Harmonization of some fuzzy subgroups

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Abstract. This paper studies a new concept, Harmonized Fuzzy Groups (HFGs), from which different types of improvements on Rosenfeld's fuzzy groups (such as Intuitionistic Fuzzy Groups (IFGs), Pythagorean Fuzzy Groups (PFGs) and a new kind of fuzzy group presented in this paper, which is called Fermatean Fuzzy Groups (FFGs)) can be realized. Also, some properties of HFGs are studied.

Keywords: harmonized fuzzy sets, harmonized fuzzy groups, harmonized fuzzy level subgroup.

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1. Introduction

Due to the failure of the classical set to address vagueness and uncertainties, Zadeh [17] introduced the notion of fuzzy sets. These sets have become acceptable and very useful among many researchers in Mathematics, Computer Science, Engineering and other areas. Fuzzy set is able to accommodate the element in a set to some degree values between $[0,1]$. Many works have been done in this area by many scientists. In order to put algebraic structure on these sets as we have in the case of classical group, Rosenfeld [12] developed the idea of fuzzy subgroups and fuzzy ideals. This became a very interesting area to many mathematicians. Mukherjee and Bhattacharya [6] introduced the notion of fuzzy normal subgroups and fuzzy cosets. Some results on fuzzy subgroups were also obtained by Onasanya and Ilori [7, 8, 9, 10].

However, Atanassov [1] demonstrated that fuzzy sets do not completely capture the imprecision in vagueness. A situation where the degree of indeterminacy is involved cannot be fully represented by Zadeh's fuzzy sets. In fuzzy sets, for a non empty set X, the degree of existence of an element $u \in X$ is represented by $\vartheta(u) \in [0,1]$, its complement, the degree of nonmembership, can be $\nu(u) = 1 - \vartheta(u)$. In this case, $\vartheta(u) + \nu(u) = 1$. This has assumed no possibility of some hesitations towards either membership or nonmembership, which usually is not the case in real life. But, Atanassov demonstrated the possibility of having $\vartheta(u) + \nu(u) \leq 1$. In that case, $\pi(u) = 1 - (\vartheta(u) + \nu(u))$ is referred to as the degree of indeterminacy of $u \in X$. Hence, the triple $(u, \vartheta(u), \nu(u))$ is referred to as Intuitionistic Fuzzy Sets (IFSs).

Furthermore, Biswas [2], like Rosenfeld, developed some group structures on the IFSs of Atanassov. He also studied some of the properties. Meanwhile, Yager [14, 15] has demonstrated the possibility of $\vartheta(u) + \nu(u) \geq 1$, in which case, the inequality was replaced with $\vartheta^2(u)+\nu^2(u) \leq 1$. In this case, if $\vartheta(u) = a$, $\vartheta^2(u) = a^2$. The triple $(u, \vartheta(u), \nu(u))$ is referred to as Pythagorean Fuzzy Sets (PFSs). The measure of indeterminacy of $u \in X$ is $\pi(u) = \sqrt{1 - (\vartheta^2(u) + \nu^2(u))}$. It turned out that every IFS is a PFS but it is not true conversely. Besides, Yager [16] has extended this set to a case of $\vartheta^n(u) + \nu^n(u) \leq 1$ and $\pi(u) = \sqrt[n]{1 - (\vartheta^n(u) + \nu^n(u))}$, where $n \geq 1$.

Recently, Bhunia and Ghoria [3] put group structure on the PFSs and developed Pythagorean Fuzzy Subgroups. They studied some other algebraic properties of these new structures as in the case of fuzzy subgroups.

As always the case, some limitations of PFSs were found and demonstrated by Senapati and Yager [13]. The case of $\vartheta^2(u) + \nu^2(u) \geq 1$ was brought to bear. Thus, the condition $\vartheta^2(u) + \nu^2(u) \leq 1$ was replaced by $\vartheta^3(u) + \nu^3(u) \leq 1$, in which case, the extent of indeterminacy is $\pi(u) = \sqrt[3]{1 - (\vartheta^3(u) + \nu^3(u))}$. Hence, triple $(u, \vartheta(u), \nu(u))$ is referred to as Fermatean Fuzzy Sets (FFSs). This is a special case of the work of Yager [16].

In this paper, $\vartheta^n(u) + \nu^n(u) \leq 1$ was considered and some group structures were put on the set. The fuzzy subgroup so obtained is called Harmonized Fuzzy Group (HFG). Finally, some of the properties of HFG were studied.

The remaining part of this paper is as follows: Section 2 contains the Preliminaries; Section 3 has the Harmonized fuzzy structure. Section 4 contains the level subgroup of the Harmonized fuzzy structure and then the Conclusion in Section 5.

2. Preliminaries

Definition 2.1 ([17]). *A fuzzy set A in X is a set of ordered pairs $A = \{(u, \vartheta_A(u)) : u \in X\}$, where $\vartheta_A(u)$ is the grade of membership of $u \in A$ and $\vartheta_A : X \longrightarrow [0,1]$ is the membership function.*

Definition 2.2 ([17]). *Let $A = \{(u, \vartheta_A(u)) : u \in X\}$ be a fuzzy set. The complement of A is defined as $A' = \{(u, \nu_A(u)) : u \in X\} = \{u \in X : \nu_A(u) = 1 - \vartheta_A(u)\}$.*

Definition 2.3 ([12]). *Let X be a group and ϑ a fuzzy subset of X. Then ϑ is called a fuzzy subgroup of X if, for any $u, v \in X$,*

$$\vartheta(uv) \geq \min\{\vartheta(u), \vartheta(v)\}$$

and

$$\vartheta(u^{-1}) \geq \vartheta(u).$$

Remark 2.1. Let ϑ and ν be fuzzy subgroups of a group X. Their intersection $\vartheta \cap \nu = \min\{\vartheta, \nu\}$ is a fuzzy subgroup [12]. But their union $\vartheta \cup \nu = \max\{\vartheta, \nu\}$ is not usually a fuzzy subgroup [4, 5]. Furthermore, let ϑ and ν be two antifuzzy subgroups of a group X. It can easily be shown that their intersection $\vartheta \cap \nu = \max\{\vartheta, \nu\}$ is an antifuzzy subgroup. Also, their union $\vartheta \cup \nu = \min\{\vartheta, \nu\}$ is an antifuzzy subgroup [7].

Theorem 2.1 ([12]). *Let X be a group and ϑ a fuzzy subset of X. Then ϑ is a fuzzy subgroup of X if and only if $\vartheta(uv^{-1}) \geq \min\{\vartheta(u), \vartheta(v)\}$ for any $u, v \in X$.*

Definition 2.4 ([1]). *An intuitionistic fuzzy set A in X is defined as*

$$A = \{\langle u, \vartheta_A(u), \nu_A(u)\rangle | u \in X\},$$

where

$$\vartheta_A : X \longrightarrow [0,1] \text{ and } \nu_A : X \longrightarrow [0,1]$$

are respectively degree of membership and degree of non-membership for every $u \in X$ with

$$0 \leq \vartheta_A(u) + \nu_A(u) \leq 1$$

and

$$\pi(u) = 1 - (\vartheta(u) + \nu(u))$$

is the degree of indeterminacy of $u \in X$.

Definition 2.5 ([2]). *Let X be a group and $A = \{\langle u, \vartheta_A(u), \nu_A(u)\rangle | u \in X\}$ an IFS. Then, A is an intuitionistic fuzzy group (IFG) if for any $u, v \in X$, the following are satisfied:*

(i) $\vartheta_A(uv) \geq \min\{\vartheta_A(u), \vartheta_A(v)\}$

(ii) $\vartheta_A(u^{-1}) \geq \vartheta_A(u)$

(iii) $\nu_A(uv) \leq \max\{\nu_A(u), \nu_A(v)\}$

(iv) $\nu_A(u^{-1}) \leq \nu_A(u)$

Proposition 2.1 ([2]). *Let X be a group and $A = \{\langle u, \vartheta_A(u), \nu_A(u)\rangle | u \in X\}$ an IFS. Then, A is an intuitionistic fuzzy group (IFG) if and only if for any $u, v \in X$, the following are satisfied:*

(i) $\vartheta_A(uv^{-1}) \geq \min\{\vartheta_A(u), \vartheta_A(v)\}$

(ii) $\nu_A(uv^{-1}) \leq \max\{\nu_A(u), \nu_A(v)\}$

Definition 2.6 ([14]). *A Pythagorean fuzzy set A in X is defined as*

$$A = \{\langle u, \vartheta_A(u), \nu_A(u)\rangle | u \in X\},$$

where

$$\vartheta_A : X \longrightarrow [0,1] \text{ and } \nu_A : X \longrightarrow [0,1]$$

are respectively degree of membership and degree of non-membership for every $u \in X$ with

$$0 \leq \vartheta_A^2(u) + \nu_A^2(u) \leq 1$$

and degree of indeterminacy of $u \in X$ is $\pi(u) = \sqrt{1 - (\vartheta^2(u) + \nu^2(u))}$.

Definition 2.7 ([3]). *Let X be a group and $A = \{\langle u, \vartheta_A(u), \nu_A(u)\rangle | u \in X\}$ a PFS. Then, A is a Pythagorean fuzzy group (PFG) if for any $u, v \in X$, the following are satisfied:*

(i) $\vartheta_A^2(uv) \geq \min\{\vartheta_A^2(u), \vartheta_A^2(v)\}$

(ii) $\vartheta_A^2(u^{-1}) \geq \vartheta_A^2(u)$

(iii) $\nu_A^2(uv) \leq \max\{\nu_A^2(u), \nu_A^2(v)\}$

(iv) $\nu_A^2(u^{-1}) \leq \nu_A^2(u)$

Proposition 2.2 ([3]). *Let X be a group and $A = \{\langle u, \vartheta_A(u), \nu_A(u)\rangle | u \in X\}$ a PFS. Then, A is a Pythagorean fuzzy group (PFG) if and only if for any $u, v \in X$,*

(i) $\vartheta_A^2(uv^{-1}) \geq \min\{\vartheta_A^2(u), \vartheta_A^2(v)\}$

(ii) $\nu_A^2(uv^{-1}) \leq \max\{\nu_A^2(u), \nu_A^2(v)\}$.

Definition 2.8 ([16]). *A generalized orthopair fuzzy set A in X is defined as*

$$A = \{\langle u, \vartheta_A(u), \nu_A(u)\rangle | u \in X\},$$

where

$$\vartheta_A : X \longrightarrow [0,1] \text{ and } \nu_A : X \longrightarrow [0,1]$$

are respectively degree of membership and degree of non-membership for every $u \in X$ with

$$0 \leq \vartheta_A^n(u) + \nu_A^n(u) \leq 1$$

and degree of indeterminacy of $u \in X$ is $\pi(u) = \sqrt[n]{1 - (\vartheta^n(u) + \nu^n(u))}$.

Remark 2.2. A special case of generalized orthopair fuzzy set for $n = 3$ was considered by [13]. In this paper, the Generalized Orthopair Fuzzy Sets in a non-empty set X are referred to as Harmonized Fuzzy Sets (HFSs) in X.

3. Harmonized fuzzy subgroups

Definition 3.1. *Let X be a group and $A = \{\langle u, \vartheta_A(u), \nu_A(u)\rangle | u \in X\}$ a HFS. Then, A is a harmonized fuzzy group (HFG), if for any $u, v \in X$ the following are satisfied:*

(i) $\vartheta_A^n(uv) \geq \min\{\vartheta_A^n(u), \vartheta_A^n(v)\}$

(ii) $\vartheta_A^n(u^{-1}) \geq \vartheta_A^n(u)$

(iii) $\nu_A^n(uv) \leq \max\{\nu_A^n(u), \nu_A^n(v)\}$

(iv) $\nu_A^n(u^{-1}) \leq \nu_A^n(u)$

Remark 3.1. For convenience we write ϑ^n and ν^n respectively for ϑ_A^n and ν_A^n, where $\vartheta^n = \underbrace{\vartheta \cdot \vartheta \cdots \vartheta}_{n\ times}$, $\nu^n = \underbrace{\nu \cdot \nu \cdots \nu}_{n\ times}$, if no confusion will arise. Note that for

$$\vartheta : X \longrightarrow [0,1] \text{ and } \nu : X \longrightarrow [0,1],$$
$$0 \leq \vartheta^n + \nu^n \leq \cdots \leq \vartheta^3 + \nu^3 \leq \vartheta^2 + \nu^2 \leq \vartheta + \nu \leq 1$$

when ϑ and ν are respectively membership and non-membership functions in HFS. Some operations on the HFS $A = \langle u, \vartheta, \nu\rangle$ can be obtained from [16]. These also could apply to the HFG $A = \langle u, \vartheta, \nu\rangle$.

The HFG

$$A = \{\langle u, \vartheta_A(u), \nu_A(u)\rangle | u \in X\}$$

is an IFG when $n = 1$ in which case ϑ_A is a fuzzy subgroup of X and ν_A is antifuzzy subgroup. It is a PFG when $n = 2$. Also, it is a Fermatean Fuzzy Group when $n = 3$ since the underlying fuzzy set is Fermatean Fuzzy Set.

Proposition 3.1. *Let X be a group and $A = \{\langle u, \vartheta(u), \nu(u)\rangle | u \in X\}$ a HFG. Then, for every $u \in X$*

(*i*) $\vartheta^n(u^{-1}) = \vartheta^n(u)$

(*ii*) $\nu^n(u^{-1}) = \nu^n(u)$

(*iii*) $\vartheta^n(e) \geq \vartheta^n(u)$

(*iv*) $\nu^n(e) \leq \nu^n(u)$

Proof. We begin the proof.

(*i*) To prove $\vartheta^n(u^{-1}) = \vartheta^n(u)$:
$\vartheta^n(u) = \vartheta^n((u^{-1})^{-1}) \geq \vartheta^n(u^{-1}) \geq \vartheta^n(u)$.

(*ii*) To prove $\nu_A^n(u^{-1}) = \nu_A^n(u)$:
$\nu^n(u) = \nu^n((u^{-1})^{-1}) \leq \nu^n(u^{-1}) \leq \nu^n(u)$.

(*iii*) To prove $\vartheta^n(e) \geq \vartheta^n(u)$:
$\vartheta^n(e) = \vartheta^n(uu^{-1}) \geq \min\{\vartheta^n(u), \vartheta^n(u^{-1})\} = \vartheta^n(u)$.

(*iv*) To prove $\nu^n(e) \leq \nu^n(u)$:
$\nu^n(e) = \nu^n(uu^{-1}) \leq \max\{\nu^n(u), \nu^n(u^{-1})\} = \nu^n(u)$. □

Proposition 3.2. *Let X be a group and $A = \{\langle u, \vartheta_A(u), \nu_A(u)\rangle | u \in X\}$ a HFS. Then, A is a harmonized fuzzy group (HFG) if and only if, for any $u, v \in X$*

(*i*) $\vartheta^n(uv^{-1}) \geq \min\{\vartheta^n(u), \vartheta^n(v)\}$

(*ii*) $\nu^n(uv^{-1}) \leq \max\{\nu^n(u), \nu^n(v)\}$

Proof. Let A be a harmonized fuzzy subgroup.

(*i*) By Definition 3.1 and Proposition 3.1,

$$\vartheta^n(uv^{-1}) \geq \min\{\vartheta^n(u), \vartheta^n(v^{-1})\} = \min\{\vartheta^n(u), \vartheta^n(v)\}.$$

(*ii*) By Definition 3.1 and Proposition 3.1,

$$\nu^n(uv^{-1}) \leq \max\{\nu^n(u), \nu^n(v^{-1})\} = \max\{\nu^n(u), \nu^n(v)\}.$$

Conversely,

(*i*) assume that

$$\vartheta^n(uv^{-1}) \geq \min\{\vartheta^n(u), \vartheta^n(v)\}.$$

Then,

$$\vartheta^n(e) = \vartheta^n(uu^{-1}) \geq \min\{\vartheta^n(u), \vartheta^n(u)\} = \vartheta^n(u).$$

This implies that $\vartheta^n(e) \geq \vartheta^n(u)$,$\forall u \in X$. Also note that, $\vartheta^n(u^{-1}) = \vartheta^n(eu^{-1}) \geq \min\{\vartheta^n(e), \vartheta^n(u)\} = \vartheta^n(u)$. This implies that $\vartheta^n(u^{-1}) \geq \vartheta^n(u)$, $\forall u \in X$. Furthermore, let

$$\vartheta^n(uv) = \vartheta^n(u(v^{-1})^{-1}) \geq \min\{\vartheta^n(u), \vartheta^n(v^{-1})\}.$$

There are two possible cases, namely: $\vartheta^n(u) \geq \vartheta^n(v^{-1})$ and $\vartheta^n(u) < \vartheta^n(v^{-1})$.

Case 1. If $\vartheta^n(u) \geq \vartheta^n(v^{-1})$, then $\vartheta^n(u) \geq \min\{\vartheta^n(u), \vartheta^n(v^{-1})\} = \vartheta^n(v^{-1}) \geq \vartheta^n(v)$. In this case, $\vartheta^n(u) \geq \vartheta^n(v)$. Then,

$$\begin{aligned} \vartheta^n(uv) &= \vartheta^n(u(v^{-1})^{-1}) \\ &\geq \min\{\vartheta^n(u), \vartheta^n(v^{-1})\} \\ &= \vartheta^n(v^{-1}) \\ &\geq \vartheta^n(v) \\ &= \min\{\vartheta^n(u), \vartheta^n(v)\}. \end{aligned}$$

Hence, $\vartheta^n(uv) \geq \min\{\vartheta^n(u), \vartheta^n(v)\}$.

Case 2. On the other hand, let $\vartheta^n(u) < \vartheta^n(v^{-1}) \Rightarrow \vartheta^n(v^{-1}) > \vartheta^n(u)$. We know that $\vartheta^n(v^{-1}) \geq \vartheta^n(v)$. Hence, $\vartheta^n(v^{-1}) \geq \min\{\vartheta^n(u), \vartheta^n(v)\}$. Then,

$$\begin{aligned} \vartheta^n(uv) &= \vartheta^n(u(v^{-1})^{-1}) \\ &\geq \min\{\vartheta^n(u), \vartheta^n(v^{-1})\} \\ &\geq \min\{\vartheta^n(u), \vartheta^n(v)\}. \end{aligned}$$

So, in both cases, $\vartheta^n(uv) \geq \min\{\vartheta^n(u), \vartheta^n(v)\}$.

(ii) assume that $\nu^n(uv^{-1}) \leq \max\{\nu^n(u), \nu^n(v)\}$. But $\nu^n(e) = \vartheta^n(uu^{-1}) \leq \max\{\nu^n(u), \nu^n(u)\} = \nu^n(u)$. This implies that $\nu^n(e) \leq \nu^n(u)$, $\forall u \in X$. Also note that, $\nu^n(u^{-1}) = \nu^n(eu^{-1}) \leq \max\{\nu^n(e), \nu^n(u)\} = \nu^n(u)$. This implies that $\nu^n(u^{-1}) \leq \nu^n(u), \forall u \in X$.

Furthermore, if $\nu^n(uv) = \nu^n(u(v^{-1})^{-1}) \leq \max\{\nu^n(u), \nu^n(v^{-1})\}$, there are two possible cases, namely: $\nu^n(u) \geq \nu^n(v^{-1})$ and $\nu^n(u) < \nu^n(v^{-1})$.

Case 1. Let $\nu^n(v^{-1}) \leq \nu^n(u)$. We know that $\nu^n(v^{-1}) \leq \nu^n(v)$. Hence, $\nu^n(v^{-1}) \leq \max\{\nu^n(u), \nu^n(v)\}$. Then,

$$\begin{aligned} \nu^n(uv) &= \nu^n(u(v^{-1})^{-1}) \\ &\leq \max\{\nu^n(u), \nu^n(v^{-1})\} \\ &\leq \max\{\nu^n(u), \nu^n(v)\}. \end{aligned}$$

Hence, $\nu^n(uv) \leq \max\{\vartheta^n(u), \vartheta^n(v)\}$.

Case 2. If $\nu^n(u) < \nu^n(v^{-1}) \Rightarrow \nu^n(v^{-1}) > \nu^n(u)$, then $\nu^n(u) < \nu^n(v^{-1}) \leq \nu^n(v)$, which implies that $\nu^n(v) = \max\{\nu^n(u), \nu^n(v)\}$. Then,

$$\begin{aligned} \nu^n(uv) &= \nu^n(u(v^{-1})^{-1}) \\ &\leq \max\{\nu^n(u), \nu^n(v^{-1})\} \\ &= \nu^n(v^{-1}) \\ &\leq \nu^n(v) \\ &= \max\{\nu^n(u), \nu^n(v)\}. \end{aligned}$$

Hence, $A = \{\langle u, \vartheta_A(u), \nu_A(u)\rangle | u \in X\}$ is a HFG.

□

Proposition 3.3. *Let X be a group and $A = \{\langle u, \vartheta(u), \nu(u)\rangle | u \in X\}$ a HFG. Then, for all $u \in X$*

(*i*) *$\{\langle u, \vartheta^n(u)\rangle | u \in X\}$ is a fuzzy subgroup of X*

(*ii*) *$\{\langle u, \nu^n(u)\rangle | u \in X\}$ is an antifuzzy subgroup of X*

Proof. The proof is straightforward by Definition 3.1. □

Example 3.1. Let the group $X = \{1, -1, i, -i\}$ such that $\vartheta(1) = 0.70$, $\vartheta(-1) = 0.60$, $\vartheta(i) = 0.40 = \vartheta(-i)$ and $\nu(1) = 0.10$, $\nu(-1) = 0.25$, $\nu(i) = 0.5 = \vartheta(-i)$. Obviously, $\langle u, \vartheta(u), \nu(u)\rangle$ is a HFG of X.

Theorem 3.1. *Let X be a group and $A = \{\langle u, \vartheta(u)\rangle | u \in X\}$ a fuzzy group (FG) of X. Then, FG $\Rightarrow$ IFG $\Rightarrow$ PFG $\Rightarrow$ FFG $\Rightarrow$ HFG.*

Proof. It has been shown in [2] that a fuzzy subgroup of a group is an IFG. Also, in [3], it was shown that an IFG is a PFG. It is only necessary to show that a PFG is a FFG and that FFG is a HFG. However, while the proof that a PFG is a FFG is left as an easy exercise, it will be shown that a FFG is a HFG.

To show that a FFG $\Rightarrow$ HFG,

(*i*) assume that $\vartheta^3(uv^{-1}) \geq \min\{\vartheta^3(u), \vartheta^3(v)\}$ and show that $\vartheta^n(uv^{-1}) \geq min\{\vartheta^n(u), \vartheta^n(v)\}$;

(*ii*) and also assume that $\nu^3(uv^{-1}) \leq \max\{\nu^3(u), \nu^3(v)\}$ and show that $\nu^n(uv^{-1}) \leq \max\{\nu^n(u), \nu^n(v)\}$.

Now,

(*i*) assume that $\vartheta^3(uv^{-1}) \geq \min\{\vartheta^3(u), \vartheta^3(v)\}$. There are two possible cases, namely: $\vartheta^3(u) \geq \vartheta^3(v)$ and $\vartheta^3(u) < \vartheta^3(v)$.

Case 1. If $\vartheta^3(u) \geq \vartheta^3(v)$, then $\vartheta^n(u) \geq \vartheta^n(v)$ which implies that $\vartheta^n(v) = \min\{\vartheta^n(u), \vartheta^n(v)\}$. Also, $\vartheta^3(uv^{-1}) \geq \min\{\vartheta^3(u), \vartheta^3(v)\} = \vartheta^3(v)$. This implies that $\vartheta^n(uv^{-1}) \geq \vartheta^n(v) = \min\{\vartheta^n(u), \vartheta^n(v)\}$. Hence, $\vartheta^n(uv^{-1}) \geq \min\{\vartheta^n(u), \vartheta^n(v)\}$.

Case 2. If $\vartheta^3(v) > \vartheta^3(u)$, then $\vartheta^n(v) > \vartheta^n(u)$. Hence, $\vartheta^n(u) = \min\{\vartheta^n(u), \vartheta^n(v)\}$. Also, $\vartheta^3(uv^{-1}) \geq \min\{\vartheta^3(u), \vartheta^3(v)\} = \vartheta^3(u)$. This implies that $\vartheta^n(uv^{-1}) \geq \vartheta^n(u) = \min\{\vartheta^n(u), \vartheta^n(v)\}$. Hence, $\vartheta^n(uv^{-1}) \geq \min\{\vartheta^n(u), \vartheta^n(v)\}$.

(ii) Furthermore, assume that $\nu^3(uv^{-1}) \leq \max\{\nu^3(u), \nu^3(v)\}$. There are also two possible cases, namely: $\nu^3(u) \leq \nu^3(v)$ and $\nu^3(u) > \nu^3(v)$.

Case 1. If $\nu^3(u) \leq \nu^3(v)$, then $\nu^n(u) \leq \nu^n(v)$ which implies that $\nu^n(v) = \max\{\nu^n(u), \nu^n(v)\}$. Also, $\nu^3(uv^{-1}) \leq \max\{\nu^3(u), \nu^3(v)\} = \nu^3(v)$. This implies that $\nu^n(uv^{-1}) \leq \nu^n(v) = \max\{\nu^n(u), \nu^n(v)\}$. Hence, $\nu^n(uv^{-1}) \leq \max\{\nu^n(u), \nu^n(v)\}$.

Case 2. If $\nu^3(v) < \nu^3(u)$, then $\nu^n(v) < \nu^n(u)$. Also, $\nu^n(u) = \max\{\nu^n(u), \nu^n(v)\}$. Also, $\nu^3(uv^{-1}) \leq \max\{\nu^3(u), \nu^3(v)\} = \nu^3(u)$. This implies that $\nu^n(uv^{-1}) \leq \nu^n(u) = \max\{\nu^n(u), \nu^n(v)\}$. Hence, $\nu^n(uv^{-1}) \leq \max\{\nu^n(u), \nu^n(v)\}$. □

Remark 3.2. In [2], it was stated that an IFS set A is an IFG if and only if ϑ and ν are respectively fuzzy and antifuzzy subgroups. But this is not so, considering Example 3.1. Besides, [3] has shown that IFG is a PFG but the converse is not always true. Hence, in general, the converse of Theorem 3.1 is not always true.

Definition 3.2. *Let X be a group such that $A_1 = \langle u, \vartheta_1, \nu_1\rangle$ and $A_2 = \langle u, \vartheta_2, \nu_1\rangle$ are harmonized fuzzy groups from X. For all $u \in X$, the intersection of A_1 and A_2 is $A = \langle u, \vartheta, \nu\rangle$, where $\vartheta^n(u) = \min\{\vartheta_1^n(u), \vartheta_2^n(u)\}$ and $\nu^n(u) = \max\{\nu_1^n(u), \nu_2^n(u)\}$.*

Definition 3.3. *Let X be a group such that $A_1 = \langle u, \vartheta_1, \nu_1\rangle$ and $A_2 = \langle u, \vartheta_2, \nu_1\rangle$ are harmonized fuzzy groups from X. For all $u \in X$, the union of A_1 and A_2 is $A = \langle u, \vartheta, \nu\rangle$, where $\vartheta^n(u) = \max\{\vartheta_1^n(u), \vartheta_2^n(u)\}$ and $\nu^n(u) = \min\{\nu_1^n(u), \nu_2^n(u)\}$.*

Theorem 3.2. *Let X be a group such that $A_1 = \langle u, \vartheta_1, \nu_1\rangle$ and $A_2 = \langle u, \vartheta_2, \nu_1\rangle$ are harmonized fuzzy groups from X. For all $u \in X$, the intersection of A_1 and A_2 is $A = \langle u, \vartheta, \nu\rangle$ is also a HFG.*

Proof.

$$\begin{aligned}
\vartheta^n(uv^{-1}) &= \min\{\vartheta_1^n(uv^{-1}), \vartheta_2^n(uv^{-1})\} \\
&\geq \min\{\min\{\vartheta_1^n(u), \vartheta_1^n(v)\}, \min\{\vartheta_2^n(u), \vartheta_2^n(v)\}\} \\
&= \min\{\min\{\vartheta_1^n(u), \vartheta_2^n(u)\}, \min\{\vartheta_1^n(v), \vartheta_2^n(v)\}\} \\
&= \min\{\vartheta^n(u), \vartheta^n(v)\}.
\end{aligned}$$

$$\begin{aligned}\nu^n(uv^{-1}) &= \max\{\nu_1^n(uv^{-1}), \nu_2^n(uv^{-1})\}\\ &\leq \max\{\max\{\nu_1^n(u), \nu_1^n(v)\}, \max\{\nu_2^n(u), \nu_2^n(v)\}\}\\ &= \max\{\max\{\nu_1^n(u), \nu_2^n(u)\}, \max\{\nu_1^n(v), \nu_2^n(v)\}\}\\ &= \max\{\nu^n(u), \nu^n(v)\}. \quad \square\end{aligned}$$

Corollary 3.1. *Let X be a group such that $\{A_i = \langle u, \vartheta_i, \nu_i\rangle\}$ is an arbitrary collection of harmonized fuzzy groups from X. For all $u \in X$, the arbitrary intersection of A_i is also a HFG.*

Proof. The same step as in Theorem 3.2 will proof the corollary. $\square$

Remark 3.3. The following example shows that the union of two HFGs is not a HFG.

Example 3.2. Consider the group $(\mathbb{Z}, +)$. Let $A_1 = \langle u, \vartheta_1, \nu_1\rangle$ and $A_2 = \langle u, \vartheta_2, \nu_2\rangle$ be two HFGs such that $A = A_1 \cup A_2 = \langle u, \vartheta, \nu\rangle$ and ϑ, ν, ϑ_1, ν_1, ϑ_2, ν_2 are as defined below:

$$\vartheta(u) = \begin{cases} 0.4, & u \in 5\mathbb{Z}\\ 0.2, & u \in (2\mathbb{Z} - 5\mathbb{Z})\\ 0.1, & elsewhere\end{cases} \qquad \nu(u) = \begin{cases} 0.1, & u \in 5\mathbb{Z}\\ 0.3, & u \in (2\mathbb{Z} - 5\mathbb{Z})\\ 0.5, & elsewhere\end{cases}$$

$$\vartheta_1(u) = \begin{cases} 0.4, & u \in 5\mathbb{Z}\\ 0.1, & elsewhere\end{cases} \qquad \nu_1(u) = \begin{cases} 0.1, & u \in 5\mathbb{Z}\\ 0.6, & elsewhere\end{cases}$$

$$\vartheta_2(u) = \begin{cases} 0.2, & u \in 2\mathbb{Z}\\ 0.1, & elsewhere\end{cases} \qquad \nu_2(u) = \begin{cases} 0.3, & u \in 5\mathbb{Z}\\ 0.5, & elsewhere\end{cases}$$

it will be shown that from the above that $\vartheta^n(15 + (-2)) = \vartheta^n(13) = 0.1^n$ and $\min\{\vartheta^n(15), \vartheta^n(-2)\} = \min\{0.4^n, 0.2^n\} = 0.2^n$. But $0.1^n \ngeq 0.2^n$. Hence, $\vartheta^n(uv) \ngeq \min\{\vartheta^n(u), \vartheta^n(v)\}$. Also, $\nu^n(15 + (-2)) = \nu^n(13) = 0.5^n$ and $\max\{\nu^n(15), \nu^n(-2)\} = \max\{0.1^n, 0.3^n\} = 0.3^n$. But $0.5^n \nleq 0.3^n$. Hence, $\nu^n(uv) \nleq \max\{\nu^n(u), \nu^n(v)\}$.

Remark 3.4. By definition, if ϑ is a fuzzy subgroup and ν an antifuzzy subgroup of a group X, $\vartheta(u^s) \geq \vartheta(u)$ and $\nu(u^s) \leq \nu(u)$ for any integer s. Hence, for HFG $A = \langle u, \vartheta, \nu\rangle$, since ϑ^n is a fuzzy subgroup by Proposition 3.3, $\vartheta^n(u^s) \geq \vartheta^n(u)$. Similarly, since ν^n is an antifuzzy subgroup by Proposition 3.3, $\nu^n(u^s) \leq \nu^n(u)$.

Proposition 3.4. *Let $A = \langle u, \vartheta, \nu\rangle$ be a HFG of a group X. Let $e, u \in X$, where e is the identity. If $\vartheta^n(u) = \vartheta^n(e)$, then $\vartheta^n(uv) = \vartheta^n(v)$, $\forall v \in X$ and if $\nu^n(u) = \nu^n(e)$, then $\nu^n(uv) = \nu^n(v)$, $\forall v \in X$.*

Proof.

$$\begin{array}{rcl}\vartheta^n(v) & = & \vartheta^n(u^{-1}uv) \\ & \geq & \min\{\vartheta^n(u), \vartheta^n(uv)\} \\ & = & \min\{\vartheta^n(e), \vartheta^n(uv)\} \\ & = & \vartheta^n(uv) \\ & \geq & \min\{\vartheta^n(u), \vartheta^n(v)\} \\ & = & \min\{\vartheta^n(e), \vartheta^n(v)\} \\ & = & \vartheta^n(v).\end{array}$$

Hence, $\vartheta^n(uv) = \vartheta^n(v)$. On the other hand,

$$\begin{array}{rcl}\nu^n(v) & = & \nu^n(u^{-1}uv)\} \\ & \leq & \max\{\nu^n(u), \nu^n(uv)\} \\ & = & \max\{\nu^n(e), \nu^n(uv)\} \\ & = & \nu^n(uv) \\ & \leq & \max\{\nu^n(u), \nu^n(v)\} \\ & = & \max\{\nu^n(e), \nu^n(v)\} \\ & = & \nu^n(v).\end{array}$$

Hence, $\nu^n(uv) = \nu^n(v)$. □

Proposition 3.5. *Let $A = \langle u, \vartheta, \nu\rangle$ be a HFG of a group X. The set $\mathbb{G} = \{u \in X : \vartheta^n(u) = \vartheta^n(e)$ and $\nu^n(u) = \nu^n(e)\}$ is a subgroup of X.*

Proof. Let $u, v^{-1} \in \mathbb{G}$. By Proposition 3.4 and since A is a HFG, $\vartheta^n(uv^{-1}) = \vartheta^n(v^{-1}) = \vartheta^n(v) = \vartheta^n(e)$. Also, by the same reasons, $\nu^n(uv^{-1}) = \nu^n(v^{-1}) = \nu^n(v) = \nu^n(e)$. Thus, $uv^{-1} \in \mathbb{G}$. Hence, $\mathbb{G}$ is a subgroup of X. □

4. (α, β)-level of a harmonized fuzzy subgroup

Bhunia *et al* [3] has discussed the Pythagorean fuzzy level subgroup but the Harmonized fuzzy level subgroup (HFLG) called (α, β)-Level of a Harmonized Fuzzy Subgroup is now being discussed.

Definition 4.1. *Let X be a set and $A = \langle u, \vartheta, \nu\rangle$ a HFS of X. Then, the set*

$$A_{(\alpha,\beta)} = \{u \in X : \vartheta(u) \geq \alpha \text{ and } \nu(u) \leq \beta, \quad \alpha, \beta \in [0, 1]\},$$

where $0 \leq \vartheta^n(u) + \nu^n(u) \leq 1$, is called Harmonized fuzzy level subset of the set X.

Definition 4.2. *Let X be a group and $A = \langle u, \vartheta, \nu\rangle$ a HFG of X. Then, the set*

$$A_{(\alpha,\beta)} = \{u \in X : \vartheta(u) \geq \alpha \text{ and } \nu(u) \leq \beta, \quad \alpha, \beta \in [0, 1]\},$$

where $0 \leq \vartheta^n(u) + \nu^n(u) \leq 1$, *is called Harmonized fuzzy level subgroup of the group* X.

Proposition 4.1. *Let* $A = \langle u, \vartheta_a, \nu_a \rangle$ *and* $B = \langle u, \vartheta_b, \nu_b \rangle$ *be HFGs of a group* X*. Then, the following hold:*

(*i*) $A_{(\theta,\beta)} \subseteq B_{(\epsilon,\alpha)}$ *if* $\epsilon \leq \theta$ *and* $\alpha \geq \beta$, $\forall \alpha, \beta, \epsilon, \theta \in [0,1]$

(*ii*) $A \subseteq B \Rightarrow A_{(\theta,\beta)} \subseteq B_{(\theta,\beta)}$, $\forall \beta, \theta \in [0,1]$.

Proof. Note that $\alpha, \beta, \epsilon, \theta \in [0,1]$.

(i) Let $u \in A_{(\theta,\beta)}$. Then, $\vartheta_a^n(u) \geq \theta \geq \epsilon$ and $\nu_a^n(u) \leq \beta \leq \alpha$. This implies that $u \in A_{(\epsilon,\alpha)}$ and thus $A_{(\theta,\beta)} \subseteq A_{(\epsilon,\alpha)}$

(ii) Assume that $A \subseteq B$, then $\vartheta_a(u) \leq \vartheta_b(u)$ and $\nu_a(u) \geq \nu_b(u)$, $\forall\ u \in X$. This implies that $\vartheta_a^n(u) \leq \vartheta_b^n(u)$ and $\nu_a^n(u) \geq \nu_b^n(u)$, $\forall\ u \in X$.

Let $u \in A_{(\theta,\beta)}$, $\vartheta_a^n(u) \geq \theta$ and $\nu_a^n(u) \leq \beta$. This implies that $\vartheta_b^n(u) \geq \vartheta_a^n(u) \geq \theta$ and $\nu_b^n(u) \leq \nu_a^n(u) \leq \beta$. Hence, $\vartheta_b^n(u) \geq \theta$ and $\nu_b^n(u) \leq \beta$, which implies that $u \in B_{(\theta,\beta)}$ and $A_{(\theta,\beta)} \subseteq B_{(\theta,\beta)}$.

□

Example 4.1. Let $X = S_3$. Then, $A = \langle u, \vartheta_a, \nu_a \rangle$ and $B = \langle u, \vartheta_b, \nu_b \rangle$ are HFGs by the definitions below:

$$\vartheta_a(u) = \begin{cases} 0.9, & u = e \\ 0.7, & u \in \{(123),(132)\} \\ 0.5, & elsewhere \end{cases} \qquad \nu_a(u) = \begin{cases} 0.2, & u = e \\ 0.5, & u \in \{(123),(132)\} \\ 0.7, & elsewhere \end{cases}$$

$$\vartheta_b(u) = \begin{cases} 0.8, & u = e \\ 0.5, & elsewhere \end{cases} \qquad \nu_b(u) = \begin{cases} 0.3, & u = e \\ 0.8, & elsewhere \end{cases}$$

$A_{(0.7,0.5)} = \{e,(123),(132)\}$ and $A_{(0.5,0.8)} = \{e,(12),(13),(23),(123),(132)\}$, so $A_{(0.7,0.5)} \subseteq A_{(0.5,0.8)}$. Note that $B \subseteq A$ and $B_{(0.7,0.5)} = \{e\} \subseteq \{e,(123),(132)\} = A_{(0.7,0.5)}$.

Proposition 4.2. *Let* $A = \langle u, \vartheta, \nu \rangle$ *be a HFG of a group* X*. Let the set* $A_{(\theta,\beta)}$ *be a HFLG of* A *for which* $\vartheta^n(e) \geq \theta$ *and* $\nu^n(e) \leq \beta$*, where* e *is the identity of* X*. Then,* $A_{(\theta,\beta)}$ *is a subgroup of* X.

Proof. Note that $A_{(\theta,\beta)} \neq \emptyset$ since $\vartheta^n(e) \geq \theta$ and $\nu^n(e) \leq \beta$ implies that $e \in A_{(\theta,\beta)}$. If no other element is in $A_{(\theta,\beta)}$, there is no need to prove anything since $\{e\} \subseteq A_{(\theta,\beta)}$. But suppose $u, v \in A_{(\theta,\beta)}$, then

$$\begin{aligned} \vartheta^n(uv^{-1}) &\geq \min\{\vartheta^n(u), \vartheta^n(v)\} \\ &= \theta \end{aligned}$$

and

$$\begin{aligned} \nu^n(uv^{-1}) &\leq \max\{\nu^n(u), \nu^n(v)\} \\ &= \beta. \end{aligned}$$

Hence, $uv^{-1} \in A_{(\theta,\beta)}$. Thus, $A_{(\theta,\beta)}$ is a subgroup of X. □

5. Conclusion

In this paper, some existing improvements on fuzzy sets and their algebraic structures have been studied, namely: Intuitionistic Fuzzy Sets (IFSs) and Intuitionistic Fuzzy Groups (IFGs); Pythagorean Fuzzy Sets (PFSs) and Pythagorean Fuzzy Groups (PFGs); Fermatean Fuzzy Sets (FFSs) on which algebraic structures are now built, Fermatean Fuzzy Groups (FFGs). All these algebraic structures are now harmonized in one structure, Harmonized Fuzzy Group (HFG). Some of the properties of these structures were studied in detail. In subsequent studies, it will be of interest to consider further properties such as Harmonized Fuzzy Cosets and Harmonized Normal Fuzzy Subgroup and some related properties.

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