Partial domination in prisms of graphs

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Abstract. For any graph $G = (V, E)$ and proportion $p \in (0, 1]$, a set $S \subseteq V$ is a p-dominating set if $\frac{|N[S]|}{|V|} \geq p$. The p-domination number $\gamma_p(G)$ equals the minimum cardinality of a p-dominating set in G. For a permutation π of the vertex set of G, the graph πG is obtained from two disjoint copies G_1 and G_2 of G by joining each v in G_1 to $\pi(v)$ in G_2. i.e., $V(\pi G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{(v, \pi(v)) : v \in V(G_1), \pi(v) \in V(G_2)\}$. The graph πG is called the prism of G with respect to π. In this paper, we find some relations between the domination and the p-domination numbers in the context of graph and its prism graph for particular values of p.
Keywords: permutation graph, algebraic graph theory, prism graph.

1. Introduction

The concept of prisms of graphs was first introduced by Chartrand and Harary [1] in 1967. They used the term *permutation graphs* to define such graphs; but their definition was different from the one we have for permutation graphs as defined in [2]. Later those graphs were named as prisms of graphs with respect to a permutation. Prisms of graphs play a great role in designing computer networks.

Partial domination [3, 4] in graphs is a variation of domination introduced in 2017. In [5], we see some algebraic properties of the partial dominating sets of a graph. Here, in this paper we study prism graphs in the context of partial domination.

2. Basic terminologies

Let $G = (V(G), E(G))$ be a finite, simple and undirected graph with $V(G)$ as its vertex set and $E(G)$ as its edge set. For any $v \in V(G)$, $N_G(v) = \{u \in$

(a) Domination (b) $\frac{2}{3}$-domination

Figure 1: Domination and Partial Domination

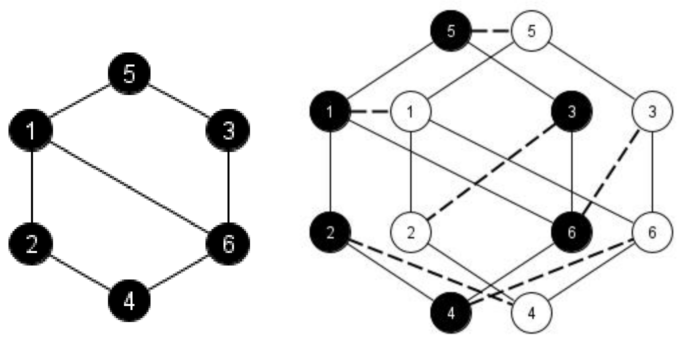

Figure 2: Graph G and its prism graph based on π=(2463)

$V(G) : uv \in E(G)\}$ and $N_G[v] = N_G(v) \cup \{v\}$ denote the open and the closed neighborhoods of v respectively. A set $S \subseteq V(G)$ is called a dominating set if every vertex in $V - S$ is adjacent to at least one vertex in S. The minimum cardinality of a dominating set is called the domination number and is denoted by $\gamma(G)$.

A subset $S \subseteq V$ is called a p-dominating set for $p \in (0, 1]$ if $\frac{|N[S]|}{|V|} \geq p$. The p-domination number, denoted by $\gamma_p(G)$ is the cardinality of the minimum p-dominating set.

In Figure 1 (a), the white vertices dominate all the vertices of the graph. In Figure 1 (b), the white vertices do not dominate the vertices of the graph. However, they dominate exactly 10 vertices of the graph. Hence, we say, the white vertices $\frac{2}{3}$-dominate the graph.

Let π be any permutation on $V(G)$. The prism πG of G with respect to π is obtained by taking two disjoint copies of G and joining each vertex v in one copy of G with $\pi(v)$ in the other copy by means of an edge. The study of domination in prisms started in 2004 [6]. Since then many works related to various domination parameters were studied [7, 8].

In [9], it has been proved that for any graph G, $\gamma(G) \leq \gamma(\pi G) \leq 2\gamma(G)$. Also, it has been defined in [6] that, G is called universal γ-fixer, if $\gamma(G) = \gamma(\pi G)$ for all permutations π of V(G) and G is called universal doubler if $\gamma(\pi G) = 2\gamma(G)$

for all permutations π of V(G). Analogous to this, in the context of partial domination we give the following definition:

Definition 2.1. *Let $p \in [0,1]$. G is called universal γ_p-fixer, if $\gamma_p(G) = \gamma_p(\pi G)$ for all permutations π of V(G) and G is called universal γ_p-doubler if $\gamma_p(\pi G) = 2\gamma_p(G)$ for all permutations π of V(G).*

Figure 2 is an example of a graph G and its prism graph with π=(2463).

3. Results

Proposition 3.1. *Let G be any n-vertex connected graph without isolated vertices and with γ=1. Then, for any permutation π of V(G) and for any $p \in (0,1]$,*

$$\gamma_p(\pi G) = \begin{cases} 1, & \text{for } p \in (0, \frac{n+1}{2n}], \\ 2, & \text{for } p \in (\frac{n+1}{2n}, 1]. \end{cases}$$

Proof. Let π be any permutation of $V(G)$. Consider πG.

Since γ(G)=1, $\exists v \in V(G)$ such that deg(v)=n-1. This v dominates n+1 vertices in πG. Hence $\frac{|N_{\pi G}[v]|}{|V(\pi G)|} = \frac{n+1}{2n}$. Thus, $\gamma_p(\pi G) = 1$, for $p \in (0, \frac{n+1}{2n}]$.

Now, consider $S = \{v, v'\}$ in πG. Let v' be the mirror image of v in the second copy of G in πG. This S dominates πG and is minimum. Thus $\gamma_p(\pi G) = 2$, for $p \in (\frac{n+1}{2n}, 1]$. □

Corollary 3.1. *Let G be any n-vertex connected graph without isolated vertices and with γ=1. Then, G is a universal γ_p-fixer for $p \in (0, \frac{n+1}{2n}]$ and is a universal γ_p-doubler for $p \in (\frac{n+1}{2n}, 1]$.*

Proposition 3.2. *Let G be any n-vertex graph. Then for any permutation π of V(G) and for any $p \in (0, \frac{n+\gamma(G)}{2n}]$, $\gamma_p(\pi G) \leq \gamma(G)$.*

Proof. Let π be any permutation of V(G). Consider πG.

Let S be a γ-set of G. Then by the definition of πG, S is a n+$\gamma(G)$ dominating set in πG. Also, if $p \leq q$ then $\gamma_p \leq \gamma_q$. Hence, for any $p \in (0, \frac{n+\gamma(G)}{2n}]$, $\gamma_p(\pi G) \leq \gamma(G)$. □

The following result shows that the above bound is sharp.

Proposition 3.3. *Let G be P_n or C_n for $n \geq 2$. Then for any permutation π on V(G), $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G) = \gamma(G)$.*

Proof. Let π be any permutation on V(G). Then by the previous proposition $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G) \leq \gamma(G)$. Hence it is enough if we prove that $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G) \geq \gamma(G)$ for any permutation π on V(G).

Let us assume the contradiction that $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G) < \gamma(G)$ for some permutation π.

WLG let $S \subseteq V(G)$ be a $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G)$-set with $\gamma(G)-1$ vertices. Let G_1 and G_2 denote the two copies of G in πG. Then two cases may arise.
Case (i). All the vertices of S are from either $V(G_1)$ or $V(G_2)$.

In this case S dominates almost $(n-1)+(\gamma(G)-1)$ vertices in πG.

$$\implies \frac{|N[S]|}{2n} \leq \frac{n-1+\gamma(G)-1}{2n} = \frac{n+\gamma(G)-2}{2n} < \frac{n+\gamma(G)}{2n}.$$

This is a contradiction to our assumption that $S \subseteq V(G)$ is a $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G)$-set. Hence the proof in this case.
Case (ii). Vertices of S are from both G_1 and G_2. WLG let there be l and m vertices from G_1 and G_2 respectively, where $l+m=\gamma(G)-1$ by our assumption. Then, we have the following:

$$\frac{|N[S]|}{2n} \leq \frac{4(l+m)}{2n} = \frac{4(\gamma(G)-1)}{2n}. \tag{1}$$

Now, $\gamma(G) = \lceil \frac{n}{3} \rceil$

$$< \frac{n}{3}+1 \implies 3\gamma(G) < n+3 < n+4. \tag{2}$$

Hence, from (1) and (2) we will get a contradiction to our assumption. Hence the proof. □

Remark 3.1. For any graph G, $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G) = 1$ if and only if $\gamma(G) = 1$.

Proposition 3.4. *Let G be any graph with u_Δ as a vertex having the maximum degree $\Delta(G)$ and u'_Δ as its mirror image in the second copy of G in πG. Then $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G) = 2$ if and only if one of the following two conditions holds for G:*

(i) $\gamma(G) = 2$;

(ii) $\gamma(G) \geq 3$ *and* $\Delta(G) \geq \frac{n+\gamma(G)-4+i}{2}$ *where* $|N[u_\Delta] \cap N[u'_\Delta]| = i$ *for* $0 \leq i \leq 2$.

Proof. Let us assume that $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G) = 2$. Then by the above remark $\gamma(G) \geq 2$. If $\gamma(G) = 2$,then the result is true. Hence we assume that $\gamma(G) \geq 3$. Let $|N[u_\Delta] \cap N[u'_\Delta]| = i$ for $0 \leq i \leq 2$.

We prove by the method of contradiction. Suppose $\Delta(G) < \frac{n+\gamma(G)-4+i}{2}$. Then, $|N_{\pi G}[u_\Delta] \cup N_{\pi G}[u'_\Delta]| < n+\gamma(G)$ which implies that $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G) > 2$ which is a contradiction. Hence the condition is necessary. For the proof of the sufficient part, let us assume that $\gamma(G) = 2$. Let S be a $\gamma(G)$-set of G. Then S is also a $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G)$-set of πG. Hence $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G) = 2$.

Now, let us assume the condition(ii). Consider $T = \{u_\Delta, u'_\Delta\}$. Then

$$N_{\pi G}[T] \geq 2\left[\frac{n+\gamma(G)-4+i+4-i}{2}\right] = n+\gamma(G).$$

Also, T is minimum in this context. Hence $\gamma_{\frac{n+\gamma(G)}{2n}}(\pi G) = 2$. □

Proposition 3.5. *Suppose* $p \in (0,1]$ *and* π *is any permutation on* $V(G)$*, where* G *is any* n*- vertex graph. Then* $\gamma_p(G) \leq \gamma_p(\pi G) \leq 2\gamma_p(G)$*.*

Proof. Let us first prove the lower bound part. Let $p \in (0,1]$ and S be a γ_p-set in πG.

$$\Longrightarrow \frac{|N[S]|}{2n} \geq p \Longrightarrow |N[S]| \geq 2np. \tag{3}$$

Let G_1 and G_2 denote the two copies of G in πG.

Now, two cases may arise.

Case (i). All the vertices of S are from either $V(G_1)$ or $V(G_2)$ Then from 3, $|N[S]| \geq 2np > np$. Hence, S is a p-dominating set in G_1 or G_2. Thus, $\gamma_p(G) \leq \gamma_p(\pi G)$ in this case.

Case (ii): Vertices of S are from both G_1 and G_2.

Let $S = X \cup Y$ where $X \subseteq V(G_1)$ and $Y \subseteq V(G_2)$. By our assumption,

$$\frac{|N[X \cup Y]|}{2n} \geq p \Longrightarrow |N[X]| + |N[Y]| \geq 2np. \tag{4}$$

Now, two cases may arise:

Subcase (i). $|N[X]| \geq np$ and $|N[Y]| \geq np$.

Let $X^* = \{w \in V(G_2)/\pi(v) = w \forall v \in X\}$ and $Y^* = \{w \in V(G_1)/\pi(w) = v \forall v \in Y\}$. Then, $X \cup Y^*$ and $X^* \cup Y$ are p-dominating sets of G. Thus,

$$\gamma_p(G) \leq |X \cup Y^*| = |X \cup Y| = \gamma_p(\pi G).$$

Subcase (ii). WLG let $|N[X]| < np$. Then $|N[Y]| \geq np + (np - |N[X]|)$ by our assumption in (4). In this case $|N[Y]| > np$. Hence, $X^* \cup Y$ is a p-dominating set of G, where X^* is defined as in the above subcase. Thus

$$\gamma_p(G) \leq |X^* \cup Y| = |X \cup Y| = \gamma_p(\pi G).$$

Hence, $\gamma_p(G) \leq \gamma_p(\pi G)$ in both the cases. Now, let us prove the upper bound part. Let S be a γ_p set of G. Let $S^* = \{v^* \in G_2/\pi(v) = v^* \forall v \in S\}$. Then, $\frac{|N[S \cup S^*]|}{2n} \geq p$. Thus, $S \cup S^*$ is a p-dominating set of πG and $|S \cup S^*| = 2\gamma_p(G)$. Hence, $\gamma_p(\pi G) \leq 2\gamma_p(G)$. □

Proposition 3.6. *Let* G *be a graph having an independent set* $M = \{v_1, v_2, ..., v_k\}$ *of k-vertices* $(k \geq 1)$ *each having the maximum degree* $\Delta(G)$*. If* $N(v_i) \cap N(v_j) = \phi \forall v_i, v_j \in M$ *then for* $i = 1, 2, ..., k$*,*

$$\gamma_p(\pi G) = i, \ for\, p \in \left(\frac{(i-1)(\Delta(G)+2)}{2n}, \frac{i(\Delta(G)+2)}{2n} \right]$$

for any permutation π *on* $V(G)$*.*

Proof. Let $N(v_i) \cap N(v_j) = \phi \forall v_i, v_j \in M$.

In this case, for $1 \leq i \leq k$, $S = \{v_1, v_2, ..., v_i\} \subseteq M$ dominates $i(\Delta(G) + 2)$ vertices in πG. Also since each vertex in M is of maximum degree, 'i' is the minimum number of vertices that are required to dominate $i(\Delta(G)+2)$ vertices in πG.

Hence, $\gamma_{\frac{i(\Delta(G)+2)}{2n}} = i$. By the same argument we can say that $\gamma_{\frac{(i-1)(\Delta(G)+2)}{2n}} = i-1$. Also, $\frac{(i-1)(\Delta(G)+2)}{2n}$ is the best proportion of domination possible with i-1 vertices. Hence the result is true in this case. □

Proposition 3.7. *Let G be a n-vertex graph having a set of k-mutually non-adjacent vertices($k \geq 1$) say $M = \{v_1, v_2, ..., v_k\}$ each having the maximum degree $\Delta(G)$. Let G and G' be the two copies of G in πG for any permutation π on V(G). Let $M' = \{v'_1, v'_2, ..., v'_k\}$ be the copy of M in G'. If for each $v_r \in M$, there exists exactly one $v_s \in M$ there exists exactly one such that $|N(v_r) \cap N(v_s)| = 1$, then for $i = 1, 2, ..., k$,*

$$\gamma_p(\pi G) = i, \text{ for } p \in \left(\frac{(i-1)(\Delta(G)+2)}{2n}, \frac{i(\Delta(G)+2)}{2n} \right]$$

for the following permutations π on V(G):

(i) π=1

(ii) $\pi(v_r) = v'_s$ *and* $\pi(v_s) = v'_r$ *for each* v_r *and* v_s *as defined above.*

(iii) $\pi \neq 1$ *and* $\pi(v_r) \neq v'_s$ *or* $\pi(v_s) \neq v'_r$ *or both and* $\pi(v_i) \in M'$ $\forall v_i \in M$.

Proof. We shall prove the theorem in three cases. For $1 \leq i \leq k$, WLG assume that i is an even number. Let $S = \{v_1, v_2, ..., v_i\} \subseteq M$ be such that there exists $\frac{i}{2}$pairs of v_r, v_s as defined above.

Case (i). $\pi = 1$.

In this case $\{v_r, v'_s\}$ for each pair of v_r, v_s will dominate $\frac{2(\Delta(G)+2)}{2n}$ vertices in πG. Thus there exists $\frac{i}{2}$ such pairs which dominate $\frac{i(\Delta(G)+2)}{2n}$ vertices in πG. Hence by an argument similar to the previous proposition, the result is proved in this case.

Case (ii). $\pi(v_r) = v'_s$ and $\pi(v_s) = v'_r$ for each v_r and v_s as defined above.

In this case $\{v_r, v'_r\}$ or $\{v_s, v'_s\}$ for each pair of v_r, v_s will serve the purpose for the required result.

Case (iii): $\pi \neq 1$ and $\pi(v_r) \neq v'_s$ or $\pi(v_s) \neq v'_r$ or both and $\pi(v_i) \in M'$ $\forall v_i \in M$. For this case we give an algorithm which returns a set T of k-vertices from $M \cup M'$ whose members are mutually non-adjacent to each other. Let $\pi(v_i) = v^*_i$ $\forall v_i \in M$ under the above permutations.

Algorithm 1 Algorithm to find T

Require: M, M', $\pi(v_x)$ $\forall v_x \in M$, $N(v_x)$ $\forall v_x$ in M and M'
Ensure: T
T=$\{\}$, $k = |M|$
For $v_x \in M$
$v_x^* = \pi(v_x)$
if $\{v_x, v_x^*\} \cap T = \phi$ **then**
    $T = T \cup \{v_x\}$
else
    if $|T| = k$ **then**
        return T
    else
        go to 2
    end if
end if
if $|N(v_x) \cap N(v_y)| = 1$ **then**
    $T = T \cup v_y^*$
end if
if $|N(v_y^*) \cap N(v_z^*)| = 1$ **then**
    $v_x = v_z$ and go to 4
end if

Each vertex in T thus got from the algorithm will dominate $\frac{\Delta(G)+2}{2n}$ vertices in πG. And for $1 \leq i \leq k$, i vertices from T will dominate $\frac{i(\Delta(G)+2)}{2n}$ vertices in πG. □

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Accepted: February 8, 2021