C^*-convex sets generated by C^*-convex maps in $*$-rings and their C^*-faces

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Abstract. In this paper, we identify some C^*-convex sets generated by C^*-affine maps and C^*-convex maps on the unital $*$-rings, and we obtain the C^*-faces of such C^*-convex sets. Specially, we show that the set $Fix(f)$ of all fixed points of the C^*-affine map f on the unital $*$-ring $\mathcal{R}$, is a C^*-convex set. Moreover, we prove that for a C^*-convex map f on some $*$-rings, the set $Fix(f)$ is a C^*-face of some C^*-convex sets.

Keywords: C^*-affine map, C^*-convex set, C^*-face, $*$-ring.

1. Introduction

One of the form of non-commutative convexity is C^*-convexity. Formal study of C^*-convexity was initiated by Loebl and Paulsen in [8]. Hopenwasser, Moore, and Paulsen obtained the set of all extreme points and C^*-extreme points of some C^*-convex sets, and the distinction of these sets in [7]. Farenick and Morenz proved that each irreducible element of the C^*-algebra M_n of complex $n \times n$ matrices, is a C^*-extreme point, the relative extreme point, of the C^*-convex set that it generates [5]. Also they studied the C^*-extreme points of the C^*-convex space $S_H(A)$ of all unital completely positive linear maps from a C^*-algebra A to the algebra $B(H)$ of continuous linear operators on a complex Hilbert space H [6]. Moreover, in [9] Morenz discovered a right analogue of linear extreme points, called structural elements to prove a generalised Krein Milman theorem for C^*-convex subsets of M_n. Also he extend the notion of face from convexity to C^*-face in C^*-convexity. The author and G. H. Esslamzadeh generalised the notion of C^*-convexity and the relative extreme points, C^*-extreme points, to $*$-rings [4]. Recently, the author has investigated the C^*-extreme points of the graph and epigraph of C^*-affine maps in [2]. Also it has shawn in [2] that for a C^*-convex map f defined on a unital $*$-ring $\mathcal{R}$ with some conditions the graph of f is a C^*-face of the epigraph of f, and some other results about the C^*-faces of C^*-convex sets in $*$-rings. Moreover, the structure of the set of all C^*-convex maps and C^*-affine maps in $*$-rings has investigated in [3].

Throughout this paper, $\mathcal{R}$ is a unital $*$-ring, that is, a ring with an involution which has an identity element. An element x in $\mathcal{R}$ is called positive, written $x \geq 0$, if $x = y_1^* y_1 + y_2^* y_2 + \cdots + y_n^* y_n$ for some $y_1, y_2, \cdots, y_n$ in $\mathcal{R}$. The self-adjoint elements of $\mathcal{R}$ can be ordered by writing $x \leq y$ in case $y - x \geq 0$. Also, the interval $[x, y]$denotes the set of all elements z in $\mathcal{R}$ such that $x \leq z \leq y$. The involution of $\mathcal{R}$ is called proper if $x^* x = 0$ implies that $x = 0$ for every $x \in \mathcal{R}$. $x \in \mathcal{R}$ is called a central element of $\mathcal{R}$, if $xa = ax$ for each $a \in \mathcal{R}$. The set of all central elements of $\mathcal{R}$ is denoted by $Z(\mathcal{R})$. The reference [1] is essential for studying the $*$-rings. We denote the graph and epigraph of a map f on $\mathcal{R}$ by *graph*(f) and *epi*(f) respectively, which are defined as following;

$$graph(f) = \{(x, y) : x \in \mathcal{R}, y = f(x)\} \subseteq \mathcal{R} \oplus \mathcal{R},$$

$$epi(f) = \{(x, y) : x \in \mathcal{R}, f(x) \leq y\} \subseteq \mathcal{R} \oplus \mathcal{R}.$$

In the next section we state the definitions of C^*-convexity, C^*-convex map, C^*-affine map, and C^*-face in $*$-rings, and the last section of this paper is devoted to the main results of this paper.

2. Definitions and preliminaries

Definition 2.1. *A subset K of a unital $*$-ring $\mathcal{R}$ is called C^*-convex, if*

$$\sum_{i=1}^{n} a_i^* x_i a_i \in K,$$

whenever $x_i \in K$, $a_i \in \mathcal{R}$ for all i and $\sum_{i=1}^{n} a_i^* a_i = 1_{\mathcal{R}}$.

In this case the summation $\sum_{i=1}^{n} a_i^* x_i a_i$ is called a C^*-convex combination of elements $x_i \in K$.

Definition 2.2. *Let K be a C^*-convex subset of $\mathcal{R}$. An element $x \in K$ is called a C^*-extreme point of K if the condition*

$$x = \sum_{i=1}^{n} a_i^* x_i a_i, \ \sum_{i=1}^{n} a_i^* a_i = 1_{\mathcal{R}}, x_i \in K, a_i \text{ is invertible in } \mathcal{R}, n \in \mathbb{N}, \tag{1}$$

implies that all x_i are unitarily equivalent to x in $\mathcal{R}$, that is, there exist unitaries $u_i \in \mathcal{R}$ such that $x_i = u_i^* x u_i$ for all i.

The set of all C^*-extreme points of K is denoted by C^*-ext(K).

In addition, if condition (1) holds, then we say that x is a proper C^*-convex combination of $x_1, \ldots, x_n$.

In [4] we defined the notion of C^*-convex maps as the following;

Definition 2.3. *Let K be a C^*-convex subset of $\mathcal{R}$. We say that a map f on K is C^*-convex if*

$$f(\sum_{i=1}^{n} a_i^* x_i a_i) \leq \sum_{i=1}^{n} a_i^* f(x_i) a_i,$$

where $n \in \mathbb{N}$, $x_i \in K$, $a_i \in \mathcal{R}$, *and* $\sum_{i=1}^{n} a_i^* a_i = 1_{\mathcal{R}}$.

If we replace the equality instead of the less or equality in this definition, we say that f *is a* C^**-affine map on* K. *Also, if* $-f$ *is* C^**-convex, we say that* f *is* C^**-concave.*

As an example $f(x) = x^*$ is a C^*-affine map on a unital $*$-ring $\mathcal{R}$. In continuation, we give some examples of C^*-convex subsets of $\mathcal{R}$.

Example 2.1. The following sets are C^*-convex sets in any unital $*$-ring $\mathcal{R}$.

(1) The set of all self adjoint elements in $\mathcal{R}$.

(2) The set of all positive elements in $\mathcal{R}$.

(3) $[0, 1_{\mathcal{R}}]$.

(4) $U(x)$ for each $x \in \mathcal{R}$, where $U(x)$ denotes the set of all elements which are unitarily equivalent to x in $\mathcal{R}$.

In the next example, we give another family of C^*-convex subsets of the unital $*$-ring $\mathcal{R}$.

Example 2.2. The following sets are C^*-convex in the unital $*$- ring $\mathcal{R}$.

(1) Every two sided ideal in $\mathcal{R}$.

(2) Every one sided $*$-ideal in $\mathcal{R}$ is a two sided ideal, and hence is a C^*-convex set.

To see this, let I be a left $*$-ideal in $\mathcal{R}$. Then $\mathcal{R}I \subseteq I$, and hence

$$I\mathcal{R} = (\mathcal{R}I)^* \subseteq I^* \subseteq I.$$

The notion of face from linear convexity has extended to C^*-face for C^*-convex subsets of a C^*-algebra in [9] by P. B. Morenz. The author and G. H. Esslamzadeh generalised this notion to $*$-rings [4].

Definition 2.4. *A nonempty subset* F *of a* C^**-convex set* $K \subseteq \mathcal{R}$ *is called a* C^**-face of* K*, if the condition* $x \in F$ *and* $x = \sum_{i=1}^{n} a_i^* x_i a_i$ *as a proper* C^**-convex combination of elements* $x_i \in K$*, implies that* $x_i \in F$ *for all* i.

Example 2.3. Let K be a C^*-convex subset of $\mathcal{R}$. Then the following sets are C^*-faces of K.

(1) The set K itself.

(2) The set $C^* - ext(K)$ of all C^*-extreme points of K, provided that it is nonempty.

3. Main results

In this section, we obtain some C^*-convex sets generated by C^*-affine maps and C^*-convex maps on the unital $*$-rings, and also we obtain C^*-faces of such C^*-convex sets.

First of all, we show that the set of all fixed points of a C^*-affine map f on a unital $*$-ring $\mathcal{R}$ is a C^*-convex subset of $\mathcal{R}$. An element $x \in \mathcal{R}$ is called a fixed point of the map f on $\mathcal{R}$ if $f(x) = x$. We denote the set of all fixed points of f by $Fix(f)$.

Proposition 3.1. *Let f be a C^*-affine map on a unital $*$-ring $\mathcal{R}$. Then, the set $Fix(f)$ of all fixed points of f is a C^*-convex subset of $\mathcal{R}$. Moreover, $Fix(f)$ is closed under the unitary orbit of its elements.*

Proof. Suppose that $x_i \in \mathcal{R}$ and $f(x_i) = x_i$ for each i $(1 \leq i \leq n)$, and $a_i \in \mathcal{R}$ such that $\sum_{i=1}^{n} a_i^* a_i = 1_{\mathcal{R}}$, then

$$f(\sum_{i=1}^{n} a_i^* x_i a_i) = \sum_{i=1}^{n} a_i^* f(x_i) a_i = \sum_{i=1}^{n} a_i^* x_i a_i.$$

So, $\sum_{i=1}^{n} a_i^* x_i a_i$ is a fixed point of f, and hence $Fix(f)$ is a C^*-convex subset of $\mathcal{R}$. Since, every C^*-convex set is closed under the unitary orbit of its elements, so for every $x \in Fix(f)$, we have $U(x) \subset Fix(f)$, i.e. every element in the unitary orbit of x, is a fixed point of f. □

Note that, if we consider the C^*-affine map $f(x) = x^*$ on $\mathcal{R}$, then $Fix(f)$ is the set $\mathcal{R}_{sa}$ of all self-adjoint elements in $\mathcal{R}$, which is a C^*-convex subset of $\mathcal{R}$. It is shown in [4] that for every C^*-convex map f on the unital $*$-algebra $\mathcal{A}$ the sets $\{x : f(x) \leq x\}$, and $\{x : f(x) \leq \alpha 1_{\mathcal{A}}\}$ are C^*-convex sets. In this section we extend this result as follows;

Theorem 3.1. *Suppose that f is a C^*-convex map on a unital $*$-ring $\mathcal{R}$, and $y \in \mathcal{R}$ such that $C^* - co(\{y\}) = \{y\}$ (specially $y \in Z(\mathcal{R})$). Then the set*

$$K = \{x \in \mathcal{R} \mid f(x) \leq y\},$$

is a C^*-convex set in $\mathcal{R}$, provided that it is nonempty.

Proof. Let $x_i \in K$ for each i, $(1 \leq i \leq n)$, and $\sum_{i=1}^{n} a_i^* x_i a_i$ be a C^*-convex combination of elements $x_i \in K$. Then $f(x_i) \leq y$, and hence $a_i^* f(x_i) a_i \leq a_i^* y a_i$ for each i, $(1 \leq i \leq n)$. Thus, by the C^*-convexity of f and using the assumption $C^* - co(\{y\}) = \{y\}$, we conclude that

$$f(\sum_{i=1}^{n} a_i^* x_i a_i) \leq \sum_{i=1}^{n} a_i^* f(x_i) a_i \leq \sum_{i=1}^{n} a_i^* y a_i = y.$$

Hence, $\sum_{i=1}^{n} a_i^* x_i a_i \in K$. Therefore, K is a C^*-convex subset of $\mathcal{R}$. □

In the next theorems we will use the following notation for every C^*-affine map $f : \mathcal{R} \longrightarrow \mathcal{R}$ and every nonempty subset B of $\mathcal{R}$.

$$f^{-1}(B) = \{x \in \mathcal{R} \mid f(x) \in B\}.$$

Theorem 3.2. *Let $f : \mathcal{R} \longrightarrow \mathcal{R}$ be a C^*-affine map on a unital $*$-ring $\mathcal{R}$. Then:*

1) $f(K)$ is a C^*-convex subset of $\mathcal{R}$, for every C^*-convex set K in $\mathcal{R}$.

2) $f^{-1}(B)$ is a C^*-convex subset of $\mathcal{R}$, for every C^*-convex set B in $\mathcal{R}$.

3) If f be a bijective C^*-affine map, then K is a C^*-convex subset of $\mathcal{R}$ iff $f(K)$ is a C^*-convex set.

Proof. We only prove the first part. The proof of the second part is similar, and 3) is a consequence of the first two parts.

Let K be a C^*-convex subset of $\mathcal{R}$, $y_i \in f(K)$ $(1 \leq i \leq n)$, and $\sum_{i=1}^n a_i^* y_i a_i$ be a C^*-convex combination of elements $y_i \in f(K)$. Then, there exists $x_i \in K$ such that $f(x_i) = y_i$ for each i $(1 \leq i \leq n)$. K is a C^*-convex set, so $\sum_{i=1}^n a_i^* x_i a_i \in K$. Since f is a C^*-affine map on $\mathcal{R}$, we have

$$\sum_{i=1}^n a_i^* y_i a_i = \sum_{i=1}^n a_i^* f(x_i) a_i = f(\sum_{i=1}^n a_i^* x_i a_i) \in f(K). \qquad \square$$

Theorem 3.3. *Suppose that f is a C^*-affine map on a unital $*$-ring $\mathcal{R}$, and $x, y \in \mathcal{R}$ such that $C^* - co(\{x\}) = \{x\}$, and $C^* - co(\{y\}) = \{y\}$ (specially $x, y \in Z(\mathcal{R})$). Then the following sets are C^*-convex sets in $\mathcal{R}$, provided that they are nonempty.*

1) $[0, y]$ *where* $y \geq 0$.
2) $[x, y]$ *where* $x \leq y$.
3) $f^{-1}(\{y\})$, *(specially* $f^{-1}(\{0\})$*, the kernel of* f*).*
4) $f^{-1}([0, y])$ *where* $y \geq 0$.
5) $f^{-1}([x, y])$ *where* $x \leq y$.

Proof. 2) Let $\sum_{i=1}^n a_i^* x_i a_i$ be a C^*-convex combination of elements $x_i \in [x, y]$. Then $x \leq x_i \leq y$ for each i $(1 \leq i \leq n)$. Hence,

$$a_i^* x a_i \leq a_i^* x_i a_i \leq a_i^* y a_i.$$

So,

$$\sum_{i=1}^n a_i^* x a_i \leq \sum_{i=1}^n a_i^* x_i a_i \leq \sum_{i=1}^n a_i^* y a_i.$$

By the assumptions $C^* - co(\{y\}) = \{y\}$, and $C^* - co(\{x\}) = \{x\}$, we have

$$x \leq \sum_{i=1}^n a_i^* x_i a_i \leq y,$$

and hence $\sum_{i=1}^n a_i^* x_i a_i \in [x, y]$. Therefore, $[x, y]$ is a C^*-convex set.

The proof of 1) is similar to the proof of 2). The other parts are the conclusions of the first two parts by applying the above theorem. $\square$

Corollary 3.1. *Let f be a C^*-affine map on a unital $*$-ring $\mathcal{R}$, and $m, n \in \mathbb{N}$ such that $m \leq n$. Then the following sets are C^*-convex sets in $\mathcal{R}$, provided that they are nonempty.*

1) $f^{-1}(n1_{\mathcal{R}})$.
2) $f^{-1}([0, 1_{\mathcal{R}}])$.
3) $[m1_{\mathcal{R}}, n1_{\mathcal{R}}]$.
4) $f^{-1}([m1_{\mathcal{R}}, n1_{\mathcal{R}}])$.

Recently, in [2] the author has proved that the graph of a C^*-convex map on a unital $*$-ring $\mathcal{R}$ is a C^*-face of its epigraph ([2], Theorem 3.4) as following:

Theorem 3.4. *Let $\mathcal{R}$ be a unital $*$-ring satisfying the positive square root axiom, and $x_1^*x_1 + x_2^*x_2 + \cdots + x_n^*x_n = 0$ implies that $x_1 = x_2 = \cdots = x_n = 0$ for every $x_i \in \mathcal{R}$ and $n \in \mathbb{N}$ and f be a C^*-convex map on $\mathcal{R}$. Then, graph(f) is a C^*-face of epi(f).*

In the next theorem, we show that the set of all fixed points of the C^*-convex map f, is a C^*-face of the set $\{x \in \mathcal{R} | f(x) \leq x\}$.

Theorem 3.5. *Let f be a C^*-convex map on a unital $*$-ring $\mathcal{R}$ such that $x_1^*x_1 + x_2^*x_2 + \cdots + x_n^*x_n = 0$ implies that $x_1 = x_2 = \cdots = x_n = 0$ for every $x_i \in \mathcal{R}$ and $n \in \mathbb{N}$. Then, the set $Fix(f) = \{x \in \mathcal{R} | f(x) = x\}$ is a C^*-face of the set $K = \{x \in \mathcal{R} | f(x) \leq x\}$, provided that it is nonempty.*

Proof. Let $x \in Fix(f)$ and $x = \sum_{i=1}^n a_i^* x_i a_i$ be a proper C^*-convex combination of elements $x_i \in K$. We must show that $x_i \in Fix(f)$ for each i $(1 \leq i \leq n)$. Since

$$x = f(x) = f(\sum_{i=1}^n a_i^* x_i a_i) \leq \sum_{i=1}^n a_i^* f(x_i) a_i \leq \sum_{i=1}^n a_i^* x_i a_i = x,$$

so $\sum_{i=1}^n a_i^* x_i a_i = \sum_{i=1}^n a_i^* f(x_i) a_i$, and hence, $\sum_{i=1}^n a_i^*(x_i - f(x_i)) a_i = 0$.

Now, since $x_i - f(x_i) \geq 0$ and using the assumption

$$x_1^*x_1 + x_2^*x_2 + \cdots + x_n^*x_n = 0 \implies x_1 = x_2 = \cdots = x_n = 0,$$

for every $x_i \in \mathcal{R}$ and $n \in \mathbb{N}$, we conclude that

$$a_i^*(x_i - f(x_i)) a_i = 0,$$

for each i $(1 \leq i \leq n)$. Thus, $x_i - f(x_i) = 0$ for each i by the invertibility of a_i. Therefore, $x_i = f(x_i)$ and $x_i \in Fix(f)$ for each i $(1 \leq i \leq n)$. Hence, $Fix(f)$ is a C^*-face of the C^*-convex set K. □

Corollary 3.2. *Let $\mathcal{R}$ be a unital $*$-ring such that $x_1^*x_1 + x_2^*x_2 + \cdots + x_n^*x_n = 0$ implies that $x_1 = x_2 = \cdots = x_n = 0$, for every $x_i \in \mathcal{R}$ and $n \in \mathbb{N}$. Then:*

1) similarly, the above theorem holds for every C^*-concave map f, and the set $K = \{x \in \mathcal{R} | f(x) \geq x\}$.

2) if f be a C^*-affine map on $\mathcal{R}$, then $Fix(f)$ is a C^*-face of the sets $K_1 = \{x \in \mathcal{R} | f(x) \leq x\}$, and $K_2 = \{x \in \mathcal{R} | f(x) \geq x\}$, provided that they are nonempty.

Theorem 3.6. *Let f be a C^*-convex map on a unital $*$-ring $\mathcal{R}$ such that $x_1^*x_1 + x_2^*x_2 + \cdots + x_n^*x_n = 0$ implies that $x_1 = x_2 = \cdots = x_n = 0$ for every $x_i \in \mathcal{R}$, $n \in \mathbb{N}$, and $y \in \mathcal{R}$ such that $C^* - co(\{y\}) = \{y\}$ (specially $y \in Z(\mathcal{R})$, the center of $\mathcal{R}$). Then $F = \{x \in \mathcal{R} | f(x) = y\}$ is a C^*-face of the C^*-convex set $K = \{x \in \mathcal{R} | f(x) \leq y\}$, provided that it is nonempty.*

Proof. The proof is similar to the proof of the above theorem. □

Corollary 3.3. *If f be a C^*-convex map on a unital $*$-ring $\mathcal{R}$ such that $x_1^*x_1 + x_2^*x_2 + \cdots + x_n^*x_n = 0$ implies that $x_1 = x_2 = \cdots = x_n = 0$ for every $x_i \in \mathcal{R}$ and $n \in \mathbb{N}$. Then $F = \{x \in \mathcal{R} | f(x) = n1_{\mathcal{R}}\}$ is a C^*-face of the C^*-convex set $K = \{x \in \mathcal{R} | f(x) \leq n1_{\mathcal{R}}\}$, provided that F be nonempty. Also, the same conclusion holds for C^*-concave map f and the C^*-convex set $K = \{x \in \mathcal{R} | f(x) \geq n1_{\mathcal{R}}\}$.*

Corollary 3.4. *If f be a C^*-convex map on a unital $*$-algebra $\mathcal{A}$ such that $x_1^*x_1 + x_2^*x_2 + \cdots + x_n^*x_n = 0$ implies that $x_1 = x_2 = \cdots = x_n = 0$, for every $x_i \in \mathcal{A}$, $n \in \mathbb{N}$, and $\alpha \in \mathbb{C}$. Then, $F = \{x \in \mathcal{A} | f(x) = \alpha 1_{\mathcal{A}}\}$ is a C^*-face of the C^*-convex set $K = \{x \in \mathcal{A} | f(x) \leq \alpha 1_{\mathcal{A}}\}$, provided that F be nonempty. Also, the same conclusion holds for C^*-concave map f and the C^*-convex set $K = \{x \in \mathcal{A} | f(x) \geq \alpha 1_{\mathcal{A}}\}$.*

Note that, since every C^*-affine map is also a C^*-convex map, so the above theorem and its corollaries hold for every C^*-affine maps.

References

[1] S. K. Berberian, *Baer $*$-rings*, Springer Verlag, New York, 1972.

[2] A. Ebrahimi Meymand, *C^*-extreme points and C^*-faces of the epigraph of C^*-affine maps in $*$-rings*, Wavelets and Linear Algebra, 5 (2018), 21-28.

[3] A. Ebrahimi Meymand, *The structure of the set of all C^*-convex maps in $*$-rings*, Wavelets and Linear Algebra, 7 (2020), 43-51.

[4] A. Ebrahimi Meymand, G. H. Esslamzadeh, *C^*-convexity and C^*-faces in $*$-rings*, Turk. J. Math., 36 (2012), 131-145.

[5] D. R. Farenick, P. B. Morenz, *C^*-extreme points of some compact C^*-convex sets*, Proc. Amer. Math. Soc., 118 (1993), 765-775.

[6] D. R. Farenick, P. B. Morenz, *C^*-extreme points in the generalised state spaces of a C^*-algebra*, Trans. Amer. Math. Soc., 349 (1997), 1725-1748.

[7] A. Hopenwasser, R. L. Moore, V. I. Paulsen, *C^*-extreme points*, Trans. Amer. Math. Soc., 163 (1981), 291-307.

[8] R. Loebl, V. I. Paulsen, *Some remarks on C^*-convexity*, Linear Algebra Appl., 35 (1981), 63-78.

[9] P. B. Morenz, *The structure of C^*-convex sets*, Canad. J. Math., 46 (1994), 1007-1026.

Accepted: April 5, 2021