Ulam stability of linear difference equations with initial conditions

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Abstract. In this paper, we prove the Hyers-Ulam stability, Hyers-Ulam-Rassias stability, Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the linear difference equations with initial conditions by applying Newton's Theorem. Also, the Ulam stability constants are obtained.
Keywords: Hyers-Ulam Stability, Hyers-Ulam-Rassias stability, Mittag-Leffler-Hyers-Ulam stability, Mittag-Leffler-Hyers-Ulam-Rassias stability, linear difference equations, initial conditions and Newton's theorem.

1. Introduction

In [44], Ulam proposed the universal Ulam stability problem: When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true? In [12], Hyers

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gave the first confirmatory answer to the question of Ulam for additive functional equations on Banach spaces. Hyers result has since then seen many significant generalizations, both in terms of the control condition used to define the concept of approximate solution [2, 3, 4, 10, 35, 36, 37, 38, 39].

A generalization of Ulam's problem was recently proposed by replacing functional equations with differential equations: The differential equation

$$\phi\left(f, x, x', x'', ...x^{(n)}\right) = 0$$

has the Hyers-Ulam stability if for a given $\epsilon > 0$ and a function x such that

$$\left|\phi\left(f, x, x', x'', ...x^{(n)}\right)\right| \leq \epsilon,$$

there exists a solution x_a of the differential equation such that $|x(t) - x_a(t)| \leq K(\epsilon)$ and $\lim_{\epsilon\to 0} K(\epsilon) = 0$. If the preceding statement is also true when we replace ϵ and $K(\epsilon)$ by $\phi(t)$ and $\varphi(t)$, where ϕ, φ are appropriate functions not depending on x and x_a explicitly, then we say that the corresponding differential equation has the generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability.

Obloza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations [31, 32]. Thereafter, In 1998, C. Alsina and R. Ger [1] were the first authors who investigated the Hyers-Ulam stability of differential equations. This result of C. Alsina and R. Ger [1] has been generalized by Takahasi [41]. They proved in [41] that the Hyers-Ulam stability holds true for the Banach Space valued differential equation $y'(t) = \lambda y(t)$. Those previous results were extended to the Hyers-Ulam stability of linear differential equation second order and higher order in [11, 16, 17, 20, 21, 22, 23, 24, 25] respectively.

Now a days, the Hyers-Ulam stability of difference equations has been given attention. In 2005, D. Popa [34] proved the Hyers-Ulam stability of the linear recurrence of first order with constant coefficient and nth order linear recurrence with constant coefficients in Banach spaces. Also the Hyers-Ulam-Rassias stability of a linear recurrences of higher order equations are established in [6, 33]. In 2007, J.Brzdek et.al [7] investigated the Hyers-Ulam stability of the nonlinear recurrences in an Abelian group with invariant metric.

In 2018, R. Murali et. al [30] proved the Hyers-Ulam stability of the forward and backward difference equations. Very recently, they proved [27, 29] the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the first and second order difference equation by using Z-transform method. These days, the Hyers-Ulam stability of difference equations has been investigated [5, 8, 13, 14, 15, 18, 27, 28, 29, 42, 43] and the investigation is going on.

In 2011, P. Gavruta, S.M. Jung and Y. Li [9] are studied the Hyers-Ulam stability for second order linear differential equations with initial and boundary conditions using Taylor formula. As well known, many different methods for solving differential equations have been used to study the Hyers-Ulam stability problem for various differential equation. But using initial conditions are have more significant advantage for solving differential equations. As a discrete case also initial conditions are have a similar advantage in solving difference equations. Motivated by the results reported in [9], our main goal is to study the Hyers-Ulam stability, Hyers-Ulam-Rassias stability, Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability linear difference equation of first order

$$z(n+1)+p(n)\ z(n)=0 \tag{1}$$

with initial condition

$$z(a)=0 \tag{2}$$

the linear difference equation of second order

$$z(n+2)+p(n)\ z(n)=0 \tag{3}$$

with initial conditions

$$z(a)=z(a+1)=0 \tag{4}$$

and the linear difference equation of kth order

$$z(n+k)+p(n)\ z(n)=0 \tag{5}$$

with initial conditions

$$z(a)=z(a+1)=z(a+2)=\cdots=z(a+k-1)=0 \tag{6}$$

by using Newton's Formula. Where $n\in(a,b+1)$, $a,b\in\mathbb{N}_0$.

2. Preliminaries

The following Definitions and theorems are very useful to prove our main results.

Definition 2.1 ([40]). *Let $t\in N_0$, $\lambda\in(-1,1)$ and $\alpha,\beta\in R^+$. The one and two parameter discrete Mittag-Leffler functions are defined by*

$$F_\alpha(\lambda,t)=\sum_{k=0}^{\infty}\lambda^k\frac{t^{\overline{\alpha k}}}{\Gamma(\alpha k+1)}, \tag{7}$$

$$F_{\alpha,\beta}(\lambda,t^{\overline{\alpha}})=\sum_{k=0}^{\infty}\lambda^k\frac{t^{\overline{\alpha k}}}{\Gamma(\alpha k+\beta)}. \tag{8}$$

Theorem 2.1. *(Newton's Theorem). If $f(n)$ is a polynomial of degree k then*

$$f(n)=f(0)+\frac{n^{(1)}}{1!}\Delta f(0)+\frac{n^{(2)}}{2!}\Delta^{(2)}f(0)+\frac{n^{(3)}}{3!}\Delta^{(3)}f(0)+\cdots+\frac{n^{(k)}}{k!}\Delta^{(k)}f(0),$$

where $n^{(k)}=\frac{n!}{(n-k)!}$.

Now, we give the definitions of Hyers-Ulam stability, Hyers-Ulam-Rassias stability, Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the linear difference equation with initial conditions.

Definition 2.2. *We say that the linear difference equation* (5) *has the Hyers-Ulam stability, if there exists a constant* $K > 0$ *with the following conditions: For every* $\epsilon > 0$ *and every* $z(n)$ *be a sequence of function of kth order difference equation satisfying the inequality*

$$|z(n+k) + p(n)\ z(n)| \leq \epsilon,$$

with initial conditions (6)*. Then, there exists some* $w(n)$ *satisfying the difference equation* $w(n+k) + p(n)\ w(n) = 0$ *with* $w(a) = w(a+1) = w(a+2) = \cdots = w(a+k-1) = 0$ *such that* $|z(n) - w(n)| \leq K\ \epsilon$*. We call such* K *as a Hyers-Ulam stability constant for the linear difference equation* (5) *with* (6)*.*

Definition 2.3. *We say that the kth order linear difference equation* (5) *has Hyers-Ulam-Rassias stability, if there exists a constant* $K > 0$ *with the following property: For every* $\epsilon > 0$*,* $z(n)$ *be a sequence of function of kth order difference equation and there exists* $\phi(n)$ *be a positive and bounded sequence satisfying the inequality* $|z(n+k) + p(n)\ z(n)| \leq \phi(n)\ \epsilon$*, with initial conditions* (6)*. Then, there exists some* $w(n)$ *satisfying the difference equation* $w(n+k)+p(n)\ w(n) = 0$ *with*

$$w(a) = w(a+1) = w(a+2) = \cdots = w(a+k-1) = 0$$

such that $|z(n) - w(n)| \leq K\ \phi(n)\ \epsilon$*. We call such* K *as a Hyers-Ulam-Rassias stability constant for the linear difference equation* (5) *with* (6)*.*

Definition 2.4. *We say that the linear difference equation* (5) *has the Mittag-Leffler-Hyers-Ulam stability, if there exists a constant* $K > 0$ *with the following conditions: For every* $\epsilon > 0$ *and every* $z(n)$ *be a sequence of function of kth order difference equation satisfying the inequality*

$$|z(n+k) + p(n)\ z(n)| \leq \epsilon F_\alpha(\lambda, n),$$

with initial conditions (6)*, where* F_α *is Mittag-Leffler function. Then, there exists some* $w(n)$ *satisfying the difference equation* $w(n+k) + p(n)\ w(n) = 0$ *with* $w(a) = w(a+1) = w(a+2) = \cdots = w(a+k-1) = 0$ *such that* $|z(n) - w(n)| \leq K\ \epsilon F_\alpha(\lambda, n)$*. We call such* K *as a Mittag-Leffler-Hyers-Ulam stability constant for the linear difference equation* (5) *with* (6)*.*

Definition 2.5. *We say that the kth order linear difference equation* (5) *has Mittag-Leffler-Hyers-Ulam-Rassias stability, if there exists a constant* $K > 0$ *with the following property: For every* $\epsilon > 0$*,* $z(n)$ *be a sequence of function of kth order difference equation and there exists* $\phi(n)$ *be a positive and bounded sequence satisfying the inequality*

$$|z(n+k) + p(n)\ z(n)| \leq \phi(n)\ \epsilon F_\alpha(\lambda, n),$$

with initial conditions (6)*, where* F_α *is Mittag-Leffler function. Then, there exists some* $w(n)$ *satisfying the difference equation* $w(n+k)+p(n)\ w(n) = 0$ *with* $w(a) = w(a+1) = w(a+2) = \cdots = w(a+k-1) = 0$ *such that* $|z(n)-w(n)| \leq K\ \phi(n)\ \epsilon F_\alpha(\lambda, n)$*. We call such* K *as a Mittag-Leffler-Hyers-Ulam-Rassias stability constant for the linear difference equation* (5) *with* (6)*.*

3. Hyers-Ulam stability

In this section, we prove the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the linear difference equations (1), (3) and (5) with initial conditions (2), (4) and (6) respectively. Firstly, we prove the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the linear difference equation (1) with (2).

Theorem 3.1. *Let* $\max|p(n)| < \frac{\gamma}{n}$ *with* $0 < \gamma < 1$ *and for every* $\epsilon > 0$*, there exists* $z(n)$*,* $n \in (a, b+1)$ *satisfies the inequality*

$$|z(n+1)+p(n)z(n)| \leq \epsilon, \tag{9}$$

with initial condition $z(a) = 0$*. Then, there exists a solution* $w(n)$ *satisfies the difference equation* $w(n+1)+p(n)w(n) = 0$ *with* $w(a) = 0$ *such that* $|z(n)-w(n)| \leq K\epsilon$*.*

Proof. Assume that for every $\epsilon > 0$, if $z(n)$ be a sequence of function satisfying the inequality (9) with $z(a) = 0$. Then, by using Newton's Theorem, we have

$$z(n) = z(a) + \frac{n^{(1)}}{1!}\Delta z(a) = z(a) + \frac{n^{(1)}}{1!}[z(a+1) - z(a)].$$

By using initial condition (2), we obtain that

$$z(n) = \frac{n!}{(n-1)!\ 1!}\ z(a+1).$$

Now, taking modulus on both sides and using the inequality (9), we get

$$\begin{aligned}
|z(n)| &= \left|\frac{n!}{(n-1)!\ 1!} z(a+1)\right| \\
&\leq \frac{n}{1!}\ \max|z(n+1)| \\
\max|z(n)| &\leq \frac{n}{1!}\ \max|z(n+1)+p(n)z(n)-p(n)z(n)| \\
&\leq \frac{n}{1!}\max|z(n+1)+p(n)z(n)| + \frac{n}{1!}\max|p(n)|\max|z(n)| \\
&\leq \frac{b}{1!}\ \epsilon + \gamma\ \max|z(n)|.
\end{aligned}$$

Let $K = \frac{b}{(1-\gamma)}$, then we have $\max|z(n)| \leq K\epsilon$. Obviously, $w(n) \equiv 0$ is a solution of the difference equation $z(n+1)+p(n)z(n) = 0$ with $w(a) = 0$ such that $|z(n)-w(n)| \leq K\epsilon$. Hence the linear difference equation (1) has the Hyers-Ulam stability with initial condition (2). □

The following corollary shows that the Hyers-Ulam-Rassias stability of the linear difference equation (1) with (2).

Corollary 3.1. *Let* $\max|p(n)| < \frac{\gamma}{n}$ *with* $0 < \gamma < 1$ *and for every* $\epsilon > 0$ *there exists* $z(n), n \in (a, b+1)$ *and* $\phi(n)$ *be a positive and bounded sequence satisfies the inequality,*

$$|z(n+1) + p(n)z(n)| \leq \phi(n)\epsilon,$$

with initial condition $z(a) = 0$. *Then, there exists a solution* $w(n)$ *satisfies the difference equation* $w(n+1) + p(n)w(n) = 0$ *with* $w(a) = 0$ *such that* $|z(n) - w(n)| \leq K\phi(n)\epsilon$ *for all* $n \in (a, b+1)$.

Proof. Assume that given $\epsilon > 0$, there exists $z(n), n \in (a, b+1)$ and $\phi(n)$ be a positive and bounded sequence satisfies the inequality,

$$|z(n+1) + p(n)z(n)| \leq \phi(n)\epsilon, \tag{10}$$

with initial condition $z(a) = 0$. By the similar way of the Theorem 3.1, we can get the result. □

Now, we investigate the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the linear difference equation (3) with initial conditions (4).

Theorem 3.2. *Let* $\max|p(n)| < \frac{2\gamma}{n(n-1)}$ *with* $0 < \gamma < 1$ *and for every* $\epsilon > 0$, $z(n)$ *be a sequence satisfies the inequality* $|z(n+2) + p(n)z(n)| \leq \epsilon$ *with initial condition* $z(a) = z(a+1) = 0$. *Then, there exists a solution* $w(n)$ *satisfies the difference equation* $w(n+2) + p(n)w(n) = 0$ *with* $w(a) = w(a+1) = 0$ *such that* $|z(n) - w(n)| \leq K\epsilon$.

Proof. Assume that for every $\epsilon > 0$, $z(n)$ be a function of second order difference equation on $(a, b+1)$ satisfying

$$|z(n+2) + p(n)z(n)| \leq \epsilon, \tag{11}$$

with $z(a) = z(a+1) = 0$. Then, by using Newton's Theorem, we have

$$\begin{aligned} z(n) &= z(a) + \frac{n^{(1)}}{1!}\Delta^{(1)}z(a) + \frac{n^{(2)}}{2!}\Delta^{(2)}z(a) \\ &= z(a) + \frac{n^{(1)}}{1!}\,[z(a+1) - z(a)] + \frac{n^{(2)}}{2!}\,[z(a+2) - 2z(a+1) + z(a)]. \end{aligned}$$

By using initial condition (4), we obtain that $z(n) = \frac{n!}{(n-2)!\;2!}\; z(a+2)$. Now, taking modulus on both sides and using the inequality (11), we get that

$$|z(n)| = \left|\frac{n!}{(n-2)!\;2!}\; z(a+2)\right| \leq \frac{n(n-1)}{2!}\; \max|z(n+2)|\,.$$

$$\begin{aligned} \max|z(n)| &\leq \frac{n(n-1)}{2!}\; \max|z(n+2) + p(n)z(n) - p(n)z(n)| \\ &\leq \frac{b(b-1)}{2}\;\epsilon + \gamma\; \max|z(n)|\,. \end{aligned}$$

Let $K = \frac{b(b-1)}{2(1-\gamma)}$, then we have $\max|z(n)| \leq K\epsilon$. Obviously, $w(n) \equiv 0$ is a solution of the difference equation

$$z(n+2) + p(n)z(n) = 0$$

with $w(a) = w(a+1) = 0$ such that $|z(n) - w(n)| \leq K\epsilon$. Hence the linear difference equation (3) has the Hyers-Ulam stability with (4). □

Corollary 3.2. *Let* $\max|p(n)| < \frac{2\gamma}{n(n-1)}$ *with* $0 < \gamma < 1$ *and for every* $\epsilon > 0$, *there exists a function* $z(n)$, $n \in (a, b+1)$ *and* $\phi(n)$ *be a positive and bounded sequence satisfies the inequality*

$$|z(n+2) + p(n)z(n)| \leq \phi(n)\epsilon, \tag{12}$$

with initial condition $z(a) = z(a+1) = 0$. *Then, there exists a solution* $w(n)$ *satisfies the difference equation* $w(n+2)+p(n)w(n) = 0$ *with* $w(a) = w(a+1) = 0$ *such that*

$$|z(n) - w(n)| \leq K\phi(n)\epsilon, \qquad \forall\ n \in (a, b+1).$$

Proof. Assume for every $\epsilon > 0$, there exists $z(n), n \in (a, b+1)$ and $\phi(n)$ be a positive and bounded sequence satisfies the inequality (18) with initial condition $z(a) = z(a+1) = 0$. Then, by using above Theorem 3.2, similarly we can obtain the Hyers-Ulam-Rassias stability of (3) with (4). □

Finally, in this section we are going to establish the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the k^{th} order linear difference equation (5) with initial conditions (6).

Theorem 3.3. *Let* $\max|p(n)| < \frac{\gamma(n-k)!\ k!}{n!}$ *with* $0 < \gamma < 1$*and for every* $\epsilon > 0$, *there exists* $z(n)$, $n \in (a, b+1)$ *satisfies the inequality*

$$|z(n+k) + p(n)z(n)| \leq \epsilon$$

with the initial conditions (6). *Then, there exists a solution* $w(n)$ *satisfies the difference equation* $w(n+k)+p(n)w(n) = 0$ *with* $w(a) = w(a+1) = w(a+2) = \cdots = w(a+k-1) = 0$ *such that* $|z(n) - w(n)| \leq K\epsilon$.

Proof. Assume that for every $\epsilon > 0$, $z(n)$ be a sequence of function of k^{th} order difference equation on $(a, b+1)$ satisfying the inequality

$$|z(n+k) + p(n)z(n)| \leq \epsilon, \tag{13}$$

with initial conditions $z(a) = z(a+1) = z(a+2) = \cdots = z(a+k-1) = 0$. Then, by using Newton's Theorem, we have

$$z(n) = z(a) + \frac{n^{(1)}}{1!}\Delta^{(1)}z(a) + \frac{n^{(2)}}{2!}\Delta^{(2)}z(a) + \frac{n^{(3)}}{3!}\Delta^{(3)}z(a) + \cdots + \frac{n^{(k)}}{k!}\Delta^{(k)}z(a).$$

By using initial condition (6), we obtain that

$$z(n) = \frac{n!}{(n-k)!\ k!}\ x(a+k).$$

Taking modulus on both sides and using the inequality (13), we get that

$$|z(n)| = \left|\frac{n!}{(n-k)!\ k!}\ x(a+k)\right| \leq \frac{n!}{(n-k)!\ k!}\ \max|z(n+k)|,$$

$$\max|z(n)| \leq \frac{n!}{(n-k)!\ k!}\ \max|z(n+k)+p(n)z(n)-p(n)z(n)|$$

choosing $K = \frac{b!}{(1-\gamma)\ (b-k)!\ k!}$. Thus we have $\max|z(n)| \leq K\epsilon$. Obviously, $w(n) \equiv 0$ is a solution of the difference equation $z(n+k)+p(n)z(n) = 0$ with $w(a) = w(a+1) = w(a+2) = \cdots = y(n+k-1) = 0$ such that $|z(n)-w(n)| \leq K\epsilon$. Hence by the virtue of Definition 2.2, the linear difference equation (5) has the Hyers-Ulam stability with (6). □

The next corollary says that the Hyers-Ulam-Rassias stability of the linear difference equation (5) with (6).

Corollary 3.3. *Let* $\max|p(n)| < \frac{\gamma(n-k)!\ k!}{n!}$ *with* $0 < \gamma < 1$ *for every* $\epsilon > 0$, *there exists a function* $z(n)$, $n \in (a, b+1)$ *and* $\phi(n)$ *be a positive and bounded sequence satisfies the inequality*

$$|z(n+k)+p(n)z(n)| \leq \phi(n)\epsilon,$$

with initial condition (6). *Then, there exists a solution* $w(n)$ *satisfies the difference equation* $w(n+k)+p(n)w(n) = 0$ *with* $w(a) = w(a+1) = w(a+2) = \cdots = w(a+k-1) = 0$ *such that* $|z(n)-w(n)| \leq K\phi(n)\epsilon$, *for all* $n \in (a, b+1)$.

Proof. If for every $\epsilon > 0$, there exists $z(n), n \in (a, b+1)$ and $\phi(n)$ be a positive and bounded sequence satisfies the inequality,

$$|z(n+k)+p(n)z(n)| \leq \phi(n)\epsilon, \tag{14}$$

with initial condition $z(a) = z(a+1) = z(a+2) = \cdots = z(a+k-1) = 0$. Then, by above Theorem 3.3, we can easily obtain the result. □

4. Mittag-Leffler-Hyers-Ulam stability

In this section, we study the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the linear difference equations (1), (3) and (5) with initial conditions (2), (4) and (6) respectively.

Theorem 4.1. *Let* $\max|p(n)| < \frac{\gamma}{n}$ *with* $0 < \gamma < 1$ *and for every* $\epsilon > 0$*, there exists* $z(n)$*,* $n \in (a, b+1)$ *satisfies the inequality*

$$|z(n+1) + p(n)z(n)| \leq \epsilon F_\alpha(\lambda, n), \tag{15}$$

with initial condition $z(a) = 0$*. Then, there exists a solution* $w(n)$ *satisfies the difference equation* $w(n+1) + p(n)w(n) = 0$ *with* $w(a) = 0$ *such that* $|z(n) - w(n)| \leq K\epsilon F_\alpha(\lambda, n)$*.*

Proof. Assume that for every $\epsilon > 0$, if $z(n)$ be a sequence of function satisfying the inequality (15) with $z(a) = 0$. Then, by using Newton's Theorem, we have

$$z(n) = z(a) + \frac{n^{(1)}}{1!}\Delta z(a) = z(a) + \frac{n^{(1)}}{1!}[z(a+1) - z(a)].$$

By using initial condition (2), we obtain that

$$z(n) = \frac{n!}{(n-1)!\ 1!}\ z(a+1).$$

Now, taking modulus on both sides and using the inequality (15), we get

$$\begin{aligned}
|z(n)| = \left|\frac{n!}{(n-1)!\ 1!} z(a+1)\right| &\leq \frac{n}{1!}\ \max|z(n+1)|, \\
\max|z(n)| &\leq \frac{n}{1!}\ \max|z(n+1) + p(n)z(n) - p(n)z(n)| \\
&\leq \frac{n}{1!}\ \max|z(n+1) + p(n)z(n)| + \frac{n}{1!}\ \max|p(n)| \max|z(n)| \\
&\leq \frac{b}{1!}\ \epsilon F_\alpha(\lambda, n) + \gamma\ \max|z(n)|.
\end{aligned}$$

Let $K = \frac{b}{(1-\gamma)}$, then we have $\max|z(n)| \leq K\epsilon F_\alpha(\lambda, n)$. Obviously, $w(n) \equiv 0$ is a solution of the difference equation $z(n+1) + p(n)z(n) = 0$ with $w(a) = 0$ such that $|z(n) - w(n)| \leq K\epsilon F_\alpha(\lambda, n)$. Hence the difference equation (1) has the Mittag-Leffler-Hyers-Ulam stability with initial condition (2). □

The following corollary shows that the Mittag-Leffler-Hyers-Ulam-Rassias stability of the linear difference equation (1) with (2).

Corollary 4.1. *Let* $\max|p(n)| < \frac{\gamma}{n}$ *with* $0 < \gamma < 1$ *and for every* $\epsilon > 0$ *there exists* $z(n), n \in (a, b+1)$ *and* $\phi(n)$ *be a positive and bounded sequence satisfies the inequality* $|z(n+1) + p(n)z(n)| \leq \phi(n)\epsilon F_\alpha(\lambda, n)$*, with initial condition* $z(a) = 0$*. Then, there exists a solution* $w(n)$ *satisfies the difference equation* $w(n+1) + p(n)w(n) = 0$ *with* $w(a) = 0$ *such that*

$$|z(n) - w(n)| \leq K\phi(n)\epsilon F_\alpha(\lambda, n),$$

for all $n \in (a, b+1)$*.*

Proof. Assume that given $\epsilon > 0$ and $z(n), n \in (a, b+1)$, there exists $\phi(n)$ be a positive and bounded sequence satisfying the inequality,

$$|z(n+1) + p(n)z(n)| \leq \phi(n)\epsilon F_\alpha(\lambda, n), \tag{16}$$

with initial condition $z(a) = 0$. Then, by the similar approach of the Theorem 4.1, we can obtain the Mittag-Leffler-Hyers-Ulam-Rassias stability of the linear difference equation (1) with initial condition (2). □

Now, we investigate the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the linear difference equation (3) with initial conditions (4).

Theorem 4.2. *Let* $\max|p(n)| < \frac{2\gamma}{n(n-1)}$ *with* $0 < \gamma < 1$ *and for every* $\epsilon > 0$, $z(n)$ *be a sequence satisfies the inequality*

$$|z(n+2) + p(n)z(n)| \leq \epsilon F_\alpha(\lambda, n)$$

with initial condition $z(a) = z(a+1) = 0$. *Then, there exists a solution* $w(n)$ *satisfies the difference equation* $w(n+2)+p(n)w(n) = 0$ *with* $w(a) = w(a+1) = 0$ *such that* $|z(n) - w(n)| \leq K\epsilon F_\alpha(\lambda, n)$.

Proof. Assume that for every $\epsilon > 0$, $z(n)$ be a function of second order difference equation on $(a, b+1)$ satisfying

$$|z(n+2) + p(n)z(n)| \leq \epsilon F_\alpha(\lambda, n), \tag{17}$$

with $z(a) = z(a+1) = 0$. Then, by using Newton's Theorem, we have

$$\begin{aligned} z(n) &= z(a) + \frac{n^{(1)}}{1!}\Delta^{(1)}z(a) + \frac{n^{(2)}}{2!}\Delta^{(2)}z(a) \\ &= z(a) + \frac{n^{(1)}}{1!}\ [z(a+1) - z(a)] + \frac{n^{(2)}}{2!}\ [z(a+2) - 2z(a+1) + z(a)]. \end{aligned}$$

By using initial condition (4), we obtain that $z(n) = \frac{n!}{(n-2)!\ 2!}\ z(a+2)$. Now, taking modulus on both sides and using the inequality (17), we get that

$$|z(n)| = \left|\frac{n!}{(n-2)!\ 2!}\ z(a+2)\right| \leq \frac{n(n-1)}{2!}\ \max|z(n+2)|$$

$$\begin{aligned} \max|z(n)| &\leq \frac{n(n-1)}{2!}\ \max|z(n+2) + p(n)z(n) - p(n)z(n)| \\ &\leq \frac{n(n-1)}{2!}\max|z(n+2)+p(n)z(n)| + \frac{n(n-1)}{2!}\max|p(n)|\max|z(n)| \\ &\leq \frac{b(b-1)}{2}\ \epsilon F_\alpha(\lambda, n) + \gamma\ \max|z(n)|\,. \end{aligned}$$

Let $K = \frac{b(b-1)}{2(1-\gamma)}$, then we have $\max|z(n)| \leq K\epsilon F_\alpha(\lambda, n)$. Obviously, $w(n) \equiv 0$ is a solution of the difference equation

$$z(n+2) + p(n)z(n) = 0$$

with $w(a) = w(a+1) = 0$ such that $|z(n) - w(n)| \leq K\epsilon F_\alpha(\lambda, n)$. Hence the linear difference equation (3) has the Mittag-Leffler-Hyers-Ulam stability with (4). □

Corollary 4.2. *Let* $\max|p(n)| < \frac{2\gamma}{n(n-1)}$ *with* $0 < \gamma < 1$ *and for every* $\epsilon > 0$*, there exists a function* $z(n)$*,* $n \in (a, b+1)$ *and* $\phi(n)$ *be a positive and bounded sequence satisfies the inequality*

$$|z(n+2) + p(n)z(n)| \leq \phi(n)\epsilon F_\alpha(\lambda, n), \tag{18}$$

with initial condition $z(a) = z(a+1) = 0$*. Then, there exists a solution* $w(n)$ *satisfies the difference equation* $w(n+2)+p(n)w(n) = 0$ *with* $w(a) = w(a+1) = 0$ *such that*

$$|z(n) - w(n)| \leq K\phi(n)\epsilon F_\alpha(\lambda, n), \qquad \forall\ n \in (a, b+1).$$

Proof. Assume for every $\epsilon > 0$, there exists $z(n), n \in (a, b+1)$ and $\phi(n)$ be a positive and bounded sequence satisfies the inequality (18) with initial condition $z(a) = z(a+1) = 0$. Then, by using above Theorem 4.2, one can easily prove the Mittag-Leffler-Hyers-Ulam-Rassias stability of (3) with initial conditions (4). □

Finally, in this section we are going to establish the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the k^{th} order linear difference equation (5) with initial conditions (6).

Theorem 4.3. *Let* $\max|p(n)| < \frac{\gamma(n-k)!\ k!}{n!}$ *with* $0 < \gamma < 1$*and for every* $\epsilon > 0$*, and every* $z(n)$*,* $n \in (a, b+1)$ *satisfies the inequality* $|z(n+k) + p(n)z(n)| \leq \epsilon F_\alpha(\lambda, n)$ *with the initial conditions* (6)*. Then, there exists a solution* $w(n)$ *satisfies the difference equation*

$$w(n+k) + p(n)w(n) = 0$$

with $w(a) = w(a+1) = w(a+2) = \cdots = w(a+k-1) = 0$ *such that*

$$|z(n) - w(n)| \leq K\epsilon F_\alpha(\lambda, n).$$

Proof. Assume that for every $\epsilon > 0$, $z(n)$ be a sequence of function of k^{th} order difference equation on $(a, b+1)$ satisfying the inequality

$$|z(n+k) + p(n)z(n)| \leq \epsilon F_\alpha(\lambda, n), \tag{19}$$

with initial conditions $z(a) = z(a+1) = z(a+2) = \cdots = z(a+k-1) = 0$. Then, by using Newton's Theorem, we have

$$z(n) = z(a) + \frac{n^{(1)}}{1!}\Delta^{(1)}z(a) + \frac{n^{(2)}}{2!}\Delta^{(2)}z(a) + \cdots + \frac{n^{(k)}}{k!}\Delta^{(k)}z(a).$$

By using initial condition (6), we obtain that

$$z(n) = \frac{n!}{(n-k)!\ k!}\ x(a+k).$$

Taking modulus on both sides and using the inequality (19), we get that

$$|z(n)| = \left|\frac{n!}{(n-k)!\ k!}\ x(a+k)\right| \leq \frac{n!}{(n-k)!\ k!}\ \max|z(n+k)|,$$

$$\begin{aligned}\max|z(n)| &\leq \frac{n!}{(n-k)!\ k!}\ \max|z(n+k)+p(n)z(n)-p(n)z(n)| \\ &\leq \frac{n!}{(n-k)!\ k!}\ \max|z(n+k)+p(n)z(n)| \\ &\qquad + \frac{n!}{(n-k)!\ k!}\ \max|p(n)|\max|z(n)|\end{aligned}$$

Let $K = \frac{b!}{(1-\gamma)\ (b-k)!\ k!}$. Thus we have $\max|z(n)| \leq K\epsilon F_\alpha(\lambda, n)$.

Obviously, $w(n) \equiv 0$ is a solution of the difference equation $z(n+k) + p(n)z(n) = 0$ with $w(a) = w(a+1) = w(a+2) = \cdots = y(n+k-1) = 0$ such that $|z(n) - w(n)| \leq K\epsilon F_\alpha(\lambda, n)$. Hence by the virtue of Definition 2.4, the linear difference equation (5) has the Mittag-Leffler-Hyers-Ulam stability with initial conditions (6). □

The next corollary says that the Mittag-Leffler-Hyers-Ulam-Rassias stability of the linear difference equation (5) with (6).

Corollary 4.3. *Let* $\max|p(n)| < \frac{\gamma(n-k)!\ k!}{n!}$ *with* $0 < \gamma < 1$ *for every* $\epsilon > 0$, $z(n)$ *be a sequence and there exists a function* $\phi(n)$ *be a positive and bounded sequence satisfies the inequality*

$$|z(n+k) + p(n)z(n)| \leq \phi(n)\epsilon F_\alpha(\lambda, n),$$

with initial condition (6). *Then, there exists a solution* $w(n)$ *satisfies the difference equation* $w(n+k) + p(n)w(n) = 0$ *with* $w(a) = w(a+1) = w(a+2) = \cdots = w(a+k-1) = 0$ *such that*

$$|z(n) - w(n)| \leq K\phi(n)\epsilon F_\alpha(\lambda, n), \qquad \forall\ n \in (a, b+1).$$

Proof. If for every $\epsilon > 0$, there exists $z(n), n \in (a, b+1)$ and $\phi(n)$ be a positive and bounded sequence satisfies the inequality,

$$|z(n+k) + p(n)z(n)| \leq \phi(n)\epsilon F_\alpha(\lambda, n), \tag{20}$$

with initial condition $z(a) = z(a+1) = z(a+2) = \cdots = z(a+k-1) = 0$. Then, by above Theorem 4.3, we can easily obtain the result. □

Conclusion: In this paper, we studied the Hyers-Ulam stability, Hyers-Ulam-Rassias stability, Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the linear difference equations of first order, second order and kth order using initial conditions with the help of Newton's Theorem. In fact the results obtained in this paper can be regarded as a discrete analogue of the stability results for linear differential equation in [9]. Additionally, this paper also provides another method to study the Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam stability of the difference equations.

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