Product of TBCH-algebras and TBCH-algebras involving ideals

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Abstract. A BCH-algebra $(H, *, 0)$ equipped with a topology τ on H (also called a BCH-topology on H) is called a topological BCH-algebra (or TBCH-algebra) if the operation $* : H \times H \to H$, defined by $*((x, y)) = x * y$ for any $x, y \in H$, is continuous, where the Cartesian product topology on $H \times H$ is furnished by τ. In this paper, we show that given two BCH-algebras $(H_1, *_1, 0_1)$ and $(H_2, *_2, 0_2)$, an operation $*$ can be defined on the product $H = H_1 \times H_2$ so that $(H, *, 0)$, where $0 = (0_1, 0_2)$, is a BCH-algebra. Moreover, if (H_1, τ_1) and (H_2, τ_2) are TBCH-algebras, then (H, τ) is a TBCH-algebra, where τ is the product topology. We also consider in this paper TBCH-algebras involving ideals.
Keywords: BCH-algebra, topology, TBCH-algebra, product, ideals.

1. Introduction

The study of BCH-algebra, an algebraic structure that generalizes the BCI-algebra, was initiated by Hu and Li [7, 8]. In their study, the concepts of ideal and closed ideal in a BCH-algebra were defined. In 2000, Hwan, R. E. and Yu, K. S. [6] defined a new type of ideal in a BCH-algebra called a translation ideal. In 2013, Borzooei, R. A., and Zahiri, O. [5] showed that a congruence relation on

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a BCH-algebra using a translation ideal can be defined. Moreover, they defined the notion of a radical of an ideal and used it together with the concept of a closed translation ideal to construct a quotient BCH-algebra.

Jun, Y. B. et al [9] defined the concept of topological BCI-algebra (or TBCI-algebra) and investigated some of its properties which include ideals in a TBCI-algebra. In 2017, Jansi, M., and Thiruveni, V. [4] introduced the concept of topological BCH-algebra (or TBCH-algebra) and investigated some of its algebraic and topological properties. The aim of this paper is to give other structural properties of topological BCH-algebras involving product of two TBCH-algebras and ideals in a TBCH-algebra.

2. Preliminary notions and known results

Definition 2.1 ([2])**.** Let (X, τ) be a topological space and let $x \in X$. Any set $U \in \tau$ containing x is called a *neighborhood* (sometimes written as *nbhd* or τ-*nbhd*) of x.

Definition 2.2 ([7])**.** A *BCH-algebra* is a nonempty set H endowed with a operation "$*$" and constant 0 satisfying the following axioms: For all $x, y, z \in H$,

(B1) $x * x = 0$,

(B2) $x * y = 0$ and $y * x = 0$ imply $x = y$,

(B3) $(x * y) * z = (x * z) * y$.

Remark 2.1 ([3, 4])**.** In any BCH-algebra $(X, *, 0)$, the following hold:

(i) $x * 0 = x$;

(ii) $x * 0 = 0$ implies $x = 0$;

(iii) $0 * (x * y) = (0 * x) * (0 * y)$; and

(iv) $(x * (x * y)) * y = 0$.

Definition 2.3 ([4])**.** Let $(X, *, 0)$ be a BCH-algebra and U, V be any nonempty subsets of X. We define a subset $U * V$ of X by $U * V = \{x * y : x \in U, y \in V\}$.

Definition 2.4 ([3, 4])**.** A nonempty subset I of a *BCH*-algebra $(X, *, 0)$ is called an *ideal* if

(i) $0 \in I$; and

(ii) $x * y \in I$ and $y \in I$ imply $x \in I$, for all $x, y \in X$.

Definition 2.5 ([6])**.** Let X be a BCH-algebra and I an ideal of X. I is a *translation ideal* if $\forall x, y, z \in X$,

$$x * y, y * x \in I \Longrightarrow (x * z) * (y * z), (z * x) * (z * y) \in I.$$

Definition 2.6 ([4])**.** Let $(H, *, 0)$ be a BCH-algebra. A topology τ furnished on H is called a *BCH-topology* on H. In addition, (H, τ) is called a *topological BCH-algebra* (or *TBCH-algebra*) if τ is a BCH-topology on H and the function $* : H \times H \to H$ defined as $*((x, y)) = x * y$ is continuous, where the Cartesian product topology on $H \times H$ is furnished by τ.

Example 2.1. Let $X = \{0, 1, 2, 3, 4\}$ and define $*$ as follows:

*	0	1	2	3	4
0	0	0	0	0	4
1	1	0	0	1	4
2	2	2	0	0	4
3	3	3	3	0	4
4	4	4	4	4	0

Then, $(X, *, 0)$ is a BCH-algebra [3]. Let $\tau = \{X, \varnothing, \{4\}, \{0, 1, 2, 3\}\}$. Then τ is a BCH-topology on X. Moreover,

$$\begin{aligned} *^{-1}(X) &= X \times X \\ *^{-1}(\varnothing) &= \varnothing \\ *^{-1}(\{4\}) &= (\{0,1,2,3\} \times \{4\}) \cup (\{4\} \times \{0,1,2,3\}) \\ *^{-1}(\{0,1,2,3\}) &= (\{0,1,2,3\} \times \{0,1,2,3\}) \cup (\{4\} \times \{4\}). \end{aligned}$$

This implies that $*$ is continuous. Thus, (X, τ) is a TBCH-algebra.

By continuity of $*$ in a TBCH-algebra (H, τ), the following result in [1] is easy:

Theorem 2.1. *Let τ be a BCH-topology on X. Then, (X, τ) is a TBCH-algebra if and only if for each $x, y \in X$ and each nbhd W of $x * y$, there exist nbhds U and V of x and y, respectively, such that $U * V \subseteq W$.*

3. Results

3.1 Product of two TBCH-algebras

Theorem 3.1. *Let $(H_1, *_1, 0_1)$ and $(H_2, *_2, 0_2)$ be BCH-algebras and $H = H_1 \times H_2$. Then $(H, *, 0)$ is a BCH-algebra, where $0 = (0_1, 0_2)$ and "$*$" is defined by $(x, y) * (a, b) = (x *_1 a, y *_2 b)$ for all $(x, y), (a, b) \in H$.*

Proof. Let $x = (x_1, x_2) \in H$. Then by (B1),

$$\begin{aligned} x * x &= (x_1, x_2) * (x_1, x_2) \\ &= (x_1 *_1 x_1, x_2 *_2 x_2) \\ &= (0_1, 0_2) \\ &= 0. \end{aligned}$$

Next, let $x = (x_1, x_2)$, $y = (y_1, y_2) \in H$. Suppose that $x * y = 0$ and $y * x = 0$. Then $0 = x * y = (x_1, x_2) * (y_1, y_2) = (x_1 *_1 y_1, x_2 *_2 y_2)$. Hence, $x_1 *_1 y_1 = 0_1$ and $x_2 *_2 y_2 = 0_2$. Similarly, $y_1 *_1 x_1 = 0_1$ and $y_2 *_2 x_2 = 0_2$. Since $(H_1, *_1, 0_1)$ and $(H_2, *_2, 0_2)$ are BCH-algebras, $x_1 = y_1$ and $x_2 = y_2$ by (B2). Thus, $x = (x_1, x_2) = (y_1, y_2) = y$.

Finally, let $x = (x_1, x_2)$, $y = (y_1, y_2)$, $z = (z_1, z_2) \in H$. Then by (B3),

$$\begin{aligned}(x * y) * z &= [(x_1, x_2) * (y_1, y_2)] * (z_1, z_2) \\ &= (x_1 *_1 y_1, x_2 *_2 y_2) * (z_1, z_2) \\ &= ((x_1 *_1 y_1) *_1 z_1, (x_2 *_2 y_2) *_2 z_2) \\ &= ((x_1 *_1 z_1) *_1 y_1, (x_2 *_2 z_2) *_2 y_2) \\ &= (x_1 *_1 z_1, x_2 *_2 z_2) * (y_1, y_2) \\ &= [(x_1, x_2) * (z_1, z_2)] * (y_1, y_2) \\ &= (x * z) * y.\end{aligned}$$

Therefore, $(H, *, 0)$ is a BCH-algebra by Definition 2.2. □

Lemma 3.1. *Let $(H_1, *_1)$ and $(H_2, *_2)$ be BCH-algebras and let $(H, *)$ be the product of H_1 and H_2 defined in Theorem* 3.1. *If $V_1, U_1 \subseteq H_1$, $V_2, U_2 \subseteq H_2$, $W_1 = V_1 \times V_2$ and $W_2 = U_1 \times U_2$, then $W_1 * W_2 = (V_1 *_1 U_1) \times (V_2 *_2 U_2)$.*

Proof. Let $(x, y) \in W_1$ and $(a, b) \in W_2$. Then $x *_1 a \in V_1 *_1 U_1$ and $y *_2 b \in V_2 *_2 U_2$. Hence, $(x, y) * (a, b) = (x *_1 a, y *_2 b) \in (V_1 *_1 U_1) \times (V_2 *_2 U_2)$ by the definition of $*$ in Theorem 3.1. Hence, $W_1 * W_2 \subseteq (V_1 *_1 U_1) \times (V_2 *_2 U_2)$.

Next, let $a_1 \in V_1$, $a_2 \in U_1$, $b_1 \in V_2$, and $b_2 \in U_2$. Then $(a_1, b_1) \in W_1$ and $(a_2, b_2) \in W_2$. Thus, by the definition of $*$ in Theorem 3.1, $(a_1 *_1 a_2, b_1 *_2 b_2) = (a_1, b_1) * (a_2, b_2) \in W_1 * W_2$. Therefore, $(V_1 *_1 U_1) \times (V_2 *_2 U_2) \subseteq W_1 * W_2$. This proves the desired equality. □

Theorem 3.2. *Let (H_1, τ_1) and (H_2, τ_2) be TBCH-algebras. Then (H, τ), where $H = H_1 \times H_2$ and τ is the product topology, is a TBCH-algebra.*

Proof. Let $(x, y), (a, b) \in H$ and let W be a τ-nbhd of $(x, y) * (a, b) = (x *_1 a, y *_2 b)$. Let $U \in \tau_1$ and $V \in \tau_2$ such that $(x *_1 a, y *_2 b) \in U \times V \subseteq W$. Since (H_1, τ_1) is a TBCH-algebra, there exist τ_1-nbhds U_x and U_a of x and a, respectively, such that $U_x *_1 U_a \subseteq U$ by Theorem 2.1. Similarly, since (H_2, τ_2) is a TBCH-algebra, there exist τ_2-nbhds V_y and V_b of y and b, respectively, such that $V_y *_2 V_b \subseteq V$. Clearly, $U_x \times V_y$ and $U_a \times V_b$ are τ-nbhds of (x, y) and (a, b), respectively. Moreover, by Lemma 3.1,

$$(U_x \times V_y) * (U_a \times V_b) = (U_x *_1 U_a) \times (V_y *_1 V_b) \subseteq U \times V \subseteq W.$$

Therefore, by Theorem 2.1, (H, τ) is a TBCH-algebra. □

3.2 Ideals in a TBCH-algebra

We denote a TBCH-algebra (X, τ) by X, unless otherwise specified.

Remark 3.1. An ideal of a TBCH-algebra may neither be an open set nor a closed set.

Example 3.1. Consider the TBCH-algebra $X = \{0, 1, 2, 3, 4\}$ with topology $\tau = \{X, \varnothing, \{4\}, \{0, 1, 2, 3\}\}$ in Example 2.1. Consider the trivial ideal $I = \{0\}$ of X. Observe that $I \notin \tau$ and $X \setminus I = \{1, 2, 3, 4\} \notin \tau$, that is, I is neither an open nor closed set in (X, τ).

The next theorem asserts that if an ideal is an open set in a TBCH-algebra, then it is also a closed set.

Theorem 3.3. *Let X be a TBCH-algebra and let A be an ideal of X. If A is an open set in X, then A is also a closed set.*

Proof. Suppose A is an open set and let $x \in X \setminus A$. Since A is an ideal of X, $x * x = 0 \in A$ by (B1) and Definition 2.4(i). By Theorem 2.1, there exist nbhds U and V of x such that $U * V \subseteq A$. Suppose $V \cap A \neq \varnothing$, say $y \in V \cap A$. Then $x * y \in U * V \subseteq A$. Since $y \in A$ and A is an ideal, $x \in A$, contrary to our assumption. Thus $V \subseteq X \setminus A$, showing that $X \setminus A$ is open. Therefore, A is a closed set in X. □

Theorem 3.4. *Let X be a TBCH-algebra and A an ideal of X. Then A is open in X if and only if 0 is an interior point of A.*

Proof. Suppose 0 is an interior point of A. Then, there exists nbhd U of 0 such that $U \subseteq A$. Let $x \in A$. By (B1), $x * x = 0 \in U$. Then, there exist nbhds G and H of x such that $G * H \subseteq U \subseteq A$ by Theorem 2.1. Let $y \in G$. Then, $y * x \in G * H \subseteq A$. Since A is an ideal and $x \in A$, $y \in A$. Hence, $x \in G \subseteq A$. Thus, A is an open set in X.
Conversely, suppose A is an open set in X. Then, $A = int(A)$. Since A is an ideal, $0 \in A = int(A)$. Therefore, 0 is an interior point of A. □

Example 3.2. Consider the *TBCH*-algebra $X = \{0, 1, 2, 3, 4\}$ with topology $\tau = \{X, \varnothing, \{4\}, \{0, 1, 2, 3\}\}$ in Example 2.1. Clearly, $\{4\}$ is an open set which is not an ideal of X. Moreover, $0 \notin int(\{4\}) = \{4\}$.

Remark 3.2. Not every open set in a *TBCH*-algebra X is an ideal of X.

The next theorem says that if every nonempty open set in a TBCH-algebra contains 0, then every nonempty open set is an ideal.

Theorem 3.5. *Let X be a TBCH-algebra such that $0 \in \bigcap_{U \in \tau \setminus \{\varnothing\}} U$. Then every nonempty open set I in X is an ideal.*

Proof. Let I be a nonempty open set in X. Then $0 \in I$ by hypothesis. Let $x, y \in X$ such that $y \in I$ and $x * y \in I$. Since I is open, there exist nbhds U and V of x and y, respectively, such that $U * V \subseteq I$ by Theorem 2.1. By the hypothesis and Remark 2.1(i), $x = x * 0 \in U * V \subseteq I$. Therefore, I is an ideal of X. □

Lemma 3.2. *Let X be a TBCH-algebra. Suppose there exists $\varnothing \neq I \subseteq X$ such that $I \subseteq U$ for each $U \in \tau$ with $0 \in U$. Then $I \subseteq V$ for any nbhd V of a point in I.*

Proof. Let V be a nbhd of some point $x \in I$. By Remark 2.1(i), $x * 0 = x \in V$. By Theorem 2.1, there exist nbhds U and W of x and 0, respectively, such that $U * W \subseteq V$. By (B1) and the hypothesis, $0 = x * x \in U * I \subseteq U * W \subseteq V$. It follows that V is an open set containing 0. Therefore, $I \subseteq V$. □

Theorem 3.6. *Let (X, τ) be a TBCH-algebra. If $I \in \tau$ and $0 \in I \subseteq U$ for each $U \in \tau$ with $0 \in U$, then I is an ideal of X.*

Proof. Let $x, y \in X$. Suppose $x * y \in I$ and $y \in I$. By Theorem 2.1, there exist nbhds U and V of x and y, respectively, such that $U * V \subseteq I$. By Lemma 3.2, $I \subseteq V$. By Remark 2.1 (i), $x = x * 0 \in U * I \subseteq U * V \subseteq I$. Therefore, I is an ideal of X. □

The following example shows that the converse of Theorem 3.6 is not necessarily true.

Example 3.3. Consider the *TBCH*-algebra $X = \{0, 1, 2, 3, 4\}$ with topology $\tau = \{X, \varnothing, \{4\}, \{0, 1, 2, 3\}\}$ in Example 2.1. Clearly, the trivial ideal $\{0\}$ is not an open set in X.

With some additional assumption, the converse of Theorem 3.3 holds.

Theorem 3.7. *Let X be a TBCH-algebra and let A be an ideal of X. Suppose there exists a nonempty open set I such that $0 \in I \subseteq U$ for each $U \in \tau$ with $0 \in U$. If A is a closed set, then it is also an open set.*

Proof. Suppose A is a closed set in X. Note that since A is an ideal of X, $0 \in A$. Assume that A is not an open set in X. By Theorem 3.4, 0 is not an interior point of A. This implies that for all nbhd U of 0, $U \nsubseteq A$. In particular, $I \nsubseteq A$. Hence, $(X \setminus A) \cap I \neq \varnothing$. Let $x \in (X \setminus A) \cap I$. Observe that $(X \setminus A) \cap I$ is an open set containing x. By Lemma 3.2, $I \subseteq (X \setminus A) \cap I \subseteq X \setminus A$. It follows that $0 \in X \setminus A$, a contradiction. Therefore, A is an open set in X. □

Corollary 3.1. *Let X be a finite TBCH-algebra and let A be an ideal of X. Then A is open if and only if it is closed.*

Proof. Suppose that A is open. Then A is closed by Theorem 3.3.

For the converse, suppose that A is closed. Let I be the intersection of all the nbhds of 0. Then I satisfies the property given in Theorem 3.7. Thus, by Theorem 3.7, A is open. □

Theorem 3.8. *Let X be a BCH-algebra and let I be a translation ideal of X. Then the relation $\cong_I$, defined on X by, $x \cong_I y$ if and only if $x * y, y * x \in I$, is a congruence relation on X. Moreover, if $x, y, z, t \in X$ and $x \cong_I y$ and $z \cong_I t$, then $z * x \cong_I t * y$.*

Proof. Let $x \in X$. By (B1) and the fact that I is an ideal of X, $x * x = 0 \in I$. Hence, $x \cong_I x$. Let $x, y \in X$ and suppose that $x \cong_I y$. Then $x * y$, $y * x \in I$. Thus, $y \cong_I x$. Now, if $x, y, z \in X$ and $x \cong_I y$ and $y \cong_I z$, then $x * y$, $y * x$, $y * z$, $z * y \in I$. Since I is a translation ideal, $(x * z) * (y * z)$, $(z * x) * (z * y) \in I$. It follows that $x * z$, $z * x \in I$ because I is an ideal of X. Hence, $x \cong_I z$. Therefore, $\cong_I$ is an equivalence relation on X.

Next, let $x, y, z, t \in X$ and suppose that $x \cong_I y$ and $z \cong_I t$. Then $x * y$, $y * x$, $z * t$, $t * z \in I$. Since I is a translation ideal, $(x*z)*(y*z)$, $(z*x)*(z*y)$, $(y*z)*(x*z)$, $(z*y)*(z*x)$, $(z*y)*(t*y)$, $(y*z)*(y*t)$, $(t*y)*(z*y)$, $(y*t)*(y*z) \in I$. Using $(y*t)$ and the fact that I is a translation ideal and $(x*z)*(y*z)$, $(y*z)*(x*z) \in I$, we have $((x * z) * (y * t))*((y * z) * (y * t)) \in I$. Hence, $(x*z)*(y*t) \in I$ because $(y * z) * (y * t) \in I$ and I is an ideal of X. Similarly, $(y * t) * (x * z) \in I$. Thus, $x * z \cong_I y * t$. Using a similar argument, it can be shown that $z * x \cong_I t * y$. Accordingly, $\cong_I$ is a congruence relation on X. □

Let X be a BCH-algebra and I a translation ideal of X. Consider the congruence relation $\cong_I$ on X defined in Theorem 3.8. The equivalence class of $x \in X$ with respect to $\cong_I$ will be denoted by I_x, that is $I_x = \{y \in X : x \cong_I y\} = \{y \in X : x * y, y * x \in I\}$.

Theorem 3.9. *Let $\mathcal{I}$ be the collection of all translation ideals in a BCH-algebra X. Then (X, τ) is a TBCH-algebra, where*

$$\tau = \{U \subseteq X : \forall x \in U, \exists I \in \mathcal{I} \text{ such that } I_x \subseteq U\}.$$

Proof. Note that for each $x \in X$ and for each $I \in \mathcal{I}$, $I_x \subseteq X$. Hence, $X \in \tau$. By definition of τ, $\varnothing \in \tau$. Let $U_1, U_2 \in \tau$. Suppose $y \in U_1 \cap U_2$. Then $y \in U_1$ and $y \in U_2$. Thus, exist $A, B \in \mathcal{I}$ such that $A_y \subseteq U_1$ and $B_y \subseteq U_2$. Let $I = A \cap B$. Let $x, y, z \in X$ and suppose that $x * y, y * x \in I$. Then $x * y, y * x \in A$ and $x * y, y * x \in B$. Since A and B are translation ideals, $(x * z) * (y * z), (z * x) * (z * y) \in A$ and $(x * z) * (y * z), (z * x) * (z * y) \in B$. Hence, $(x * z) * (y * z), (z * x) * (z * y) \in A \cap B = I$. It follows that $I \in \mathcal{I}$.

Let $x \in I_y$. Then $y * x, x * y \in I \subseteq A$. Hence, $y \cong_A x$, that is, $x \in A_y$. Similarly, $x \in B_y$. It follows that $I_y \subseteq A_y \cap B_y \subseteq U_1 \cap U_2$. This shows that $U_1 \cap U_2 \in \tau$.

Let $\{U_\alpha : \alpha \in \mathcal{A}\} \subseteq \tau$ and let $y \in \bigcup_{\alpha \in \mathcal{A}} U_\alpha$. Then $y \in U_\beta$ for some $\beta \in \mathcal{A}$. This implies that there exists $I \in \mathcal{I}$ such that $I_y \subseteq U_\beta \subseteq \bigcup_{\alpha \in \mathcal{A}} U_\alpha$. Hence, $\bigcup_{\alpha \in \mathcal{A}} U_\alpha \in \tau$. Therefore, τ is a *BCH*-topology on X.

Next, let $I \in \mathcal{I}$ and let $x \in X$. Then clearly, $I_x \in \tau$. Now, let $x, y \in X$ and let U be a nbhd of $x * y$. Then there exists $I \in \mathcal{I}$ such that $I_{x*y} \subseteq U$. By

Theorem 3.8, $\cong_I$ is a congruence relation on X and so

$$\begin{aligned} I_x * I_y &= \{a * b \in X : a \in I_x \text{ and } b \in I_y\} \\ &= \{a * b \in X : x \cong_I a \text{ and } y \cong_I b\} \\ &= \{a * b \in X : x * y \cong_I a * b\} \\ &\subseteq \{z \in X : x * y \cong_I z\} \\ &= I_{x*y}. \end{aligned}$$

Thus, I_x and I_y are τ-nbhds of x and y, respectively and $I_x * I_y \subseteq I_{x*y} \subseteq U$. By Theorem 2.1, $(X, *, \tau)$ is a TBCH-algebra. □

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