Weakly semi ω-continuous functions

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Abstract. In this paper we will use the semi ω-open sets to introduce a new type of weaker form of continuous function, namely weakly semi ω-continuous. Then we discuss various properties of this form of continuous functions. Finally, we study some separation axioms by using the semi ω-open subsets and we obtain some theorems linking the concepts of $s\omega$-strongly closed graph, $s\omega$-compact spaces and $s\omega$-connected with weakly semi ω-continuous functions.

Keywords: semi-open, ω-open, semi ω-open, weakly semi ω-continuous.

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1. Introduction

Throughout this work a space will always mean a topological space in which no separation axioms is assumed unless explicitly stated. If E is a subset of a space (S, τ) then the closure of E, the interior of E and the relative topology on E in (S, τ) will be denoted by $Cl(E)$, $Int(E)$ and τ_E, respectively. A space (S, τ) is called anti-locally countable [6] if each non-empty open subset of (S, τ) is uncountable.

The point $s \in S$ is called a condensation point of E [8] if for each $G \in \tau$ with $s \in G$, the set $G \cap E$ is uncountable. The set E is called ω-closed if it contains all its condensation points [9]. The complement of an ω-closed set is called ω-open or equivalently E is ω-open [6] if for each $s \in E$, there exists an open set G containing s such that $G - E$ is countable. The family of all ω-open subsets of a space (S, τ), denoted by τ_ω, forms a topology on S finer that τ. The ω-closure of a subset E of a space (S, τ) is the closure of E in the space (S, τ_ω) and it is denoted by $Cl_\omega(E)$. Many researchers have invested this type of open sets to present many works as [1, 2, 12].

A subset E of a space (S, τ) is called semi open if there exists an open set G such that $G \subseteq E \subseteq Cl(G)$. In [5], Al-Zoubi and Al-Nashef introduced the class of semi ω-open subsets of a topological space (S, τ) by generalizing ω-open subsets in the same way that the semi-open sets has generalized open sets, where a subset E of a space (S, τ) is called semi ω-open if there exists an ω-open set G such that $G \subseteq E \subseteq Cl(G)$. The family of all semi ω-open subsets of $(S,\tau)(resp.,$ the family of all semi ω-open subsets of (S, τ) containing s) will be denoted by $S\omega O(S, \tau)(resp., S\omega O(S, s))$. The semi ω-closure of a subset E of a space (S, τ) is the intersection of all semi ω-closed subsets of (S, τ) containing E and denoted by $Cl_{s\omega}(E)$. The union of all semi ω-open subsets of (S, τ) which is subset of E is the semi ω-interior of E and denoted by $Int_{s\omega}(E)$.

In this paper, we use the class of semi ω-open subsets to introduce and study a generalization of continuity which called weakly semi ω-continuous. In sections 2 and 3 of this work, we present the definition of weakly semi ω-continuous and investigate some characterizations of this type of functions and study some of there basic properties. Finally in section 4, also we present some separation axioms by using the semi ω-open subsets then we introduce the concept of $s\omega$-strongly closed graph, $s\omega$-compact and $s\omega$-connected and some theorems linking these concepts with weakly semi ω-continuous will be shown.

In this paper $\mathbb{R}, \mathbb{Q}$ and $\mathbb{N}$ denote, respectively the set of real numbers, the set of rational numbers and the set of natural numbers.

Now, we begin with some notations, definitions, and result will be used in this work.

Theorem 1.1 ([5]). *A subset E of a space (S, τ) is semi ω-open iff $E \subseteq Cl(Int_\omega(E))$.*

Lemma 1.1 ([5]). *If H is an open subset of a space (S, τ) and $E \in S\omega O(S, \tau)$, then $H \cap E \in S\omega O(S, \tau)$.*

Theorem 1.2 ([5]). *Let E be a subset of a space (S,τ). Then $Int_{s\omega}(E) = E \cap Cl(Int_{\omega}(E))$.*

Lemma 1.2 ([5]). *Let (Z, τ_Z) be a subspace of a space (S, τ) and $E \subseteq Z$. Then:*
(i) $E \in S\omega O(Z, \tau_Z)$ if $E \in S\omega O(S, \tau)$.
(ii) $E \in S\omega O(S, \tau)$ if $Z \in \omega O(S, \tau)$ and $E \in S\omega O(Z, \tau_Z)$.

Corollary 1.1 ([5]). *Let (S, τ) be a space and let $E \subseteq S$. Then:*
(i) $Cl_{s\omega}(S - E) = S - Int_{s\omega}(E)$.
(ii) $Int_{s\omega}(S - E) = S - Cl_{s\omega}(E)$.

Definition 1.1 ([10]). *A function $\psi : (S, \tau) \rightarrow (Z, \sigma)$ is said to be weakly continuous if for each point $s \in S$ and open set H in (Z, σ) containing $\psi(s)$, there is an open set G in (S, τ) with $s \in G$ and $\psi(G) \subseteq Cl(H)$.*

Definition 1.2 ([3]). *A function $\psi : (S, \tau) \rightarrow (Z, \sigma)$ is called semi ω-continuous, if for every open subset H of Z, $\psi^{-1}(H) \in S\omega O(S)$.*

Lemma 1.3 ([3]). *Let $\psi : (S, \tau) \rightarrow (Z, \sigma)$ be an open continuous surjection. If $G \in S\omega O(S, \tau)$, then $\psi(G) \in S\omega O(Z, \sigma)$.*

Definition 1.3 ([13]). *A space (S, τ) is called a Urysohn space iff whenever $s \neq t$ in S, there are two non-empty open set G and H in (S, τ) with $s \in G$, $t \in H$ and $Cl(G) \cap Cl(H) = \phi$.*

Definition 1.4 ([7]). *A topological space (S, τ) is said to be almost compact if whenever $\mathcal{G} = \{G_\alpha : \alpha \in I\}$ is open cover of (S, τ) there is a finite subsets $I_\circ$ of I with $S = \bigcup_{\alpha \in I_\circ} Cl(G_\alpha)$.*

2. Weakly semi ω-continuous

In this section we will define the class of weakly semi ω-continuous functions and establish some properties of this class.

Definition 2.1. *A function $\psi : (S, \tau) \rightarrow (Z, \sigma)$ is said to be weakly semi ω-continuous if for each $s \in S$ and for each open set H in (Z, σ) containing $\psi(s)$ there is a set $G \in S\omega O(S, s)$ with $\psi(G) \subseteq Cl(H)$.*

Observe that if S is a countable set then every function $\psi : (S, \tau) \rightarrow (Z, \sigma)$ is weakly semi ω-continuous. The following diagram follows immediately from the definitions and the fact that every open set is semi ω-open.

$$\begin{array}{ccc} \text{Continuous} & \rightarrow & \text{Weakly continuous} \\ \downarrow & & \downarrow \\ \text{Semi } \omega\text{-continuous} & \rightarrow & \text{Weakly Semi } \omega\text{-continuous} \end{array}$$

The following examples show that the converses in the above diagram need not be true in general.

Example 2.1. Let $S = \mathbb{R}$ with the topology $\tau = \tau_u$ and let $Z = \{0, 1\}$ with the topology $\sigma = \{\phi, Z, \{1\}\}$. Let $\psi : (S, \tau) \to (Z, \sigma)$ be defined by $\psi(s) = \begin{Bmatrix} 1 & , s \in \mathbb{Q} \\ 0 & , s \in \mathbb{R} - \mathbb{Q} \end{Bmatrix}$. It is clear that ψ is weakly semi ω-continuous. However ψ is not semi ω-continuous since $\psi^{-1}(\{1\}) = \mathbb{Q} \notin S\omega O(\mathbb{R}, \tau_u)$.

Example 2.2. Let $S = \{0, 1, 2\}$ with the topology $\tau = \{\phi, S, \{0\}, \{1, 2\}\}$. Let $\psi : (S, \tau) \to (S, \tau)$ be the function defined by $\psi(s) = \begin{Bmatrix} 0 & , s = 0, 1 \\ 2 & , s = 2 \end{Bmatrix}$. Then ψ is weakly semi ω-continuous but it is not weakly continuous, since $\psi(1) = 0 \in \{0\}$ which is an open set in (S, τ) with $Cl(\{0\}) = \{0\}$ and there is no open set G in (S, τ) contains 1 with $\psi(G) \subseteq Cl(\{0\}) = \{0\}$.

In the following two theorems we give some characterizations of weakly semi ω-continuous functions.

Theorem 2.1. *The following are equivalent for a function $\psi : (S, \tau) \to (Z, \sigma)$:*

(i) ψ is weakly semi ω-continuous.

(ii) for each $s \in S$ and open set H in (Z, σ) containing $\psi(s)$, we have $s \in Cl(Int_\omega(\psi^{-1}(Cl(H))))$.

(iii) $\psi^{-1}(H) \subseteq Int_{s\omega}(\psi^{-1}(Cl(H)))$ for each open set H in (Z, σ).

Proof. ($i \to ii$) Let $s \in S$ and H be an open set in (Z, σ) containing $\psi(s)$. Then there is $G \in S\omega O(S, s)$ with $G \subseteq \psi^{-1}(Cl(H))$. Since $G \in S\omega O(S, s)$ then $s \in G \subseteq Cl(Int_\omega(G)) \subseteq Cl(Int_\omega(\psi^{-1}(Cl(H))))$.

($ii \to iii$) Let H be an open set in (Z,σ) containing $\psi(s)$. Since $s \in Cl(Int_\omega(\psi^{-1}(Cl(H)))) \cap \psi^{-1}(Cl(H))$ and by Theorem 1.2, $s \in Int_{s\omega}(\psi^{-1}(Cl(H))$.

($iii \to i$) Let $s \in S$ and H be open set in (Z, σ) containing $\psi(s)$. Put $G = Int_{s\omega}(\psi^{-1}(Cl(H))$. Then $G \in S\omega O(S, s)$ with $\psi(G) \subseteq Cl(H)$. □

Theorem 2.2. *The following are equivalent for a function $\psi : (S, \tau) \to (Z, \sigma)$:*

(i) ψ is weakly semi ω-continuous.

(ii) $Cl_{s\omega}(\psi^{-1}(Int(Cl(E)))) \subseteq \psi^{-1}(Cl(E))$ for any subset E of Z.

(iii) $Cl_{s\omega}(\psi^{-1}(Int(F))) \subseteq \psi^{-1}(F)$ for every closed subset F of Z.

(iv) $Cl_{s\omega}(\psi^{-1}(H)) \subseteq \psi^{-1}(Cl(H))$ for every open subset $H$of Z.

Proof. ($i \to ii$) Let E be a subset of Z. Suppose that $s \in S - \psi^{-1}(Cl(E))$. Then there is an open set H in (Z, σ) with $\psi(s) \in H$ and $H \cap E = \phi$, so $Cl(H) \cap Int(Cl(E)) = \phi$. Since ψ is weakly semi ω-continuous, then there is $G \in S\omega O(S, s)$ with $\psi(G) \subseteq Cl(H)$ hence $G \cap \psi^{-1}(Int(Cl(E))) = \phi$. Thus $Cl_{s\omega}(\psi^{-1}(Int(Cl(E)))) \subseteq S - G$ and so $s \in S - Cl_{s\omega}(\psi^{-1}(Int(Cl(E))))$.

($ii \to iii$) The proof is clear since $Cl(F) = F$.

($iii \to iv$) Let H be an open in Z. Since $H \subseteq Int(Cl(H))$, then $Cl_{s\omega}(\psi^{-1}(H)) \subseteq Cl_{s\omega}(\psi^{-1}(Int(Cl(H)))) \subseteq \psi^{-1}(Cl(H))$.

($iv \to i$) Let H be an open set in Z containing $\psi(s)$. Then, by Corollary 1.1, $S - Int_{s\omega}(\psi^{-1}(Cl(H))) = Cl_{s\omega}(S - \psi^{-1}(Cl(H))) = Cl_{s\omega}(\psi^{-1}(Z - Cl(H))) \subseteq$

$\psi^{-1}(Cl(Z - Cl(H))) \subseteq S - \psi^{-1}(H)$. Thus $\psi^{-1}(H) \subseteq Int_{s\omega}(\psi^{-1}(Cl(H)))$ for each H open set in Z containing $\psi(s)$. Hence, by Theorem 2.1, ψ is weakly semi ω-continuous. □

3. Some properties of weakly semi ω-continuous

In this section we obtain some properties of weakly semi ω-continuous functions. The composition $\varphi \circ \psi : (S, \tau) \to (W, \theta)$ of a continuous function $\psi : (S, \tau) \to (Z, \sigma)$ and a weakly semi ω-continuous function $\varphi : (Z, \sigma) \to (W, \theta)$ is not necessarily weakly semi ω-continuous as the following example shows. Thus, the composition of weakly semi ω-continuous functions need not be weakly semi ω-continuous.

Example 3.1. Let $S = \mathbb{R}$ with the topology $\tau = \tau_u$ and let $Z = \{0, 1, 2\}$ with the topologies $\sigma = \tau_{ind}$ and $\theta = \{\phi, Z, \{1\}, \{0, 2\}\}$. Let $\psi(s) : (\mathbb{R}, \tau_u) \to (Z, \tau_{ind})$ be the function defined by $\psi(s) = \left\{\begin{matrix} 1 & , s \in \mathbb{Q} \\ 0 & , s \in \mathbb{R} - \mathbb{Q} \end{matrix}\right\}$ and $\varphi(s) : (Z, \tau_{ind}) \to (W, \theta)$ be the identity function. Then ψ is continuous and φ is weakly semi ω-continuous but $(\varphi \circ \psi)(s)$ is not weakly semi ω-continuous. Note that if $s \in \mathbb{Q}$, then $(\varphi \circ \psi)(s) = 1 \in H = \{1\}$. Suppose that there is $G \in S\omega O(\mathbb{R}, \tau_u)$ with $s \in G$ and $(\varphi \circ \psi)(G) \subseteq Cl(H) = \{1\}$. Then $G \subseteq \mathbb{Q}$, i.e. G is countable, a contradiction, since whenever G is a countable subset of $(\mathbb{R}, \tau_u)$, $Int_\omega(G) = \phi$. Therefore, $\varphi \circ \psi$ is not weakly semi ω-continuous.

The proof of the following result follows immediately from the definitions.

Theorem 3.1. *Let $\psi : (S, \tau) \to (Z, \sigma)$ and $\varphi : (Z, \sigma) \to (W, \theta)$ be two functions. Then:*

(i) $\varphi \circ \psi$ is weakly semi ω-continuous if φ is weakly continuous and ψ is semi ω-continuous.

(ii) $\varphi \circ \psi$ is weakly semi ω-continuous if φ is continuous and ψ is weakly semi ω-continuous.

Note that Theorem 3.1 is not true if φ is assumed to be only semi ω-continuous as it is shown in the following example.

Example 3.2. Let $Z = \{0, 1, 2\}$ with the topology $\sigma = \{\phi, Z, \{0\}, \{0, 1\}\}$, and let $W = \{a, b\}$ with the topology $\theta = \{\phi, W, \{a\}, \{b\}\}$. Let $\psi : (\mathbb{R}, \tau_u) \to (Z, \sigma)$ be the function defied by $\psi(s) = \left\{\begin{matrix} 2 & , s \in \mathbb{Q} \\ 1 & , s \in \mathbb{R} - \mathbb{Q} \end{matrix}\right\}$ and $\varphi : (Z, \sigma) \to (W, \theta)$ be defined as follows $\varphi(0) = \varphi(2) = a$ and $\varphi(1) = b$. Then ψ is weakly semi ω-continuous function and φ is ω-continuous function but $\varphi \circ \psi$ is not weakly semi ω-continuous at $s \in \mathbb{Q}$.

Corollary 3.1. *If $\psi : (S, \tau) \to \prod_{\alpha \in \Delta} S_\alpha$ is a weakly semi ω-continuous function then for each $\alpha \in \Delta$, $p_\alpha \circ \psi$ is weakly semi ω-continuous function, where p_α is the projection function from the product space $\prod_{\alpha \in \Delta} S_\alpha$ onto the space S_α.*

Theorem 3.2. *Let $\psi : (S,\tau) \to (Z,\sigma)$ be an open continuous surjection function. If $\varphi : (Z,\sigma) \to (W,\theta)$ is a function with $\varphi \circ \psi : (S,\tau) \to (W,\theta)$ is weakly semi ω-continuous, then φ is weakly semi ω-continuous.*

Proof. Let $t \in Z$ and $H \in \theta$ with $\varphi(t) \in H$. Choose $s \in S$ with $\psi(s) = t$. Since $\varphi \circ \psi$ is weakly semi ω-continuous, there is $G \in S\omega O(S,\tau)$ with $s \in G$ and $\varphi(\psi(G)) \subseteq Cl(H)$. But ψ is an open continuous surjection, therefore by Lemma 1.3, $\psi(G) \in S\omega O(Z,\sigma)$ with $\psi(s) \in \psi(G)$ and the result follows. □

Corollary 3.2. *Let $\psi_\alpha : (S_\alpha,\tau_\alpha) \to (Z_\alpha,\sigma_\alpha)$ be a function for each $\alpha \in \Delta$. If the product function $\psi = \prod_{\alpha\in\Delta} \psi_\alpha : \prod_{\alpha\in\Delta} S_\alpha \to \prod_{\alpha\in\Delta} Z_\alpha$ is weakly semi ω-continuous, then ψ_α is weakly semi ω-continuous for each $\alpha \in \Delta$.*

Proof. For each $\beta \in \Delta$, let $p_\beta : \prod_{\alpha\in\Delta} S_\alpha \to S_\beta$ and $q_\beta : \prod_{\alpha\in\Delta} Z_\alpha \to Z_\beta$ be the projections. Then, we have $q_\beta \circ \psi = \psi_\beta \circ p_\beta$ for each $\beta \in \Delta$. Then, by Theorems 3.2 ψ_β, is weakly semi ω-continuous. □

The following example shows that the converse is not true.

Example 3.3. Let $S = \mathbb{R}$ with the topology $\tau = \{G : G \subseteq \mathbb{Q}\} \cup \{\mathbb{R}\}$ and $Z = \{0,1\}$ with τ_{dis}. Let $\psi : (S,\tau) \to (Z,\tau_{dis})$ be the function defined by $\psi(s) = \begin{Bmatrix} 0 & , s \in \mathbb{Q} \\ 1 & , s \in \mathbb{R} - \mathbb{Q} \end{Bmatrix}$. Note that $\mathbb{Q}$ and $\mathbb{R} - \mathbb{Q}$ are semi ω-open sets in (S,τ) and so ψ is weakly semi ω- continuous function. However, the product function $h = \psi \times \psi : (\mathbb{R} \times \mathbb{R}) \to (Z \times Z)$ defined by $h(s,t) = (\psi(s),\psi(t))$ for all $(s,t) \in (\mathbb{R} \times \mathbb{R})$ is not weakly semi ω-continuous. Let $(s,t) \in (\mathbb{R} - \mathbb{Q}) \times (\mathbb{R} - \mathbb{Q})$. Then $h(s,t) = (1,1)$. Put $H = \{(1,1)\}$. Then H is an open set in $((Z \times Z),(\tau_{dis} \times \tau_{dis}))$ with $h(s,t) \in H$. Now, suppose there is a semi ω-open set E in $(S \times S)$ with $h(E) \subseteq Cl(H) = H$. There is $M \in (\tau \times \tau)_\omega$ with $M \subseteq E \subseteq Cl(M)$. Then $h(M) \subseteq H$ and so $M \subseteq (\mathbb{R} - \mathbb{Q}) \times (\mathbb{R} - \mathbb{Q})$. Since $M \neq \phi$, choose $(r,t) \in M$. Note that the only open set containing (r,t) is $(\mathbb{R} \times \mathbb{R})$ therefore $(\mathbb{R} \times \mathbb{R}) - M$ is countable. Thus $(\mathbb{R}\times\mathbb{Q}) \cup (\mathbb{R}\times\mathbb{Q}) = (\mathbb{R}\times\mathbb{R}) - ((\mathbb{R} - \mathbb{Q}) \times (\mathbb{R} - \mathbb{Q})) \subseteq (\mathbb{R}\times\mathbb{R}) - M$, a contradiction.

Theorem 3.3. *Let $\psi : (S,\tau) \to (Z,\sigma)$ be a mapping and let $\varphi : (S,\tau) \to ((S \times Z),(\tau \times \sigma))$ be the graph mapping of ψ given by $\varphi(s) = (s,\psi(s))$ for every point $s \in S$. Then $\varphi : S \to (S \times Z)$ is weakly semi ω-continuous iff ψ is weakly semi ω-continuous.*

Proof. Note that $\psi = p_Z \circ \varphi$, where $p_Z : S \times Z \to Z$ is the projection function. Now, assume that φ is weakly semi ω-continuous. Then, by Theorem 3.1, ψ is weakly semi ω-continuous.

Conversely, assume that ψ is semi ω-continuous. Let $s \in S$ and H be any open set in $(S \times Z)$ containing $\varphi(s)$. Then there are open sets $F \subseteq S$ and $B \subseteq Z$ with $\varphi(s) = (s,\psi(s)) \in F \times B \subseteq H$. Since ψ is weakly semi ω-continuous, there is $M \in S\omega O(S,\tau)$ with $s \in M$ and $\psi(M) \subseteq Cl(B)$. Put $G = F \cap M$. Then

$G \in S\omega O(S,\tau)$, by Lemma 1.1, such that $s \in G$ and $\psi(G) \subseteq Cl(B)$. Therefore, we have $\varphi(G) \subseteq G \times \psi(G) \subseteq F \times Cl(B) \subseteq Cl(F) \times Cl(B) \subseteq Cl(H)$. □

Theorem 3.4. *(i) Let $\psi : (S,\tau) \to (Z,\sigma)$ be any function. If ψ is weakly semi ω-continuous and E is an open subset of S, then the restriction $\psi|_E : (E,\tau_E) \to (Z,\sigma)$ is weakly semi ω-continuous.*

(ii) Let $s \in S$. If there is an ω-open subset E of S containing s with $\psi|_E : (S,\tau) \to (Z,\sigma)$ is weakly semi ω-continuous at s, then ψ is weakly semi ω-continuous at s.

Proof. (i) Let $s \in E$ and H be any open set in (Z,σ) containing $(\psi|_E)(s) = \psi(s)$. Since ψ is weakly semi ω-continuous, there is $G \in S\omega O(S,\tau)$ containing s and $\psi(G) \subseteq Cl(H)$. Put $G_E = G \cap E$. Then by Lemma 1.1, $G_E \in S\omega O(S,\tau)$ and by Lemma 1.2, $G_E \in S\omega O(E,\tau_E)$ with $s \in G_E$ and $(\psi|_E)(G_E) = \psi(G_E) = \psi(G \cap E) \subseteq \psi(G) \cap \psi(E) \subseteq Cl(H)$. This indicates that $\psi|_E$ is weakly semi ω-continuous at s.

(ii) Let H be any open set in (Z,σ) with $\psi(s) \in H$. Since $\psi|_E$ is weakly semi ω-continuous, there is $M \in S\omega O(E,\tau_E)$ with $s \in M$ and $\psi(M) = \psi|_E(M) \subseteq Cl(H)$. Since E is an ω-open in (S,τ), then by Lemma 1.2(ii), M is a semi ω-open in (S,τ) and the result follows. □

Corollary 3.3. *If $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ is an open cover of S, then ψ is weakly semi ω-continuous iff $\psi|_{G_\alpha}$ is weakly semi ω-continuous for every $\alpha \in \Delta$.*

Proof. Follows from Theorem 3.4. □

The following example shows that Theorem 3.4 part (i) is not true if E is ω-open subset of (S,τ).

Example 3.4. Let $S = \mathbb{R}$ with the topology $\tau = \{G \subseteq \mathbb{R} : 1 \notin G$ and $\mathbb{R} - G$ is finite$\}$. The function $\psi : (S,\tau) \to (\{0,1\},\tau_{dis})$ defined by $\psi(s) = \left\{\begin{array}{ll} 1, & s \in \mathbb{R}-\mathbb{Q} \\ 0, & s \in \mathbb{Q} \end{array}\right\}$ is semi ω-continuous (Example 2.2 of [3]) and so it is weakly semi ω-continuous. Put $E = (\mathbb{R} - \mathbb{Q}) \cup \{1\}$. Then E is ω-open in (S,τ) but it is not open. Now, the resection function $\psi|_E : (E,\tau_E) \to (\{0,1\},\tau_{dis})$ is not weakly semi ω-continuous at $s = 1$ since $\{1\} \notin S\omega O(S,\tau)$.

4. Some separation axioms

In this section we study some separation axioms by using the semi ω-open subsets then we introduce the concept of $s\omega$-strongly closed graph, $s\omega$-compact spaces and $s\omega$-connected and then we obtain some theorems linking these concepts with weakly semi ω-continuous functions.

Definition 4.1. *A topological space (S,τ) is said to be:*

(i) $s\omega - T_1$ if for each pair of distinct points s and t there are $G \in S\omega O(S,s)$, $O \in S\omega O(S,t)$ with $s \notin O$ and $t \notin G$.

(ii) $s\omega-T_2$ *if for each pair of distinct points* s *and* t *there are* $G \in S\omega O(S,s)$, $O \in S\omega O(S,t)$ *with* $G \cap O = \phi$.

In the following definition we will present the definition of $s\omega$-strongly closed graph, where the graph of the function $\psi : (S,\tau) \rightarrow (Z,\sigma)$ is the subset $L_\psi = \{(s,\psi(s)) : s \in S\}$ of the product space $(S \times Z)$.

Definition 4.2. *A function* $\psi : (S,\tau) \rightarrow (Z,\sigma)$ *is said to have a* $s\omega$*-strongly closed graph if for each* $(s,t) \in (S \times Z) - L_\psi$, *there are* $G \in S\omega O(S)$ *and open set* $H \in \sigma$ *containing* t *with* $(s,t) \in (G \times H)$ *and* $(G \times Cl(H)) \cap L_\psi = \phi$.

The following lemma follows immediately from above definition

Lemma 4.1. *A function* $\psi : (S,\tau) \rightarrow (Z,\sigma)$ *has a* $s\omega$*-strongly closed graph iff for each point* $(s,t) \in (S \times Z) - L_\psi$ *there are* $G \in S\omega O(S,s)$ *and open set* $H \in \sigma$ *containing* t *with* $\psi(G) \cap Cl(H) = \phi$.

Theorem 4.1. *Let* $\psi : (S,\tau) \rightarrow (Z,\sigma)$ *be a function having a* $s\omega$*-strongly closed graph. Then:*
(i) (S,τ) *is* $s\omega - T_1$ *if* ψ *is injective.*
(ii) (Z,σ) *is* T_2 *if* ψ *is surjective.*

Proof. (i) Let $s, m \in S$ be two distinct points. Since ψ has a $s\omega$-strongly closed graph there is $G \in S\omega O(S,s)$ and open set $H \in \sigma$ containing $\psi(m)$ with $\psi(G) \cap Cl(H) = \phi$. Therefore, $m \notin G$ and hence (S,τ) is $s\omega - T_1$.

(ii) Let $t, z \in Z$ be two distinct points. Since ψ is surjective there is $s \in S$ with $\psi(s) = z$. Since ψ has a $s\omega$-strongly closed graph then there are $G \in S\omega O(S,s)$ and open set $H \in \sigma$ containing t with $\psi(G) \cap Cl(H) = \phi$. Therefore, $z \in \psi(G) \subseteq Z - Cl(H)$. □

Theorem 4.2. *Let* $\psi : (S,\tau) \rightarrow (Z,\sigma)$ *be a weakly semi* ω*-continuous injective function. Then:*
(i) (S,τ) *is* $s\omega - T_2$ *if* (Z,σ) *is Urysohn.*
(ii) (S,τ) *is* $s\omega - T_1$ *if* (Z,σ) *is* T_2.

Proof. (i) Let $s, m \in S$ be two distinct points. Since $\psi(s) \neq \psi(m)$ then there are open sets $H, M \in \sigma$ with $\psi(s) \in H$, $\psi(m) \in M$ and $Cl(H) \cap Cl(M) = \phi$. Since ψ is weakly semi ω-continuous, there are $G \in S\omega O(S,s), O \in S\omega O(S,m)$ with $\psi(G) \subseteq Cl(H)$ and $\psi(O) \subseteq Cl(M)$. As $\psi^{-1}(Cl(H)) \cap \psi^{-1}(Cl(M)) = \phi$, then G and O are the requested sets. Therefore, (S,τ) is $s\omega - T_2$.

(ii) Let $s, m \in S$ be two distinct points. Since $\psi(s) \neq \psi(m)$ then there are open sets $H, M \in \sigma$ with $\psi(s) \in H$, $\psi(m) \in M$ and $\psi(m) \notin Cl(H)$, $\psi(s) \notin Cl(M)$. Since ψ is weakly semi ω-continuous, there are $G \in S\omega O(S,s), O \in S\omega O(S,m)$ with $\psi(G) \subseteq Cl(H)$ and $\psi(O) \subseteq Cl(M)$. Since $m \notin G$ and $s \notin O$ so (S,τ) is $s\omega - T_1$. □

Theorem 4.3. *Let* $\psi : (S,\tau) \rightarrow (Z,\sigma)$ *be a weakly semi* ω*-continuous. If* (Z,σ) *is a Urysohn space, then* $\psi : (S,\tau) \rightarrow (Z,\sigma)$ *has a* $s\omega$*-strongly closed graph.*

Proof. Let $(s,t) \in (S \times Z) - L_\psi$. Then there are open sets $M_1, M_2 \in \sigma$ with $t \in M_1$, $\psi(s) \in M_2$ and $Cl(M_1) \cap Cl(M_2) = \phi$. Since ψ is weakly semi ω-continuous, there is $G \in S\omega O(S,s)$ with $\psi(G) \subseteq Cl(M_1)$. It follows $\psi(G) \cap Cl(M_2) = \phi$ and so $(G \times Cl(M_2)) \cap L_\psi = \phi$. □

Theorem 4.4. *If $\psi : (S,\tau) \to (Z,\sigma)$ is weakly semi ω-continuous, $\varphi : (S,\tau) \to (Z,\sigma)$ is weakly continuous and Z is Urysohn, then the set $\{s \in S : \psi(s) = \varphi(s)\}$ is semi ω-closed in S.*

Proof. Let $E = \{s \in S : \psi(s) = \varphi(s)\}$ and $s \in S - E$. Then $\psi(s) \neq \varphi(s)$. Since Z is Urysohn, there are open sets $H, M \in \sigma$ with $\psi(s) \in H$, $\varphi(s) \in M$ and $Cl(H) \cap Cl(M) = \phi$. Since ψ is weakly semi ω-continuous there is $G \in S\omega O(S,\tau)$ containing s with $\psi(G) \subseteq Cl(H)$. Since φ is weakly continuous there is an open set $O \in \tau$ containing s with $\varphi(O) \subseteq Cl(M)$. Put $U = G \cap O$. Then $U \in S\omega O(S,\tau)$ containing s and $\psi(U) \cap \varphi(U) \subseteq Cl(H) \cap Cl(M) = \phi$. We obtain $U \cap E = \phi$ and hence E is semi ω-closed in S. □

Theorem 4.5. *Let $\psi : (S,\tau) \to (Z,\sigma)$ be weakly semi ω-continuous. If ψ is injective and has $s\omega$-strongly closed graph, then (S,τ) is $s\omega - T_2$.*

Proof. Let $s, m \in S$ be two distinct points. Then $\psi(s) \neq \psi(m)$ and $(s, \psi(m)) \notin L_\psi$. Since ψ has $s\omega$-strongly closed graph, there are $G \in S\omega O(S,s)$ and $H \in \sigma$ containing $\psi(m)$ with $\psi(G) \cap Cl(H) = \phi$. On other hand, ψ is weakly semi ω-continuous and so there is $M \in S\omega O(S,m)$ with $\psi(M) \subseteq Cl(H)$. Thus $\psi(G) \cap \psi(M) = \phi$ and hence $G \cap M = \phi$. Thus (S,τ) is $s\omega - T_2$. □

In the following definition we will present the concept of $s\omega$-compact.

Definition 4.3. *A topological space (S,τ) is said to be $s\omega$-compact if every $s\omega$-open cover of S has a finite subcover.*

It is note that every $s\omega$-compact is compact space but the converse is not true as the following example shows.

Example 4.1. The space $(\mathbb{N}, \tau_{cof})$ is compact but it is not $s\omega$-compact since $\tau_\omega = \tau_{dis} \subseteq S\omega O(S)$.

Theorem 4.6. *Let $\psi : (S,\tau) \to (Z,\sigma)$ be a weakly semi ω-continuous surjective function. If (S,τ) is $s\omega$-compact, then (Z,σ) is almost compact.*

Proof. Let $\{H_\alpha : \alpha \in \Delta\}$ be an open cover of Z. For each $s \in S$ there is $\alpha(s) \in \Delta$ with $\psi(s) \in H_{\alpha(s)}$. Since ψ is weakly semi ω-continuous, there is $G_s \in S\omega O(S,s)$ with $\psi(G_s) \subseteq Cl(H_{\alpha(s)})$. Therefore, $\{G_s : s \in S\}$ is a semi ω-open cover of (S,τ) and so there is $\{s_1, s_2, ...s_n\} \subseteq S$ with $S = \bigcup_{i=1}^{n} G_{s_i}$. Thus $Z = \psi(S) = \psi(\bigcup_{i=1}^{n} G_{s_i}) \subseteq \bigcup_{i=1}^{n} Cl(H_{\alpha(s_i)})$ and hence (Z,σ) is almost compact. □

Definition 4.4. *A topological space (S,τ) is called to be $s\omega$-connected iff S can not be written as a union of two non-empty disjoint semi ω-open sets.*

Theorem 4.7. *If* (S,τ) *is a* $s\omega$*-connected space and* $\psi : (S,\tau) \to (Z,\sigma)$ *is a weakly semi* ω*-continuous surjection function, then* (Z,σ) *is connected.*

Proof. At first we show that if H is a clopen subset in (Z,σ) then $\psi^{-1}(H)$ is semi ω-clopen in (S,τ). Let H be clopen subset in (Z,σ). Then by Theorem 2.1, $\psi^{-1}(H) \subseteq Int_{s\omega}(\psi^{-1}(Cl_\sigma(H))) = Int_{s\omega}(\psi^{-1}(H))$, thus $\psi^{-1}(H) \in S\omega O(S,\tau)$.

Now, we show that $\psi^{-1}(H)$ is semi ω-closed in (S,τ). Suppose by contrary that there is $s \in Cl_{s\omega}(\psi^{-1}(H)) - \psi^{-1}(H)$. Since ψ is weakly semi ω-continuous and $(Z - H) \in \sigma$ containing $\psi(s)$, there is $G \in S\omega O(S,\tau)$ with $s \in G$ and $\psi(G) \subseteq Cl(Z - H) = Z - H$. But $s \in Cl_{s\omega}(\psi^{-1}(H))$ and so $G \cap \psi^{-1}(H) \neq \phi$. Therefore, $\phi \neq \psi(G) \cap H \subseteq (Z - H) \cap H$ a contradiction. Thus $\psi^{-1}(H)$ is semi ω-closed in (S,τ). Now, suppose that (Z,σ) is not connected. Then there are non-empty open sets H_1,$H_2 \in \sigma$ with $H_1 \cap H_2 = \phi$ and $H_1 \cup H_2 = Z$. Hence we have $\psi^{-1}(H_1) \cap \psi^{-1}(H_2) = \phi$ and $\psi^{-1}(H_1) \cup \psi^{-1}(H_2) = S$. Now, H_1 and H_2 are clopen in (Z,σ), then $\psi^{-1}(H_1)$ and $\psi^{-1}(H_2)$ are semi ω-clopen in (S,τ). This implies that (S,τ) is semi ω-disconnected, a contradiction. Therefore, (Z,σ) is connected. □

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