Some differential subordinations and fuzzy differential subordinations using generalized integral operator

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Abstract. S.S. Miller and P.T. Mocanu introduced differential subordination and derived some properties associated with it. Motivated by this studies the aim of this paper is to establish some properties of differential subordination and fuzzy differential subordination associated with generalized integral operator which defined in the open unit disk.
Keywords: differential subordination, fuzzy subordination, fuzzy set, fuzzy best dominant.

1. Introduction

The theory of differential subordination was introduced by S.S. Miller and P.T.Mocanu in [7], then developed by many authors see also [9, 10]. In [11]

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authors have extended the notion of subordination from the geometric theory of analytic functions of one complex variable to the fuzzy set theory. In [12] the authors have defined the notion of fuzzy differential subordination. In this paper we will study differential subordination and fuzzy differential subordination for certain classes of holomorphic functions.

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ and $H(U)$ denote the class of holomorphic functions in U. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, we denote by $H[a, n] = \{f \in H(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ..., z \in U\}$ and $\mathcal{A}_n$=$\{f \in H(U) : f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + ..., z \in U\}$ with $\mathcal{A}_1$=$\mathcal{A}$.

Definition 1.1 ([7])**.** *For the functions f and g analytic in U, we say that the function f is subordinate to g in U and written as $f \prec g$, if there exist a Schwartz function w analytic in U with $w(0) = 0, |w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$ $(z \in U)$. In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0) = g(0),\ f(U) \subset g(U)$.*

Definition 1.2 ([19])**.** *Let X be a non-empty set. An application $F : X \to [0, 1]$ is called fuzzy subset. An alternate definition, more precise would be the following:*
A pair (S, F_S), where F_S: $X \to [0, 1]$ and $supp(S, F_S) = \{x \in X : 0 < F_S(x) \le 1\}$ is called fuzzy subset. The function F_S is called membership function of the fuzzy subset (S, F_S).

Definition 1.3 ([11])**.** *Let two fuzzy subsets of X be (M, F_M) and (N, F_N). We say that the fuzzy subsets M and N are equal if and only if $F_M(x) = F_N(x),\ x \in X$ and we denote this by $(M, F_M) = (N, F_N)$. The fuzzy subset (M, F_M) is contained in the fuzzy subset (N, F_N) if and only if $F_M(x) \le F_N(x),\ x \in X$ and we denote the inclusion relation by $(M, F_M) \subseteq (N, F_N)$.*
Assume that D is a set in $\mathbb{C}$ and f, g are holomorphic functions. We indicate by

$$f(D) = supp(f(D), F_{f(D)}) = \{f(z) : 0 < F_{f(D)}(f(z)) \le 1, z \in D\}$$

and

$$g(D) = supp(g(D), F_{g(D)}) = \{g(z) : 0 < F_{g(D)}(g(z)) \le 1, z \in D\}.$$

Definition 1.4 ([11])**.** *Suppose that D is a set in $\mathbb{C}$, $z_0 \in D$ is a fixed point and let the functions $f, g \in H(D)$. The function f is named a fuzzy subordinate to g and write $f \prec_F g$ or $f(z) \prec_F g(z)$ if*
(1) $f(z_0) = g(z_0)$,
(2) $F_{f(D)}(f(z)) \le F_{g(D)}(g(z)),\ z \in D$.

Definition 1.5 ([12])**.** *Let h be univalent in U and $\Psi : \mathbb{C}^3 \times U \to \mathbb{C}$. If $\mathcal{P}$ holomorphic in U satisfies the fuzzy differential subordination*

$$F_{\psi(\mathbb{C}^3.U)}(\psi(\mathcal{P}(z), z\mathcal{P}'(z), z^2\mathcal{P}''(z); z)) \le F_{h(U)}(h(z)), \tag{1}$$

i.e.,

$$\psi(\mathcal{P}(z), z\mathcal{P}'(z), z^2\mathcal{P}''(z); z) \prec_F h(z),\ z \in U$$

then $\mathcal{P}$ is called a fuzzy solution of the fuzzy differential subordination. The univalent function q is called a fuzzy dominant of the fuzzy solutions of the fuzzy differential subordination, or more simple a fuzzy dominant, if $\mathcal{P}(z) \prec_F q(z),\ z \in U$ for all $\mathcal{P}$ satisfying (1). A fuzzy dominant $\tilde{q}$ that satisfies $\tilde{q}(z) \prec_F q(z),\ z \in U$ for all fuzzy dominant q of (1)is said to be the fuzzy best dominant of (1).

Definition 1.6. *Raina and Sharma [14] (see, also, [1], [2]) defined the integral operator for $\mu > 0$ and $a, c \in \mathbb{C}$ are such that $\Re(c-a) \geq 0$, $\mathcal{J}_\mu^{a,c} : T \to T$ as*
i) For $\Re(c-a) \geq 0$ and $\Re(a) > -\mu$ by

$$\mathcal{J}_\mu^{a,c} f(z) = \frac{\Gamma(c+\mu)}{\Gamma(a+\mu)\Gamma(c-a)} \int_0^1 (1-t)^{c-a-1} t^{a-1} f(zt^\mu) dt. \tag{2}$$

ii) For a=c by

$$\mathcal{J}_\mu^{a,a} f(z) = f(z). \tag{3}$$

Where Γ stands for Euler's Gamma function (which is valid for all complex number except the non-positive integer) for $f(z)$, it follows easily from (2) and (3) that

$$\mathcal{J}_\mu^{a,c} f(z) = z + \frac{\Gamma(c+\mu)}{\Gamma(a+\mu)} \sum_{k=2}^{\infty} \frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)} a_k z^k, (\mu > 0, \Re(c) \geq \Re(a) > -\mu). \tag{4}$$

We need the following lemmas in investigating our main results.

Lemma 1.1 ([7]). *Let h be a convex function with $h(0) = a$, and let $\gamma \in \mathbb{C}^*$ be a complex number with $\Re(\gamma) \geq 0$. If $\mathcal{P} \in H[a,n]$ and*

$$\mathcal{P}(z) + \frac{1}{\gamma} z\mathcal{P}'(z) \prec h(z),\ z \in U$$

then

$$\mathcal{P}(z) \prec g(z) \prec h(z),\ z \in U$$

where

$$g(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t) t^{\frac{\gamma}{n}-1} dt,\ z \in U.$$

Lemma 1.2 ([7]). *Let g be a convex function in U and let $h(z) = g(z) + n\alpha z g'(z)$, for $z \in U$, where $\alpha > 0$ and n is positive integer. If $\mathcal{P}(z) = g(0) + \mathcal{P}_n z^n + \mathcal{P}_{n+1} z^{n+1} + \ldots,\ z \in U$ is holomorphic in U and*

$$\mathcal{P}(z) + \alpha z\mathcal{P}'(z) \prec h(z),\ z \in U$$

then

$$\mathcal{P}(z) \prec g(z),\ z \in U,$$

and the result is sharp.

Lemma 1.3 ([13]). *Let h be a convex function with $h(0) = a$, and let $\gamma \in \mathbb{C}^*$ be a complex number with $\Re(\gamma) \geq 0$. If $\mathcal{P} \in H[a, n]$ with $\mathcal{P}(0) = a$ and $\Psi : \mathbb{C}^2.U \to \mathbb{C}, \psi(\mathcal{P}(z), z\mathcal{P}'(z)) = \mathcal{P}(z) + \frac{1}{\gamma} z\mathcal{P}'(z)$ is holomorphic in U, then*

$$F_{\psi(\mathbb{C}^2.U)}\left[\mathcal{P}(z) + \frac{1}{\gamma} z\mathcal{P}'(z))\right] \leq F_{h(U)}(h(z)),$$

implies

$$F_{\mathcal{P}(U)}(\mathcal{P}(z)) \leq F_{q(U)}(q(z)) \leq F_{h(U)}(h(z)), z \in U,$$

i.e.,

$$\mathcal{P}(z) \prec_F q(z) \prec_F h(z),$$

where

$$q(z) = \frac{\gamma}{n z^{\frac{\gamma}{n}}} \int_0^z h(t) t^{\frac{\gamma}{n} - 1} dt$$

the function q is convex and is the fuzzy best (a,n)-dominant.

Lemma 1.4 ([13]). *Suppose that q is convex function in U, let $h(z) = g(z) + n\gamma z g'(z)$, $\gamma > 0$ and $n \in \mathbb{N}$. If $\mathcal{P} \in H[q(0), n]$ and $\Psi : \mathbb{C}^2 \times U \to \mathbb{C}, \psi(\mathcal{P}(z), z\mathcal{P}'(z)) = \mathcal{P}(z) + \gamma z\mathcal{P}'(z)$ is holomorphic in U, then*

$$F_{\psi(\mathbb{C}^2.U)}\left[\mathcal{P}(z) + \gamma z\mathcal{P}'(z)\right] \leq F_{h(U)}(h(z)),$$

implies

$$F_{\mathcal{P}(U)}(\mathcal{P}(z)) \leq F_{q(U)}(q(z)), z \in U,$$

i.e.,

$$\mathcal{P}(z) \prec_F q(z)$$

and q is the fuzzy best dominant.

Recently Oros and Oros [12, 13], Lupas[4, 5, 6], Hyder [8], Wanas [17], Wanas and Bulut [16], Altınkaya and Wanas [3] and Wanas and Majeed [15, 18] have obtained fuzzy differential subordination results for certain classes of holomorphic functions.Using the integral operator defined in (4), we study differential subordinations and fuzzy differential subordinations.

2. Results based on subordination

Theorem 2.1. *Let g be a convex function, $g(0) = 1$ and let h be the function $h(z) = g(z) + \frac{z}{\lambda} g'(z)$, $z \in U$, if $a, \lambda > 0, n \in \mathbb{N}, f \in \mathcal{A}$ and satisfies the differential subordination*

$$\left(\frac{\mathcal{J}_{\mu}^{a,c} f(z)}{z}\right)^{\lambda - 1} \left(\mathcal{J}_{\mu}^{a,c} f(z)\right)' \prec h(z), z \in U, \tag{5}$$

then

$$\left(\frac{\mathcal{J}_{\mu}^{a,c} f(z)}{z}\right)^{\lambda} \prec g(z), z \in U,$$

and this result is sharp.

Proof. Let

$$\mathcal{J}_{\mu}^{a,c}f(z) = z + \frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}\sum_{k=2}^{\infty}\frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)}a_k z^k, (\mu > 0, \Re(c) \geq \Re(a) > -\mu).$$

Consider

$$\mathcal{P}(z) = \left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda} = 1 + \mathcal{P}_{\lambda}z^{\lambda} + \mathcal{P}_{\lambda+1}z^{\lambda+1} + ...,\ z \in U.$$

We deduce that $\mathcal{P} \in H[1,\lambda]$. Differentiating above equation, we get

$$\left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda-1}\left(\mathcal{J}_{\mu}^{a,c}f(z)\right)' = \mathcal{P}(z) + \frac{1}{\lambda}z\mathcal{P}'(z),\ z \in U,$$

then (5) becomes

$$\mathcal{P}(z) + \frac{1}{\lambda}z\mathcal{P}'(z) \prec h(z) = g(z) + \frac{z}{\lambda}g'(z), z \in U.$$

By using Lemma 1.2, we have

$$\mathcal{P}(z) \prec g(z), z \in U,$$

i.e.,

$$\left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda} \prec g(z),\ z \in U.$$

It can be observed that this result is sharp. □

Theorem 2.2. *Let h be a holomorphic function which satisfies the inequality $Re(1+\frac{zh''(z)}{h'(z)}) > -\frac{1}{2}, z \in U$, and $h(0) = 1$. if $a, \lambda > 0, n \in \mathbb{N}, f \in \mathcal{A}$ and satisfies the differential subordination*

$$\left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda-1}\left(\mathcal{J}_{\mu}^{a,c}f(z)\right)' \prec h(z), z \in U, \tag{6}$$

then

$$\left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda} \prec q(z), z \in U,$$

where

$$q(z) = \frac{\lambda}{z^{\lambda}}\int_0^z h(t)t^{\lambda-1}dt.$$

The function q is convex and it is the best dominant.

Proof. Let

$$\begin{aligned}\mathcal{P}(z) &= \left(\frac{\mathcal{J}_{\mu}^{a,c} f(z)}{z}\right)^{\lambda}\\ &= \left(\frac{z+\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}\sum_{k=2}^{\infty}\frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)}a_k z^k}{z}\right)^{\lambda}\\ &= \left(1+\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}\sum_{k=2}^{\infty}\frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)}a_k z^{k-1}\right)^{\lambda}\\ &=1+\sum_{j=\lambda+1}^{\infty}\mathcal{P}_j z^{j-1}\\ &=1+\lambda\left(\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}\sum_{k=2}^{\infty}\frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)}a_k z^{k-1}\right)+\\ &+\frac{\lambda(\lambda-1)}{2!}\left(\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}\sum_{k=2}^{\infty}\frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)}a_k z^{k-1}\right)^2+...,\end{aligned}$$

for $z \in U,\ \mathcal{P} \in H[1,\lambda]$. Differentiating $\mathcal{P}$, we obtain

$$\left(\frac{\mathcal{J}_{\mu}^{a,c} f(z)}{z}\right)^{\lambda-1}\left(\mathcal{J}_{\mu}^{a,c} f(z)\right)' = \mathcal{P}(z)+\frac{1}{\lambda}z\mathcal{P}'(z), z\in U,$$

and (6) becomes

$$\mathcal{P}(z)+\frac{1}{\lambda}z\mathcal{P}'(z) \prec h(z), z\in U.$$

Using Lemma 1.1, we have

$$\mathcal{P}(z) \prec q(z), z\in U,$$

i.e.,

$$\left(\frac{\mathcal{J}_{\mu}^{a,c} f(z)}{z}\right)^{\lambda} \prec q(z) = \frac{\lambda}{z^{\lambda}}\int_0^z h(t)t^{\lambda-1}dt,\ z\in U,$$

and q is the best fuzzy dominant. □

Corollary 2.1. *Let* $h(z)=\frac{1+(2\beta-1)z}{1+z}$ *be a convex function in* U*, where* $0\le\beta<1$*. If* $a,\lambda\ge 0, n\in\mathbb{N}, f\in\mathcal{A}$ *and satisfies the differential subordination*

$$\left(\frac{\mathcal{J}_{\mu}^{a,c} f(z)}{z}\right)^{\lambda-1}\left(\mathcal{J}_{\mu}^{a,c} f(z)\right)' \prec h(z), z\in U, \tag{7}$$

then

$$\left(\frac{\mathcal{J}_{\mu}^{a,c} f(z)}{z}\right)^{\lambda} \prec q(z), z\in U,$$

where q is given by

$$q(z) = (2\beta - 1) + \frac{2(1-\beta)\lambda}{z^{\lambda}} \int_0^z \frac{t^{\lambda-1}}{1+t} dt, z \in U.$$

The function q is convex and it is the best dominant.

Proof. Following the same steps as in the proof of the Theorem 2.2 and considering $\mathcal{P}(z) = \left(\frac{\mathcal{J}_{\mu}^{a,c} f(z)}{z}\right)^{\lambda}$, the differential subordination (7) becomes

$$\mathcal{P}(z) + \frac{1}{\lambda} z\mathcal{P}'(z) \prec h(z) = \frac{1+(2\beta-1)z}{1+z}, z \in U.$$

By using Lemma 1.2, we have $\mathcal{P}(z) \prec q(z)$ i.e.,

$$\begin{aligned} \left(\frac{\mathcal{J}_{\mu}^{a,c} f(z)}{z}\right)^{\lambda} \prec q(z) =& \frac{\lambda}{z^{\lambda}} \int_0^z h(t) t^{\lambda-1} dt \\ =& \frac{\lambda}{z^{\lambda}} \int_0^z t^{\lambda-1} \frac{(1+(2\beta-1)t)}{1+t} dt \\ =& \frac{\lambda}{z^{\lambda}} \int_0^z \left[(2\beta-1) t^{\lambda-1} + 2(1-\beta) \frac{t^{\lambda-1}}{1+t} \right] dt. \end{aligned}$$

Therefore,

$$\left(\frac{\mathcal{J}_{\mu}^{a,c} f(z)}{z}\right)^{\lambda} \prec q(z) = (2\beta-1) + \frac{2(1-\beta)\lambda}{z^{\lambda}} \int_0^z \frac{t^{\lambda-1}}{1+t} dt, z \in U. \qquad \square$$

3. Results based on fuzzy subordination

Theorem 3.1. *Suppose that the convex function h satisfies $h(0) = 1$. Let $f \in \mathcal{A}$ and*

$$\frac{1}{z} \mathcal{J}_{\mu}^{a,c} f(z) + \frac{\Gamma(c+\mu)}{\Gamma(a+\mu)} \sum_{k=2}^{\infty} \frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)} a_k z^{k-1} + z(\mathcal{J}_{\mu}^{a,c} f(z))''$$

is holomorphic in U. If

$$F_{\psi(\mathbb{C}^2.U)} \left[\frac{1}{z} \mathcal{J}_{\mu}^{a,c} f(z) + \frac{\Gamma(c+\mu)}{\Gamma(a+\mu)} \sum_{k=2}^{\infty} \frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)} a_k z^{k-1} + z(\mathcal{J}_{\mu}^{a,c} f(z))'' \right]$$

$$\leq F_{h(U)}(h(z)), \tag{8}$$

then

$$F_{(\mathcal{J}_{\mu}^{a,c} f)'(U)} (\mathcal{J}_{\mu}^{a,c} f(z))' \leq F_{q(U)} q(z) \leq F_{h(U)} h(z),$$

i.e.,

$$(\mathcal{J}_{\mu}^{a,c} f(z))' \prec_F q(z) \prec_F h(z),$$

where $q(z) = \frac{1}{z} \int_0^z h(t) dt$ is convex and is the fuzzy best dominant.

Proof. Assume that

$$\mathcal{P}(z) = \left(\mathcal{J}_\mu^{a,c} f(z)\right)'. \tag{9}$$

Then $\mathcal{P} \in H[1,1]$ and $\mathcal{P}(0) = 1$. therefore, we have

(10)

$$\begin{aligned}
\mathcal{P}(z) + z\mathcal{P}'(z) = &\left(\mathcal{J}_\mu^{a,c} f(z)\right)' + z\left(\mathcal{J}_\mu^{a,c} f(z)\right)'' \\
=&1 + \frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}\sum_{k=2}^{\infty}\frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)}a_k k z^{k-1} + \\
&+ z\left(\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}\sum_{k=2}^{\infty}\frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)}a_k k(k-1) z^{k-2}\right) \\
=&1 + \frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}\sum_{k=2}^{\infty}k^2\frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)}a_k z^{k-1} \\
=&\frac{1}{z}\mathcal{J}_\mu^{a,c} f(z) + \frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}\sum_{k=2}^{\infty}\frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)}a_k z^{k-1} + z(\mathcal{J}_\mu^{a,c} f(z))''.
\end{aligned}$$

According to (8)and (10), we deduce that

$$F_{\psi(\mathbb{C}^2.U)}\left[\mathcal{P}(z) + z\mathcal{P}'(z)\right] \leq F_{h(U)}(h(z)).$$

Thus, by applying Lemma 1.3 with $\gamma = 1$, we obtain

$$F_{\mathcal{P}(U)}\mathcal{P}(z) \leq F_{q(U)}q(z) \leq F_{h(U)}h(z), z \in U.$$

From (9), we find that

$$F_{(\mathcal{J}_\mu^{a,c} f)'U}(\mathcal{J}_\mu^{a,c} f(z))' \leq F_{q(U)}q(z) \leq F_{h(U)}h(z),$$

i.e.,

$$(\mathcal{J}_\mu^{a,c} f(z))' \prec_F q(z) \prec_F h(z),$$

where $q(z) = \frac{1}{z}\int_0^z h(t)dt$ is convex and is the fuzzy best dominant. □

For $a = c$ and $h(z) = \frac{1+(2p-1)z}{1+z}(0 \leq p < 1)$ in Theorem 1.3, we obtain the following corollary:

Corollary 3.1. *Let $f \in \mathcal{A}$ and $zf''(z) + f'(z)$ is holomorphic in U. If*

$$zf''(z) + f'(z) \prec_F \frac{1 + (2p-1)z}{1+z},$$

then

$$f'(z) \prec_F q(z) \prec_F \frac{1 + (2p-1)z}{1+z},$$

where $q(z) = 2p - 1 + \frac{2(1-p)}{z}ln(1+z)$ is convex and the fuzzy best dominant.

Theorem 3.2. *Suppose that the convex function h satisfies $h(0) = 1$. Let $f \in \mathcal{A}$ and $(\mathcal{J}_{\mu}^{a,c} f(z))'$ is holomorphic in U. If*

$$F_{\psi(\mathbb{C}^2.U)}\left[(\mathcal{J}_{\mu}^{a,c} f(z))'\right] \leq F_{h(U)} h(z), \tag{11}$$

then

$$F_{(\mathcal{J}_{\mu}^{a,c} f)(U)}\left(\frac{\mathcal{J}_{\mu}^{a,c} f(z)}{z}\right) \leq F_{q(U)} q(z) \leq F_{h(U)} h(z),$$

i.e.,

$$\left(\frac{\mathcal{J}_{\mu}^{a,c} f(z)}{z}\right) \prec_F q(z) \prec_F h(z),$$

where $q(z) = \frac{1}{z}\int_0^z h(t)dt$ is convex and is the fuzzy best dominant.

Proof. Assume that

$$\mathcal{P}(z) = \frac{\mathcal{J}_{\mu}^{a,c} f(z)}{z}. \tag{12}$$

It is clear that $\mathcal{P} \in H[1,1]$ and $\mathcal{P}(0) = 1$, we find that

$$\mathcal{P}(z) + z\mathcal{P}'(z) = (\mathcal{J}_{\mu}^{a,c} f(z))'. \tag{13}$$

In view of (13), the fuzzy differential subordination (11) becomes

$$F_{\psi(\mathbb{C}^2.U)}\left[\mathcal{P}(z) + z\mathcal{P}'(z)\right] \leq F_{h(U)}(h(z)).$$

Thus, by applying Lemma 1.3 with $\gamma = 1$, we obtain

$$F_{\mathcal{P}(U)}\mathcal{P}(z) \leq F_{q(U)} q(z) \leq F_{h(U)} h(z), z \in U.$$

From (12), we get

$$F_{(\mathcal{J}_{\mu}^{a,c} f)(U)}\left(\frac{\mathcal{J}_{\mu}^{a,c} f(z)}{z}\right) \leq F_{q(U)} q(z) \leq F_{h(U)} h(z),$$

i.e.,

$$\left(\frac{\mathcal{J}_{\mu}^{a,c} f(z)}{z}\right) \prec_F q(z) \prec_F h(z),$$

where $q(z) = \frac{1}{z}\int_0^z h(t)dt$ is convex and is the fuzzy best dominant. □

For $a = c$ and $h(z) = e^{bz}, |b| \leq 1$ in Theorem 3.2, we obtain the following

Corollary 3.2. *Let $f \in \mathcal{A}$, $f'(z)$ is holomorphic in U. If $f'(z) \prec_F e^{bz}$, then*

$$\frac{f(z)}{z} \prec_F q(z) \prec_F e^{bz},$$

where $q(z) = \frac{e^{bz}-1}{bz}$ is convex and the fuzzy best dominant.

Theorem 3.3. *Let g be a convex function, g(0)=1 and let h be the function* $h(z)= g(z)+\frac{z}{\lambda}g'(z), z\in U$. *If* $a,\lambda>0, n\in\mathbb{N}, f\in\mathcal{A}$ *and* $\left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda-1}\left(\mathcal{J}_{\mu}^{a,c}f(z)\right)'$ *is holomorphic in U. If*

$$F_{\psi(\mathbb{C}^2.U)}\left[\left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda-1}\left(\mathcal{J}_{\mu}^{a,c}f(z)\right)'\right]\le F_{h(U)}h(z), \tag{14}$$

then

$$F_{(\mathcal{J}_{\mu}^{a,c}f)^{\lambda}(U)}\left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda}\le F_{g(U)}g(z),$$

i.e.,

$$\left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda}\prec_F g(z),$$

and this result is sharp.

Proof. Assume that

$$\mathcal{P}(z)=\left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda}. \tag{15}$$

Then $\mathcal{P}\in H[1,1]$ and $\mathcal{P}(0)=1$, therefore in view of (4) and (15), we have

$$\mathcal{P}(z)+\frac{1}{\lambda}z\mathcal{P}'(z)=\left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda-1}\left(\mathcal{J}_{\mu}^{a,c}f(z)\right)'. \tag{16}$$

According to (14) and (16), We obtained

$$F_{\psi(\mathbb{C}^2.U)}\left[\mathcal{P}(z)+\frac{1}{\lambda}z\mathcal{P}'(z)\right]\le F_{h(U)}h(z),$$

then by applying Lemma 4 with $\gamma=\lambda$ we have

$$F_{\mathcal{P}(U)}(\mathcal{P}(z))\le F_{q(U)}(q(z))\le F_{h(U)}(h(z)), z\in U.$$

From (15) we obtain

$$F_{(\mathcal{J}_{\mu}^{a,c}f)^{\lambda}(U)}\left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda}\le F_{g(U)}g(z),$$

i.e.,

$$\left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda}\prec_F g(z),\ \ z\in U,$$

and this result is sharp. □

Theorem 3.4. *Let h be a holomorphic function which satisfies the inequality $Re\left(1+\frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, $z \in U$ and $h(0) = 1$ if $a, \lambda > 0$, $n \in \mathbb{N}, f \in \mathcal{A}$ and $\left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda-1}\left(\mathcal{J}_{\mu}^{a,c}f(z)\right)'$ is holomorphic in U. If*

$$F_{\psi(\mathbb{C}^2.U)}\left[\left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda-1}\left(\mathcal{J}_{\mu}^{a,c}f(z)\right)'\right] \leq F_{h(U)}h(z), \tag{17}$$

then

$$F_{(\mathcal{J}_{\mu}^{a,c}f)^{\lambda}(U)}\left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda} \prec F_{q(U)}q(z), z \in U,$$

i.e.,

$$\left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda} \prec_F q(z), z \in U,$$

where $q(z) = \frac{\lambda}{z^{\lambda}}\int_0^z h(t)t^{\lambda-1}dt$ the function q is convex and it is the best dominant.

Proof. Assume that

$$\mathcal{P}(z) = \left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda}. \tag{18}$$

It is clear that $\mathcal{P} \in H[1,1]$ and $\mathcal{P}(0) = 1$, we find that

$$\mathcal{P}(z) + \frac{1}{\lambda}z\mathcal{P}'(z) = \left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda-1}\left(\mathcal{J}_{\mu}^{a,c}f(z)\right)'. \tag{19}$$

According to (17) and (19), we obtain

$$F_{\psi(\mathbb{C}^2.U)}\left[\mathcal{P}(z) + \frac{1}{\lambda}z\mathcal{P}'(z)\right] \leq F_{h(U)}h(z).$$

Then by applying Lemma 3 with $\gamma = \lambda$, we have

$$F_{\mathcal{P}(U)}\mathcal{P}(z) \leq F_{q(U)}q(z) \leq F_{h(U)}h(z), z \in U.$$

From (18), we get

$$F_{\psi(\mathbb{C}^2.U)}\left[\left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda-1}\left(\mathcal{J}_{\mu}^{a,c}f(z)\right)'\right] \leq F_{q(U)}q(z) \leq F_{h(U)}h(z),$$

and

$$\left(\frac{\mathcal{J}_{\mu}^{a,c}f(z)}{z}\right)^{\lambda} \prec_F q(z),\ z \in U,$$

where $q(z) = \frac{1}{z}\int_0^z h(t)dt$ is convex and is the fuzzy best dominant. □

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Accepted: August 26, 2021