Extensions of e-reversible rings

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Abstract. Kose, et al. in [12] defined and studied the right (left) e -reversible rings. We introduce a strong condition on the Kose's notion and we call it e-strongly reversible rings, we define as follows: if $ab = 0$ implies $bea = 0$ for any $a, b \in R$, and e is an idempotent element in R. We show that e-reversible ring need not be e-strongly reversible. Also, we study some ring extensions over right (left) e-reversible rings e.g., Morita context and the Jordan construction.

Keywords: reversible ring, idempotent element, ring extensions and matrix ring.

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1. Introduction

Throughout this paper all rings R are associative with identity. Let $T_2(R)$, $E(R)$, $N(R)$, and $C(R)$ denote the 2×2 upper triangular matrix ring over R, the set of all idempotent elements, the set of nilpotent elements, and the center of R, respectively. An idempotent e of a ring R is called left (right) semicentral if $ae = eae$ ($ea = eae$) for each a in R. A ring R is called *reduced* if it has no nonzero nilpotent elements. According to Lambek [14], a ring R is *symmetric* if $abc = 0$ implies $acb = 0$ for $a, b, c \in R$. Anderson and Camillo [2] showed that a ring R is *symmetric* if and only if $r_1 r_2 ... r_n = 0$ implies $r_{\sigma(1)} r_{\sigma(2)} ... r_{\sigma(n)} = 0$, for any permutation of the set $\{1, 2, ..., n\}$ and $r_i \in R$. According to [6] a ring R is said to be *reversible* if for any $a, b \in R$, $ab = 0$ implies $ba = 0$. Anderson and Camillo [14], called zero symmetric rings and denote it by ZC_2 for what is called reversible. A ring R is called *semicommutative* if for any $a, b \in R$ such that $ab = 0$ implies $aRb = 0$. Moreover, a ring R is said to be *central reversible* if for every $a, b \in R$ such that $ab = 0$ implies ba is central in R. The study of reversible and central reversible rings can be found in [6] and [10]. *Central reversible* rings generalize *reversible* rings as shown in [10]. Finally a ring R is called *reflexive* if $aRb = 0$ implies $bRa = 0$ for $a, b \in R$, see [13]. A ring R is called *Abelian* if all of its idempotent elements are central. Another direction for generalizing above rings can be found in [17]. A ring R is said to be *right (left) e-reduced* if $N(R)e = 0$ ($eN(R) = 0$), and R is *e-symmetric* if whenever $abc = 0$, then $acbe = 0$ for $a, b, c \in R$, e is an idempotent element in R. Also according to [11], *any ring* R is said to be right (left) *e-semicommutative* if for any $a, b \in R$ such that $ab = 0$ implies $aRbe = 0$ ($eaRb = 0$). In [8] a ring R is called right *idempotent reflexive* if $aRe = 0$ implies $eRa = 0$ for $a \in R$, left *idempotent reflexive* can be defined similarly. If a ring R is both left and right idempotent reflexive then the ring is called an *idempotent reflexive* ring. Later reversible rings extended to e-reversible rings for an idempotent $e \in R$. A ring R is said to be right (left) *e-reversible* if $ab = 0$ implies $bae = 0$ ($eba = 0$), for $a, b \in R$ and $e \in E(R)$. See [12].

Motivated by the above, we define *e-strong reversible* ring for an idempotent $e \in R$, that is if for any $a, b \in R$, $ab = 0$ implies $bea = 0$, and study its properties. The properties of e-reversible and e-strongly reversible are independent. Suitable examples are given. We examined the transfer of *e-reversibility* over some various ring extensions. Namely, we study *e-reversibility* over the ring of polynomials, the classical ring of quotient, Jordan construction and finally the Morita context is discussed.

2. e-reversible rings

In this section we discuss the property of e–reversibility and related rings.

Definition 2.1 ([12]). A ring R is called right (left) e-reversible if $ab = 0$ implies $bae = 0$ ($eba = 0$), for $a, b \in R$ and $e \in E(R)$.

It is clear that for an idempotent $e \in R$, if R is reversible then R is e-reversible, since $ab = 0$ implies $bae = 0$ and $(eba) = 0$, for any $a, b \in R$.

Proposition 2.1. *Let R be a reduced ring. Then*

$$T(R) = \left\{ \begin{bmatrix} a & b & c \\ 0 & d & g \\ 0 & 0 & f \end{bmatrix} \quad \textit{such that} \quad a, b, c, d, g, f \in R \right\}$$

is right E-reversible, where $E = \begin{bmatrix} e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ *and e in $E(R)$.*

Proof. Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ 0 & d_1 & g_1 \\ 0 & 0 & f_1 \end{bmatrix}$, $B = \begin{bmatrix} a_2 & b_2 & c_2 \\ 0 & d_2 & g_2 \\ 0 & 0 & f_2 \end{bmatrix}$, in R such that $AB = 0$. Then $a_1 a_2 = 0$, since R is a reduced ring, we get $(a_2 a_1)^2 = 0$, $a_2 a_1 = 0$. This leads to $a_2 a_1 e = 0$. For $E = \begin{bmatrix} e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then $BAE = 0$. Therefore, $T(R)$ is right E-reversible. □

We use $S_l(R)$ (resp., $S_r(R)$) for the set of right (resp., left) semicentral idempotents and $Z(R)$ for the set of central idempotents of R. Note that $S_l(R) \cap S_r(R) = Z(R)$ and if R is a semiprime ring, then $S_l(R) = S_r(R) = Z(R)$.

Theorem 2.2 ([11]). *Let R be a ring and $e = e^2 \in R$. Then R is right (left) e-reversible iff e is left (right) semicentral and eRe is reversible.*

Proposition 2.3. *For any ring R, if R is a right e-reversible, then R is left idempotent reflexive.*

Proof. Assume R is right *e-reversible* and $eRa = 0$, then $era = 0$. Since R is right e-reversible, $aere = 0$ and as e is left semicentral, then $are = 0$ for all $r \in R$. Concluding, $aRe = 0$, thus R is *left idempotent reflexive.* □

The implication in Proposition 2.3 is irreversible as shown in the upcoming example.

Example 2.1. Consider the ring
$$R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{Z}, a + c \equiv 0 \ (mod 2) \right\}.$$

R is idempotent reflexive [5]. For $a = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$, we have $ab = 0$, but $ba \neq 0$. Hence R is not reversible ring. Thus R is not e-reversible ring for any non zero idempotent element in R.

Proposition 2.4. *For any ring R, if eR is right e-reversible and $(1-e)R$ is right $(1-e)$-reversible, then R is reversible.*

Proof. Let $ab = 0$ then $eab = 0$. Since eR is a right e-reversible and e is a left semicentral, $beae = bae = 0$. By the assumption that $(1-e)R$ is right $(1-e)$-reversible, we have $(1-e)ab = 0$ which leads to $b(1-e)a(1-e) = 0$, and $(1-e)$ is a left semicentral implies that $(1-e)$ is a left semicentral implying that, $ba(1-e) = 0$ and hence $ba - bae = 0$. Now, since $bae = 0$, we get $ba = 0$. Therefore, R is a reversible ring. □

In the next result we study some properties of subsets and ideals of e-reversible rings.

Lemma 2.5. *For any ring R, consider the following conditions:*

(1) *R is right e-reversible.*

(2) *$AB = 0$ implies $BAe = 0$ for any nonempty subsets A and B of R.*

(3) *$IJ = 0$ implies $JI = 0$ for all left ideals I, J of R where I is a left ideal generated by an idempotent $e \in E(R)$.*

(4) *$IJ = 0$ implies $JI = 0$ for all ideals I, J of R where I is an ideal generated by an idempotent $e \in E(R)$. Then we have the following implication* (1) $\Leftrightarrow$ (2), (2) $\Leftrightarrow$ (3), (3) $\Rightarrow$ (4).

Proof. (1) $\Leftrightarrow$ (2) Let $AB = 0$ where A, B are subsets of R. For any $a \in A$, $b \in B$ with $ab = 0$, since R is right e-reversible we have $bae = 0$, and $BAe = 0$.

(2) $\Rightarrow$ (3) Let I and J be left ideals of R such that I is generated by $e \in E(R)$. Suppose $IJ = 0$. Since I is left ideal, $RI \subset I$, then $RIJ \subset IJ = 0$ and $RIJ = 0$. Since $e \in I$ we have $ReJ = 0$. By e-reversible property of R we have $JRe = 0$. Since I is generated by e we conclude that $I = Re$. Therefore, $JI = 0$.

(3) $\Rightarrow$ (2) Clearly.

(3) $\Rightarrow$ (4) Let I, J be ideals of R where I is the ideal generated by e and $IJ = 0$. Since $IRJ \subset IJ = 0$ we get $eRJ = 0$. By e-reversibility of R, we get $JeRe = 0$ and then $JRe = 0$, $JReR = 0$. Since I is generated by e, $I = ReR$ and so $JI = 0$. □

It is shown in [9] that for a reversible ring R the homomorphic image of R need not be reversible. We now study the homomorphic image property of e-reversible rings.

The authors in [12, Theorem 2.27] show that if R/I is right $\bar{e}$-reversible, then R is right e-reversible where I is a reduced ideal of R. Moreover in [12, Proposition 2.28] they show that if R is e-symmetric ring and I an ideal of R with $I = r_R(J)$ for some subset J of R, then R/I is right $\bar{e}$-reversible. While we have the following obvious results.

Lemma 2.6. *Let R be right e-reversible ring. Then R/I is right $\bar{e}$-reversible, for any ideal I in R.*

Proposition 2.7. *For any ring R, if R/I is right $\overline{e}$-reversible with I is right e-reduced ideal in R, then R is right e-reversible.*

Proof. Let $ab = 0$ and $\overline{a}, \overline{b} \in R/I$ with $\overline{a} = a + I$, $\overline{b} = b + I$. Suppose $I = \overline{a}\overline{b} = \overline{0}$. Since R/I is right $\overline{e}$-reversible, $(b+I)(a+I)(e+I) = I$. Hence $bae \in I$. But ba is a nilpotent element since $(ba)^2 = baba = 0$, e is an idempotent and I is e-reduced, then $bae = 0$. Therefore R is right e-reversible. □

Recall that an ideal P of a ring R is called strongly prime if $a^2 \in P$ implies $a \in P$.

Proposition 2.8. *For a ring R, the quotient ring R/I by a strongly prime ideal I is a right $\overline{e}$-reversible ring for every idempotent $e \in R$.*

Proof. Let $\overline{a}\overline{b} = 0$, $ab \in I$, $baba = (ba)^2 \in I$. Since I is strongly prime, then $ba \in I$ and hence $bae \in I$. Finally $\overline{b}\overline{a}\overline{e} = 0$ and it follows that R/I is right $\overline{e}$-reversible. □

Lemma 2.9. *Let R be a right e-reversible ring and $f \in E(R)$. If R satisfies one of the following conditions, then R is right f-reversible;*

(1) $eR + (1 - f)R = R$;
(2) $ea + 1 - f \in U(R)$ for some $a \in R$;
(3) $Re + R(1 - f) = R$;
(4) $ae + 1 - f \in U(R)$ for some $a \in R$.

Proof. (1) Let $ab = 0$, R is right e-reversible,therefore e is left semicentral. Noting that $eR + (1 - f)R = R$, then $fR = feR = efeR \subseteq eR$. Since R has unity, it follows that $f = ef$. Since R is right e-reversible, then $bae = 0 = baef$, $baf = 0$. Hence, R is a right f-reversible ring.

(2) Let $ea+1-f = u \in U(R)$. Then $fu = fea$ and one obtains $f = feau^{-1}$. Noting that e is left semicentral, then $f = ef$, and this gives that R is right f-reversible ring.

(3) If $Re + R(1 - f) = R$, then $Rf = Ref$. Set $f = cef$ for some $c \in R$. Then $f = ecef = ef$ since e is left semicentral, therefore R is right f-reversible.

(4) Set $ae + 1 - f = v \in U(R)$. Then $fv = fae$ and one obtains $f = faev^{-1}$. Noting that e is left semicentral, then $f = ef$, therefore R is right f-reversible. □

3. e-strong reversible rings

In this section, we introduce the definition of e-strong reversible rings and show that e-reversible ring need not be e-strongly reversible.

Definition 3.1. Let R be a ring. Then R is called an e-strong reversible ring if for any $a, b \in R$, $ab = 0$ implies $bea = 0$.

It is clear that, R is reversible if and only if R is strongly 1-reversible. Also if R is e-strong reversible ring, $e \in Z(R)$ then R is e-reversible.

The following Example shows that e-reversible ring need not to be e-strong reversible.

Example 3.1. [[15]] Let $S = \{a, b\}$ be the semigroup with multiplication $a^2 = ab = a$ and $b^2 = ba = b$. Consider the semigroup ring $R = \mathbb{Z}_2[S] = \{0, a, b, a + b\}$, and $E(R) = \{0, a, b\}$. It is easy to check that, R is left b-reversible and R is not b-strong reversible.

Example 3.2. Consider the ring $T_2(R)$. $T_2(R)$ is $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$-strong reversible but not right $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$-reversible. Since, for $x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $e = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, we have $xy = 0$, and $yex = 0$ but $yxe \neq 0$.

We discuss the relation between central reversible and e-strong reversible properties through the following Lemma.

Lemma 3.1. *Let R be a semiprime ring. Then every central reversible is e-strong reversible.*

Proof. Let $ab = 0$, then $abe = 0$ for any $e \in R$. Since R is central reversible, $bea \in C(R)$. On the other hand $beasbea = beabeas = 0$ for any $s \in R$. Since R is semiprime, we have $bea = 0$. Therefore, R is e-strong reversible. □

Theorem 3.2. *Any ring R is an e-strong reversible ring if and only if $e \in Z(R)$ and eRe is a reversible ring.*

Proof. Assume $e \in C(R)$ and eRe is a reversible ring. By Theorem 2.2, R is a right e-reversible ring, then R is e-strong reversible. Conversely, assume that R is an e-strong reversible ring. For each $a \in R$, set $g = e + ea(1 - e)$ and $f = e + (1 - e)ae \in E(R)$. Then $eg = g$, $ge = e$, $fe = f$ and $ef = e$. Since R is an e-strong reversible we have $(1 - g)g = 0$ then $ge(1 - g) = 0$ hence $e(1 - g) = 0$ therefore $e = eg = g$. Also $e(1 - f) = 0$ then $(1 - f)e = 0$ therefore $e = fe = f$. Hence $g = e = f$. One obtains $ea(1 - e) = 0 = (1 - e)ae$ for each $a \in R$. Thus $e \in C(R)$. Further, R is a right e-reversible ring. By Theorem 2.2, eRe is a reversible ring. □

4. Some extensions of e-reversible rings

In this section we observe the e-reversibility of some important kinds of ring extensions.

For a monomorphism α of a ring R, we consider the relation between a ring R and the Jordan construction of the ring $A(R, \alpha)$. Let $A(R, \alpha) = \{x^{-i} r x^i : r \in R, i \geq 0\}$ be a subset of the skew laurent polynomial ring $R[x, x^{-1}; \alpha]$, with respect to the relations $xr = \alpha(r)x$ and $rx^{-1} = x^{-1}\alpha(r)$ for all $r \in$

R. Then for each $j \geq 0$, we can write $x^{-i}rx^i = x^{-(i+j)}\alpha^j(r)x^{(i+j)}$. It follows that $A(R,\alpha)$ forms a subring of $R[x,x^{-1};\alpha]$ with the following operations: $(x^{-i}rx^i)+(x^{-j}sx^j) = x^{-(i+j)}(\alpha^j(r)+\alpha^i(s))x^{(i+j)}$, $(x^{-i}rx^i)(x^{-j}sx^j) = x^{-(i+j)}(\alpha^j(r)\alpha^i(s))x^{(i+j)}$, for $r,s \in R$ and $i,j \geq 0$ (see, [7]).

Lemma 4.1 ([7], Lemma 2.1)**.** *Let α be a monomorphism of a ring R. Then $E(A) = \{x^{-i}ex^i : e \in E(R)$ and $i \geq 0\}$.*

Lemma 4.2. *Let α be a monomorphism of a ring R with $\alpha(e) = e$. Then A is right $(x^{-k}ex^k)$-reversible if and only if R is right e-reversible.*

Proof. Let R be right e-reversible and $(x^{-i}rx^i)(x^{-j}sx^j) = 0$, for some $r,s \in R$ and nonnegative integers i,j. Then $x^{-(i+j)}\alpha^j(r)\alpha^i(s)x^{(i+j)} = 0$ and so $\alpha^j(r)\alpha^i(s) = 0$. For each $k \geq 0$ we have $\alpha^{j+k}(r)\alpha^{i+k}(s) = 0$. Since R is right e-reversible, we have $\alpha^{i+k}(s)\alpha^{j+k}(r)\alpha^{i+j}(e) = 0$, for each $k \geq 0$. Therefore,

$$x^{-(i+j+k)}\alpha^{i+k}(s)\alpha^{j+k}(r)\alpha^{i+j}(e)x^{i+j+k} = 0$$

and then $(x^{-j}sx^j)(x^{-i}rx^i)(x^{-k}ex^k) = 0$. Hence A is right $(x^{-k}ex^k)$-reversible. The other direction follows easy. □

An element u of a ring R is called right regular if $ur = 0$ implies $r = 0$ for $r \in R$, similarly left regular elements can be defined. An element is regular if it is both left and right regular(and hence not a zero divisor). The multiplicative closed subsets Δ of a ring R consisting of central regular elements. The elements of $\Delta^{-1}R$ is of the form $u^{-1}r$ such that $u \in \Delta$, $r \in R\}$. We suppose that every idempotent in $\Delta^{-1} R$ is of the form $u^{-1}e$ with $e \in E(R)$ and $u \in \Delta$.

Lemma 4.3. *Let Δ be a multiplicative closed subsets of a ring R consisting of central regular elements.*

(i) If $\Delta^{-1} R$ is right $u^{-1}e$-reversible, then R is right e-reversible.

(ii) If R is right e-reversible, then $\Delta^{-1} R$ is right $u^{-1}e$-reversible.

Proof. (i) Let $\Delta^{-1}R$ be a right e-reversible ring, and $ab = 0$ where $a,b \in R$. Then $u^{-1}w^{-1}ab = 0$, and $(u^{-1}a)(w^{-1}b) = 0$. Since $\Delta^{-1}R$ is right $u^{-1}e$-reversible, we have $(w^{-1}b)(u^{-1}a)(v^{-1}e) = 0$, and then $w^{-1}u^{-1}v^{-1}bae = 0$. Finally, we have $bae = 0$. Therefor R is right e-reversible.

(ii) Conversely, let R be right e-reversible and $ab = 0$, then $bae = 0$. Multiply by $v^{-1}w^{-1}u^{-1}$ then we have $(v^{-1}b)(w^{-1}a)(u^{-1}e) = 0$. Hence $\Delta^{-1}R$ is right $w^{-1}e$-reversible. □

In [1] a ring R is called reversible Armendariz if $f(x)\,g(x) = 0$, then $b_ja_i = 0$, $\forall i,\ j$, whenever $f(x) = \sum_{i=0}^{m} a_ix^i, g(x) = \sum_{j=0}^{n} b_jx^j$ are polynomials in the polynomial ring $R[x]$. In [9, Example 2.1] shows that a ring R being reversible need not imply $R[x]$ being reversible. While we have the following Theorem.

Theorem 4.4. *Let R be reversible Armendariz. Then $R[x]$ is $e(x)$-reversible, where $e(x)$ is an idempotent element in $R[x]$.*

Proof. Assume that $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j$, $e(x) = \sum_{k=0}^{l} e_k x^k \in R[x]$, where e_k are idempotent elements in R. Let $f(x)g(x) = 0$. Since R is reversible Armendariz, then $b_j a_i = 0$ $\forall i, j$. Then $b_j a_i e_k = 0$ $\forall i, j, k$. Then we have $g(x)f(x)e(x) = 0$. There for $R[x]$ is right $e(x)$-reversible Armendariz. □

Corollary 4.5. *For a ring R, $R[x]$ is a right $e(x)$-reversible ring if and only if $R[x, x^{-1}]$ is right $e(x)x^n$-reversible, where $e(x)$ is an idempotent element in $R[x]$ and $e(x)x^n$ is an idempotent element in $R[x, x^{-1}]$.*

Proof. Let $\Delta = \{1, x, x^2, ...\}$. Then Δ is a multiplicative closed central subset of the ring $R[x]$ since $R[x, x^{-1}] = \Delta^{-1}R[x]$. Thus $R[x, x^{-1}]$ is $e(x)$-reversible. The sufficient condition is straight forward since a subring of e-reversible ring is also e-reversible ring and $R[x] \subseteq R[x, x^{-1}]$. □

5. Matrix ring extensions

We devote this section to study the property of right (left) e-reversible ring over some types of matrix rings.

Given a ring R and an R-bimodule ${}_RM_R$, the trivial extension of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the multiplication

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2), \text{ where } r_1, r_2 \in R \text{ and } m_1, m_2 \in M.$$

This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ with the usual matrix operations. Therefore, we have

Theorem 5.1. (1) *If $T(R, R)$ is a right $E = \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix}$-reversible then R is a right e-reversible for each $e = e^2 \in R$.*

(2) *Suppose that R is a reduced ring. If R is right e-reversible, then $T(R, R)$ is right $E = \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix}$-reversible.*

Proof. Suppose $T(R, R)$ is right E-reversible and let $ab = 0$. Let $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, $B = \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}$ in $T(R, R)$, hence $AB = 0$. Since $T(R, R)$ is E-reversible $BAE = \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix} = \begin{bmatrix} bae & 0 \\ 0 & bae \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, this leads to $bae = 0$. Thus R is right e-reversible.

Conversely, let R be right e-reversible and $AB = 0$ for any $A, B \in T(R, R)$. Then $AB = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \begin{bmatrix} c & d \\ 0 & c \end{bmatrix} = \begin{bmatrix} ac & ad + bc \\ 0 & ac \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Therefore, $ac = 0 = ad + bc$, and since R is reduced we get $(ca)^2 = caca = 0$, $ca = 0$ and then

$cae = 0$. Since $ad + bc = 0$, then $ada + bca = 0$ which implies $ada = 0$, then $adad = 0 = (ad)^2$. Since R is a reduced ring and then $ad = 0$, also R is right e-reversible hence $dae = 0$. Therefore, $bc = 0$. Moreover, since R is a right e-reversible, $cbe = 0$. Hence $BAE = \begin{bmatrix} c & d \\ 0 & c \end{bmatrix} \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix} = \begin{bmatrix} cae & cbe + dae \\ 0 & cae \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. So $T(R, R)$ is a right $\begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix}$-reversible ring. □

Proposition 5.2. *Let R be a ring and $e \in E(R)$, $r \in R$. Then we have the following results:*

(1) *$T_3(R)$ is right $\begin{bmatrix} 1 & r & r \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$-reversible if and only if R is reversible.*

(2) *$T_3(R)$ is right $\begin{bmatrix} e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$-reversible if and only if R is right e-reversible.*

(3) *$T_3(R)$ is right $\begin{bmatrix} e & e & e \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$-reversible ring if and only if R is right e-reversible.*

Proof. (1) Let $a, b \in T_3(R)$ and $ab = 0$. Then we have $\begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} b & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Since $T_3(R)$ is right $\begin{bmatrix} 1 & r & r \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$-reversible,

then $\begin{bmatrix} b & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & r & r \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} ba & bar & bar \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. This leads to $ba = 0$, therefor R is a reversible ring. Conversely, let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ 0 & d_1 & e_1 \\ 0 & 0 & f_1 \end{bmatrix}$, $B = \begin{bmatrix} a_2 & b_2 & c_2 \\ 0 & d_2 & e_2 \\ 0 & 0 & f_2 \end{bmatrix} \in T_3(R)$. $AB = \begin{bmatrix} a_1a_2 & a_1b_2 + b_1d_2 & a_1c_2 + b_1e_2 + c_1f_2 \\ 0 & d_1d_2 & d_1e_2 + e_1f_2 \\ 0 & 0 & f_1f_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $a_1a_2 = d_1d_2 = f_1f_2 = 0$. Since R is a reversible ring, then $a_2a_1 = d_2d_1 = 0 = f_2f_1$. $BA \begin{bmatrix} 1 & r & r \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a_2a_1 & a_2a_1r & a_2a_1r \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. So $T_2(R)$ is right $\begin{bmatrix} 1 & r \\ 0 & 0 \end{bmatrix}$-reversible. Similarly, we can prove (2) and (3). □

Recall that a Morita context is a 4-tuble $T = (A, W, V, B)$ and two bimodule homomorphisms ϕ, θ where A, B are rings, ${}_AW_B$ and ${}_BV_A$ are bimodules, and there exists a context product $W \times V \to A$ and $V \times W \to B$ written multiplicatively as $(w, v) \longmapsto wv$, $(v, w) \longmapsto vw$, such that $\begin{bmatrix} A & W \\ V & B \end{bmatrix}$ is an associative ring with matrix operations. Morita context were introduced in [18]. A Morita context $\begin{bmatrix} A & W \\ V & B \end{bmatrix}$ is called trivial if the context products are trivial, i.e $WV = 0$ and $VW = 0$.

According to [4], the matrix $\begin{bmatrix} r & w \\ v & s \end{bmatrix}$ is an idempotent element in the trivial Morita context if and only if $r^2 = r$, $s^2 = s$, $rw + ws = w$ and $vr + sv = v$. Therefore, the trivial idempotents are $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Other idempotents of T are of the form $\begin{bmatrix} 1 & w \\ v & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & w \\ v & 1 \end{bmatrix}$.

We study now the e-reversibility over the trivial Morita context.

Theorem 5.3. *Let* $T = \begin{bmatrix} R & W \\ V & S \end{bmatrix}$ *be a trivial Morita context ring, if* $Wf = W$, $Ve = V$ *for any* $e^2 = e \in R$ *and* $f^2 = f \in S$. *Then* T *is right* $\begin{bmatrix} e & w \\ v & f \end{bmatrix}$*-reversible if and only if* R *is right* e*-reversible and* S *is right* f*-reversible, and* $V = 0, W = 0$.

Proof. Assume that T is $\begin{bmatrix} e & w \\ v & f \end{bmatrix}$-reversible. If we take $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $f = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, then clearly e and f are idempotents of T. Since $eTe \cong R$ and $fTf \cong S$, using Theorem [2.2], R is right e-reversible and S is right f-reversible. Conversely, Let R be right e-reversible and S be right f-reversible rings and $\begin{bmatrix} r_1 & w_1 \\ v_1 & s_1 \end{bmatrix} \begin{bmatrix} r_2 & w_2 \\ v_2 & s_2 \end{bmatrix} = 0$, then $\begin{bmatrix} r_1r_2 & w_1w_2 + w_1s_2 \\ v_1r_2 + s_1v_2 & s_1s_2 \end{bmatrix} = 0$. Since R is right e-reversible and S is right f-reversible, then $r_2r_1e = 0$ and $s_2s_1f = 0$. Hence $\begin{bmatrix} r_2 & w_2 \\ v_2 & s_2 \end{bmatrix} \begin{bmatrix} r_1 & w_1 \\ v_1 & s_1 \end{bmatrix} \begin{bmatrix} e & w \\ v & f \end{bmatrix} = \begin{bmatrix} r_2r_1e & w^* \\ v^* & s_2s_1f \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and consequently T is right $\begin{bmatrix} e & w \\ v & f \end{bmatrix}$-reversible. □

Theorem 5.4. *Let* $T = \begin{bmatrix} r & w \\ v & s \end{bmatrix}$ *be right* $\begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}$*-reversible. Then* R *is right* e*-reversible.*

Proof. Let T be right $\begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}$-reversible, $r_1r_2 = 0$ and $s_1s_2 = 0$. Since $0 = \begin{bmatrix} r_1 & 0 \\ 0 & s_1 \end{bmatrix}\begin{bmatrix} r_2 & 0 \\ 0 & s_2 \end{bmatrix} = \begin{bmatrix} r_1r_2 & 0 \\ 0 & s_1s_2 \end{bmatrix}$ then $\begin{bmatrix} r_2 & 0 \\ 0 & s_2 \end{bmatrix}\begin{bmatrix} r_1 & 0 \\ 0 & s_1 \end{bmatrix}\begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} r_2r_1e & 0 \\ 0 & 0 \end{bmatrix}$. Therefore, R is right e-reversible. □

Theorem 5.5. *Let $T = \begin{bmatrix} r & w \\ v & s \end{bmatrix}$ be left $\begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix}$-reversible. Then S is left e-reversible.*

Proof. As in Theorem [5.4]. □

Let S and T be any rings, ${}_SM_T$ an S-T-bimodule. Then $R = \begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$ is the formal triangular matrix ring. Now, we give the following Corollary as a direct consequence of Theorem 5.3.

Corollary 5.6. *Let $T = \begin{bmatrix} R & W \\ 0 & S \end{bmatrix}$ be the formal triangular matrix ring. Then the following conditions are equivalent*

(1) *T is right $\begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}$-reversible.*

(2) *R is right e-reversible, S is right f-reversible and $f \in r_S(W)$.*

Proof. (1) → (2) Let $r_1r_2 = 0$ and $s_1s_2 = 0$. Let $\begin{bmatrix} r_1 & 0 \\ 0 & s_1 \end{bmatrix}, \begin{bmatrix} r_2 & 0 \\ 0 & s_2 \end{bmatrix} \in T$. Then $0 = \begin{bmatrix} r_1 & 0 \\ 0 & s_1 \end{bmatrix}\begin{bmatrix} r_2 & 0 \\ 0 & s_2 \end{bmatrix} = \begin{bmatrix} r_1r_2 & 0 \\ 0 & s_{1s_2} \end{bmatrix}$. Since T is right $\begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}$-reversible, $0 = \begin{bmatrix} r_2 & 0 \\ 0 & s_2 \end{bmatrix}\begin{bmatrix} r_1 & 0 \\ 0 & s_1 \end{bmatrix}\begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix} = \begin{bmatrix} r_2r_1e & 0 \\ 0 & s_2s_1f \end{bmatrix}$. Therefore, $r_2r_1e = 0$ and $s_2s_1f = 0$. We conclude that R is right e-reversible and S is right f-reversible.

(2) → (1) Let $r_1r_2 = 0$ and $s_1s_2 = 0$. Assume $\begin{bmatrix} r_1 & w_1 \\ 0 & s_1 \end{bmatrix}, \begin{bmatrix} r_2 & w_2 \\ 0 & s_2 \end{bmatrix} \in T$, such that $0 = \begin{bmatrix} r_1 & w_1 \\ 0 & s_1 \end{bmatrix}\begin{bmatrix} r_2 & w_2 \\ 0 & s_2 \end{bmatrix} = \begin{bmatrix} r_1r_2 & r_1w_2 + w_1s_2 \\ 0 & s_1s_2 \end{bmatrix}$. Since R is right e-reversible and S is right f-reversible. We have $r_2r_1e = 0$ and $s_2s_1f = 0$. Therefore, $\begin{bmatrix} r_2 & w_2 \\ 0 & s_2 \end{bmatrix}\begin{bmatrix} r_1 & w_1 \\ 0 & s_1 \end{bmatrix}\begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix} = \begin{bmatrix} t_2t_1e & t_2w_1f + w_2s_1f \\ 0 & s_2s_1f \end{bmatrix}$. Since $f \in r_S(W)$, then $t_2w_1f + w_2s_1f = 0$. Hence, T is right $\begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}$-reversible. □

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