3.1. Consider $H := [0, \infty)$ with the hyperoperation "$\circledast$", defined in [9]:

$$x \circledast y := \begin{cases} [0, x], & \text{if } x \leq y, \\ (0, y], & \text{if } x > y \neq 0, \\ \{x\}, & \text{if } y = 0 \end{cases}$$

for all $x, y \in H$. Then $(H, \circledast, 0)$ is a hyper BCI-algebra. Now, let $k \in H$. Then $R_H(k) = [k, \infty)$. Let $\emptyset \neq A \subseteq H$. First, suppose that A is bounded, and let $p = \sup A$. By Theorem 3.2(i), $R_H(A) = \bigcap_{a\in A} R_H(a) = \bigcap_{a\in A}[a, \infty)$. Since $a \leq p$ for all $a \in A$, $[p, \infty) \subseteq R_H(A)$. Now, let $r \in R_H(A)$. Then $a \leq r$ for all $a \in A$; hence, $p \leq r$. It follows that $r \in [p, \infty)$. Thus, $R_H(A) = [p, \infty)$. Suppose that A is not bounded. Let $m \in H$. Since A is not bounded, there exists $z \in A$ such that $m < z$. This implies that $m \notin R_H(z)$. Thus, $m \notin \bigcap_{w\in A} R_H(w) = R_H(A)$.

Hence, $R_H(A) = \emptyset$. Accordingly, $\mathcal{B}_R(H) = \{R_H(A) : A \subseteq H\} = \{R_H(p) : p \in H\} = \{[p, \infty) : p \in H\}$.

Next, let $G \in \tau_R(H) \setminus \{\emptyset\}$. Then $G = \bigcup_{a \in \mathcal{K}} R_H(a)$ for some $\mathcal{K} \subseteq H$. Since $a \in R_H(a) \subseteq G$ for all $a \in \mathcal{K}$, it follows that $\mathcal{K} \subseteq G$. Let $q = \inf G$ and consider the following cases:

Case 1. Suppose $q \in G$. Since $q \leq a$ for all $a \in \mathcal{K}$, it follows that $R_H(a) \subseteq R_H(q)$ for all $a \in \mathcal{K}$. This implies that $G \subseteq R_H(q)$. Further, since $q \in G$, there exists $a \in \mathcal{K}$ such that $q \in R_H(a)$. This implies that $a \leq q$. Since $q = \inf G$, $a = q$. Thus, $R_H(a) = R_H(q) \subseteq G$, showing that $G = R_H(q)$.

Case 2. Suppose $q \notin G$. Since $q \leq a$ for all $a \in \mathcal{K}$, $R_H(a) \subseteq (q, \infty)$ for all $a \in \mathcal{K}$. Hence, $G \subseteq (q, \infty)$. Now, let $p \in (q, \infty)$. Then $q < p$. This implies that there exists $z \in G$ such that $q < z < p$. Let $b \in \mathcal{K}$ such that $z \in R_H(b)$. Then $q < b \leq z < p$. Thus, $p \in R_H(b) \subseteq G$. Therefore, $G = (q, \infty)$.

Accordingly, $\tau_R(H) = \{\emptyset\} \cup \{R_H(x) : x \in H\} \cup \{(x, \infty) : x \in H\}$.

Example 3.2. Consider the hyper BCI-algebra defined in Example 2.3. Then by Theorem 3.5, $\mathcal{B}_R(H) = \{R_H(A) : A \subseteq H\} = \{\{0, a\}, \{a\}, \{b\}, \emptyset\}$. Thus, $\tau_R(H) = \{\emptyset, H, \{0, a\}, \{a\}, \{b\}, \{a, b\}\}$.

Observe that in Example 3.1, $(H, \tau_R(H))$ is connected, however, in Example 3.2, $H = \{0, a\} \cup \{b\}$. Hence, $(H, \tau_R(H))$ is disconnected.

Theorem 3.6. *Let H be a hyper BCI-algebra. If $R_H(0) = H$, then the topological space $(H, \tau_R(H))$ is connected.*

Proof of Theorem 3.6. Suppose that $R_H(0) = H$. It follows from Corollary 3.1(i) that there does not exist $G \in \tau_R(H) \setminus \{H\}$ such that $0 \in G$. Consequently, H cannot have a decomposition. Thus, $(H, \tau_R(H))$ is connected. □

Theorem 3.7. *If H is a finite hyper BCI-algebra, then the family $\mathcal{S}_R(H) = \{R_H(a) : a \in H\}$ is a subbase of $\tau_R(H)$.*

Proof of Theorem 3.7. Clearly, $\mathcal{S}_R(H) \subseteq \tau_R(H)$. By Theorem 3.2(i) and the assumption that H is finite, it follows that every element of $\mathcal{B}_R(H) \setminus \{\emptyset\}$ is a finite intersection of members of $\mathcal{S}_R(H)$. Therefore, $\mathcal{S}_R(H)$ is a subbase of $\tau_R(H)$. □

Remark 3.1. The converse of Theorem 3.7 is not always true. Consider Example 3.1. Notice that $\mathcal{B}_R(H) = \{R_H(A) : \emptyset \neq A \subseteq H\} = \{R_H(p) : p \in H\}$ is also a subbase of $\tau_R(H)$. However, H is an infinite hyper BCI-algebra.

Theorem 3.8. *Let H be a hyper BCI-algebra. The following statements are equivalent:*

(i) *H is hyperatomic.*

(ii) *$R_H(a) = \{a\}$ for all $a \in H \setminus \{0\}$.*

(iii) $a \notin R_H(A)$ *for all* $a \in H \setminus \{0\}$ *and* $A \notin \mathscr{P}(\{0,a\})$.

Proof of Theorem 3.8. $(i) \Rightarrow (ii)$ Suppose that H is hyperatomic. Let $a \in H \setminus \{0\} = A^*(H)$. Suppose further that $R_H(a) \neq \{a\}$ and let $x \in R_H(a) \setminus \{a\}$. Then $a \ll x$. Since a is a hyperatom, $a = 0$ or $a = x$, a contradiction. Hence, $R_H(a) = \{a\}$ for all $a \in H \setminus \{0\}$.

$(ii) \Rightarrow (iii)$ Suppose that $R_H(a) = \{a\}$ for all $a \in H \setminus \{0\}$. Let $b \in H \setminus \{0\}$ and let $A \in \mathscr{P}(H) \setminus \mathscr{P}(\{0,b\})$. Suppose that $b \in R_H(A)$. Then $x \ll b$ for all $x \in A$. Pick $x \in A \setminus \{b\}$. Then $b \in R_H(x)$, a contradiction. Hence, $b \notin R_H(A)$. This shows that (iii) holds.

$(iii) \Rightarrow (i)$ Let $b \in H \setminus \{0\}$. Then $b \notin R_H(A)$ for all $A \in \mathscr{P}(H) \setminus \mathscr{P}(\{0,b\})$ by assumption. Suppose that $x \ll b$. Then $b \in R_H(x)$. By assumption, $x \in \{0,b\}$. Hence, $x = 0$ or $x = b$. Therefore, H is hyperatomic. □

Corollary 3.2. *If* H *is a hyperatomic hyper BCI-algebra such that* $R_H(0) = H$*, then* $R_H(\{x,0\}) = \{x\}$ *for all* $x \in H \setminus \{0\}$.

Proof of Corollary 3.2. Let $x \in H \setminus \{0\}$. By Theorem 3.2(i) and Theorem 3.8, $R_H(\{x,0\}) = R_H(x) \cap R_H(0) = \{x\} \cap H = \{x\}$. □

Lemma 3.3. *Let* H *be a hyper BCI-algebra. If* H *is hyperatomic, then* $\mathcal{B}_R(H) = \{R_H(0)\} \cup \{\{a\} : a \in H \setminus \{0\}\}$.

Proof of Lemma 3.3. Suppose that H is hyperatomic and let $\emptyset \neq A \subseteq H$. Now, suppose that $|A \cap A^*(H)| \geq 2$. Let $a, b \in A \cap A^*(H)$ with $a \neq b$. Then by Theorem 3.1(ii) and Theorem 3.8, $R_H(A) = \emptyset$. Consequently,
$\mathcal{B}_R(H) = \{R_H(0)\} \cup \{R_H(a) : a \in H \setminus \{0\}\} = \{R_H(0)\} \cup \{\{a\} : a \in H \setminus \{0\}\}$.
This proves the assertion. □

Theorem 3.9. *Let* H *be a hyperatomic hyper BCI-algebra and* $A \subseteq H$*. Then* $A \in \tau_R(H)$ *if and only if* $0 \notin A$ *or* $R_H(0) \subseteq A$.

Proof of Theorem 3.9. Suppose $A \in \tau_R(H)$ and suppose that $0 \in A$. Then, by Lemma 3.3, $R_H(0) \subseteq A$.

For the converse, suppose first that $0 \notin A$. Then $\{a\} \in \mathcal{B}_R(H)$ for all $a \in A$ by Lemma 3.3. Hence, $A = \bigcup_{a \in A} \{a\} \in \tau_R(H)$. Next, suppose that $R_H(0) \subseteq A$. If $A = R_H(0)$, then $A \in \tau_R(H)$. Suppose that $A \neq R_H(0)$. Let $D = A \setminus R_H(0)$. Since $\{a\} \in \mathcal{B}_R(H)$ for all $a \in D$, it follows that $A = \left(\bigcup_{a \in D} \{a\}\right) \cup R_H(0) \in \tau_R(H)$. □

Corollary 3.3. *Let* H *be a hyperatomic hyper BCI-algebra and* $F \subseteq H$*. Then* F *is* $\tau_R(H)$*-closed if and only if* $0 \in F$ *or* $R_H(0) \cap F = \emptyset$.

Proof of Corollary 3.3. Suppose that F is $\tau_R(H)$-closed. Then $F^c \in \tau_R(H)$. Thus, $0 \notin F^c$ or $R_H(0) \subseteq F^c$ by Theorem 3.9. This implies that $0 \in F$ or $R_H(0) \cap F = \emptyset$.

Next, suppose that $0 \in F$. Then $0 \notin F^c$. If $R_H(0) \cap F = \emptyset$, then $R_H(0) \subseteq F^c$. In both cases, we find that $F^c \in \tau_R(H)$, that is, F is $\tau_R(H)$-closed. □

Theorem 3.10. *Let H be a hyperatomic hyper BCI-algebra with $|H| \geq 2$. Then $(H, \tau_R(H))$ is connected if and only if $R_H(0) = H$.*

Proof of Theorem 3.10. Suppose $(H, \tau_R(H))$ is connected. Suppose further that $R_H(0) \neq H$. Then $A = R_H(0)$ and $B = H \setminus R_H(0)$ are nonempty disjoint $\tau_R(H)$-open sets by Theorem 3.9. Further, since $A \cup B = H$, $A \cup B$ is a decomposition of H, contrary to our assumption that $(H, \tau_R(H))$ is connected. Therefore, $R_H(0) = H$.

Conversely, suppose that $R_H(0) = H$. Suppose further that $(H, \tau_R(H))$ is disconnected. Then there exist nonempty disjoint $\tau_R(H)$-open sets P and Q such that $H = P \cup Q$. Assume, without loss of generality, that $0 \in P$. Then, by Theorem 3.9, $R_H(0) \subseteq P$. Since $R_H(0) = H$, $P = H$ and $Q = \emptyset$, a contradiction. Therefore, $(H, \tau_R(H))$ is connected. $\square$

Theorem 3.11. *Let H be hyperatomic hyper BCI-algebra such that $(H, \tau_R(H))$ is connected. Then $\tau_R(H) = \tau^0(H) = \{H\} \cup \{A \subseteq H : 0 \notin A\}$.*

Proof of Theorem 3.11. By Lemma 3.3 and Theorem 3.10, $\mathcal{B}_R(H) = \{H\} \cup \{\{a\} : a \in H \setminus \{0\}\}$. Let $G \in \tau_R(H)$. Then $G = H$ or $G = \emptyset$ or G is the union of singletons $\{a\}$ where $a \in H \setminus \{0\}$. Thus, $0 \notin G \neq H$. Therefore, $\tau_R(H) = \tau^0(H)$. $\square$

Theorem 3.12. *Let H be an ordered hyper BCI-algebra and $D \subseteq H$. Then D is dense if and only if for each $x \in H \setminus D$, there exists $y \in D$ such that $x \ll y$.*

Proof of Theorem 3.12. Suppose that D is dense in H and let $x \in H \setminus D$. Then $x \in \overline{D} = H$. Since by Theorem 3.2(*iii*), $x \in R_H(x) \in \tau_R(H)$, we have $R_H(x) \cap D \neq \emptyset$. Let $y \in R_H(x) \cap D$. Then $y \in D$ and $x \ll y$.

Conversely, let $z \in H \setminus D$ and let $\emptyset \neq A \subseteq H$ such that $z \in R_H(A)$. Then $a \ll z$ for all $a \in A$. Since $z \in H \setminus D$, by assumption, there exists $y \in D$ such that $z \ll y$. Since H is ordered, it follows that $a \ll y$ for all $a \in A$. Consequently, $y \in R_H(A)$ implying that $R_H(A) \cap D \neq \emptyset$. Hence, $z \in \overline{D}$. Since z was arbitrarily chosen, it follows that D is dense in H. $\square$

Theorem 3.13. *Let H be a hyperatomic hyper BCI-algebra and $A, F \subseteq H$. Then with respect to $\tau_R(H)$,*

(*i*)

$$int(A) = \begin{cases} A, & \text{if } 0 \notin A \text{ or } R_H(0) \subseteq A \\ A \setminus \{0\}, & \text{otherwise} \end{cases}$$

(*ii*)

$$\overline{F} = \begin{cases} F, & \text{if } 0 \in F \text{ or } R_H(0) \cap F = \emptyset \\ F \cup \{0\}, & \text{otherwise} \end{cases}$$

(*iii*) *F is dense if and only if $F = H$ or $F = H \setminus \{0\}$ and $R_H(0) \neq \{0\}$.*

Proof of Theorem 3.13. (*i*) If $0 \notin A$ or $R_H(0) \subseteq A$, then $A \in \tau_R(H)$ by Theorem 3.9. It follows that $intA = A$. Suppose that $A \notin \tau_R(H)$. Then $0 \in A$ and $A \setminus \{0\} \in \tau_R(H)$ by Theorem 3.9. Thus, $intA = A \setminus \{0\}$.

(*ii*) If $0 \in F$ or $R_H(0) \cap F = \emptyset$, then F is $\tau_R(H)$-closed by Corollary 3.3. Hence, $\overline{F} = F$. Suppose that F is not $\tau_R(H)$-closed. Then $0 \notin F$ and $F \cup \{0\}$ is $\tau_R(H)$-closed by Corollary 3.3. Therefore, $\overline{F} = F \cup \{0\}$.

(*iii*) Suppose that F is dense and $F \neq H$. Since F is not $\tau_R(H)$-closed, $0 \notin F$ and $R_H(0) \cap F \neq \emptyset$ by Corollary 3.3. Hence, $R_H(0) \neq \{0\}$ and $\overline{F} = H = F \cup \{0\}$ by (*ii*). Therefore, $F = H \setminus \{0\}$.

Conversely, suppose that $F = H \setminus \{0\}$ and $R_H(0) \neq \{0\}$. Since $R_H(0) \neq \{0\}$ and $0 \notin F$, F is not $\tau_R(H)$-closed. Hence, $\overline{F} = H$ by (*ii*). □

Lemma 3.4. *Let K be a hyper subalgebra of a hyper BCI-algebra H and let $D \subseteq H$. Then $R_H(D) \cap K \subseteq R_K(D \cap K)$. Moreover, if $D \subseteq K$, then $R_H(D) \cap K = R_K(D)$.*

Proof of Lemma 3.4. Let $D \subseteq H$ and let $z \in R_H(D) \cap K$. Then $z \in K$ and $d \ll z$ for all $d \in D$. In particular, $d \ll z$ for all $d \in D \cap K$. Thus, $z \in R_K(D \cap K)$. Hence, $R_H(D) \cap K \subseteq R_K(D \cap K)$.

If $D \subseteq K$, then $R_H(D) \cap K \subseteq R_K(D \cap K) = R_K(D)$. Now, let $z \in R_K(D)$. Then $z \in K$ and $d \ll z$ for all $d \in D \subseteq K \subseteq H$. Hence, $z \in R_H(D) \cap K$. Therefore, $R_H(D) \cap K = R_K(D)$. □

Lemma 3.5. *Let K be a hyper subalgebra of an ordered hyper BCI-algebra H. Then for any $\emptyset \neq A \subseteq H$, $R_H(A) \cap K = \bigcup_{x \in R_H(A) \cap K} R_K(x)$.*

Proof of Lemma 3.5. Let $x \in R_H(A) \cap K$ where $\emptyset \neq A \subseteq H$. Then $a \ll x$ for all $a \in A$ and $x \in K$. Let $y \in R_H(x) \cap K$. Then $x \ll y$ and $y \in K$. Since H is ordered, $a \ll y$ for all $a \in A$. Hence, $y \in R_H(A) \cap K$. That is, $R_H(x) \cap K \subseteq R_H(A) \cap K$. Consequently, $\bigcup_{x \in R_H(A) \cap K} (R_H(x) \cap K) \subseteq R_H(A) \cap K$.

Now, let $w \in R_H(A) \cap K$. By Theorem 3.2(iii), $w \in R_H(w)$. It follows that $w \in R_H(w) \cap K$. Hence, $R_H(A) \cap K \subseteq R_H(w) \cap K$. Thus, we have $R_H(A) \cap K \subseteq \bigcup_{x \in R_H(A) \cap K} (R_H(x) \cap K)$. Therefore, by Lemma 3.4,

$$R_H(A) \cap K = \bigcup_{x \in R_H(A) \cap K} (R_H(x) \cap K) = \bigcup_{x \in R_H(A) \cap K} R_K(x).$$

This proves the assertion. □

Theorem 3.14. *Let K be a hyper subalgebra of an ordered hyper BCI-algebra H. Then $\tau_R(K)$ coincides with the relative topology τ_K on K.*

Proof of Theorem 3.14. By Theorem 3.5 and Theorem 3.1, the families $\mathcal{B}_R(K) = \{R_K(A) : \emptyset \neq A \subseteq K\}$ and $\mathcal{B}_K = \{R_H(A) \cap K : \emptyset \neq A \subseteq H\}$ are bases for $\tau_R(K)$ and the relative topology τ_K, respectively.

Let $U \in \mathcal{B}_K$ and let $x \in U$. Then there exists $A \subseteq H$ such that $U = R_H(A) \cap K$. Since $x \in R_H(x)$ and $x \in K$, $x \in R_H(x) \cap K = R_K(x)$ by Lemma 3.4. Since H is ordered, $R_K(x) \subseteq R_H(A) \cap K$, by Lemma 3.5. Take $U' = R_K(x)$. Then for each $U \in \mathcal{B}_K$ and $x \in U$, there exists $U' \in \mathcal{B}_R(K)$ such that $x \in U' \subseteq U$. Thus, $\tau_K \subseteq \tau_R(K)$.

On the other hand, let $U \in \mathcal{B}_R(K)$. Then there exists $B \subseteq K$ such that $U = R_K(B)$. By Lemma 3.4, $U = R_K(B) = R_H(B) \cap K \in \mathcal{B}_K$. Hence, $\mathcal{B}_R(K) \subseteq \mathcal{B}_K$, that is, $\tau_L(K) \subseteq \tau_K$. Consequently, $\tau_R(K) = \tau_K$. □

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References

[1] K. Iséki, *An algebra related with a propositional calculus*, Proceedings of the Japan Academy, 42 (1966), 26-29.

[2] Y.B Jun, X.L. Xin, D.S. Lee, *On topological BCI-algebras*, Inform. Sci., 116 (1999), 253-261.

[3] F. Marty, *Sur une generalization de la notion de groupe*, 8th Congress Math. Scandinaves, Stockholm, 1934, 45-49.

[4] F. Nisar, R.S. Tariq, S.A. Bhatti, *Fuzzy ideals in hyper BCI-algebras*, World Applied Science Journal, 12 (2012), 1771-1777.

[5] M.T. Panganduyon, S.R. Canoy Jr., *On a graph induced by a hyper BCI-algebra*, European Journal of Pure and Applied Mathematics, 12 (2019), 146-158.

[6] R. Patangan, S.R. Canoy Jr., *A topology on a hyper BCK-algebra via left application of a hyper order*, JP Journal of Algebra, Number Theory and Applications, 40 (2018), 321-332.

[7] R. Patangan, S.R. Canoy Jr., *A topology on a hyper BCK-algebra*, JP Journal of Algebra, Number Theory and Applications, 40 (2018), 787-797.

[8] L.A. Steen, J.A. Seebach Jr., *Counterexamples in topology*, Berlin, New York: Springer-Verlag, 1978.

[9] X.L. Xin, *Hyper BCI-algebras*, Discussiones Mathematicae, 26 (2006), 5-19.

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