Fuzzy structures in hyper GR-algebras

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Abstract. In this paper, we consider hyper GR-ideals of a hyper GR-algebra applied to fuzzy sets such as fuzzy hyper GR-ideal of type 2 and fuzzy distributive hyper GR-ideal of type 2. These concepts are introduced including the concept of distributive hyper GR-ideal. Several properties and characterizations of distributive hyper GR-ideals, fuzzy hyper GR-ideals of type 2 and fuzzy distributive hyper GR-ideals of type 2 are obtained. Moreover, a relationship between fuzzy hyper GR-ideals of type 2 and hyper GR-ideals and fuzzy distributive hyper GR-ideals of type 2 and distributive hyper GR-ideals with respect to its level subset are presented. Furthermore, it is obtained that the family of fuzzy distributive hyper GR-ideals of type 2 of a hyper GR-algebra is a complete distributive lattice with respect to join and meet.

Keywords: hyper GR-algebras, hyper GR-ideals, distributive hyper GR-ideal, fuzzy hyper GR-ideals of type 2, fuzzy distributive hyper GR-ideal of type.

1. Introduction

Hyperstructure theory was born in 1934 at the 8th Congress of Scandinavian Mathematicians, where Marty [7] introduced the hypergroup notion as a generalization of groups and proved its utility in solving some problems of groups, algebraic functions and rational fractions. Surveys of the theory can be found in the books of Corsini [18], Vougiouklis [22], Corsini and Leoreanu [19]. Hypergroups are studied from the theoretical point of view and for their applications to many areas of pure and applied mathematics, such as: geometry, topology, cryptography and coding theory, graphs and hypergraphs, probability theory, theory of fuzzy and rough sets, automata theory, and artificial intelligence. Moreover, hyperstructures can be associated with elementary particles in physical theory [2]. It was also used in inheritance issues in genetics [3].

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In the following years, several reaserchers worked on this area. One of the well developed hyperstructure is the hyper BCI-algebra [23]. By following this hyperstructure, Indangan and Petalcorin [11, 1] introduced a new hyperstructure which is called a hyper GR-algebra. They established some results on hyper GR-ideals of a hyper GR-algebra and hyper homomorphic properties on hyper GR-algebras. Recently, a pseudo hyper GR-algebra was established by Manzano and Petalcorin [10]. Pseudo hyper GR-ideals were also defined and classified to determine their relationship to each other.

In the real world, the classes of objects encountered may not have precisely defined criteria of membership. For example, the class of blue colors clearly includes sky blue and navy blue as its members and clearly excludes colors such as red and orange. However, colors such as blue violet and cyan have an ambiguous status with respect to the class of blue colors. With this ambiguity, the theory of fuzzy set arises. When Zadeh [15] first introduced the fuzzy set, unlike sets where the objects are precise, he defined it as a class of objects where objects are given grades of membership. In other words, a fuzzy set A is characterized by a membership function μ where each object is assigned a grade of membership between 0 to 1. The closer the value of the function at an object x to 1, the higher the membership of x in A. When A is a set in the ordinary sense, the membership function is simply the characteristic function. This concept of fuzzy sets are extremely useful for many people involved in research and development including engineers, mathematicians, computer software developers and researchers, natural scientists, medical researchers, social scientists, public policy analysts, business analysts, and jurists [4]. Moreover, it provides a natural framework for generating many concepts of general mathematics. Specifically, it had been applied to some algebras and hyperstructure algebras. Some of these studies are fuzzy hyper BCK-ideals of hyper BCK-algebras[14], fuzzy implicative hyper BCK-ideals of hyper BCK-algebras [13], and fuzzy ideals in hyper BCI-algebras [8], hyper K-subalgebras based on fuzzy points [16], on fuzzy hyper B-ideals of hyper B-algebras [12], some results on fuzzy implicative hyper GR-ideals [6], and on intuitionistic fuzzy hyper GR-ideals in hyper GR-algebras [5].

In this study, the fuzzification of a hyper GR-algebras are considered. In particular, this paper seeks to introduce the concepts of distributive hyper GR-ideal, fuzzy hyper GR-ideal of type 2 and fuzzy distributive hyper GR-ideal of type 2, and aims to establish some of their characterizations and relationships.

2. Preliminaries

In this section, we gather some basic notions and properties relevant to this study.

Let H be a nonempty set, $x \in H$ and A, B a nonempty subsets of H. Consider a hyperoperation $\circledast$. We define $A \circledast B = \bigcup_{a \in A, b \in B} a \circledast b$, $A \circledast x = A \circledast \{x\}$, and $x \circledast B = \{x\} \circledast B$. Moreover, $x \ll y$ is defined by $0 \in x \circledast y$ and $A \ll B$ is defined by for any $a \in A$, there exists $b \in B$ such that $a \ll b$. We call " $\ll$ " a *hyperorder* on H.

Definition 2.1 ([11])**.** *Let H be a nonempty set and $\circledast$ be a hyperoperation on H. Then $(H; \circledast, 0)$ is called a hyper GR-algebra if it contains a constant $0 \in H$ and it satisfies the following conditions, for all $x, y, z \in H$:*

(HGR1) $(x \circledast z) \circledast (y \circledast z) \ll x \circledast y$;

(HGR2) $(x \circledast y) \circledast z = (x \circledast z) \circledast y$;

(HGR3) $x \ll x$;

(HGR4) $0 \circledast (0 \circledast x) \ll x, x \neq 0$; *and*

(HGR5) $(x \circledast y) \circledast z \ll y \circledast z$.

For the sake of simplicity, we also call H a hyper GR-algebra.

Example 2.1. Let $H = [0, 1]$ such that for any $a, b \in [0, 1]$,

$$a \circledast b = \begin{cases} [0, 0.3], & \text{if } b \neq 0 \text{ or } a = 0 = b; \\ \{a\}, & \text{if } a \neq 0 \text{ and } b = 0. \end{cases}$$

It can be seen that H is a hyper GR-algebra.

Definition 2.2 ([11])**.** *Let S be a subset of a hyper GR-algebra H containing 0. We say that S is a hyper subGR-algebra on H if it is a hyper GR-algebra with respect to the hyperoperation $\circledast$ on H.*

Definition 2.3 ([11])**.** *A subset I of a hyper GR-algebra H is said to be a hyper GR-ideal of H if*

(I1) $0 \in I$, *and*

(I2) *for all $x, y \in H$, $x \circledast y \subseteq I$ and $y \in I$ imply that $x \in I$.*

Definition 2.4 ([6])**.** *A nonempty subset I of a hyper GR-algebra H is called an implicative hyper GR-ideal of H if for any $x, y, z \in H$,*

(IH1) $0 \in I$, *and*

(IH2) $(x \circledast z) \circledast (y \circledast x) \subseteq I$ *and* $z \in I$ *imply* $x \in I$.

Theorem 2.1 ([6])**.** *Let H be a hyper GR-algebra such that $x \in x \circledast y$ for any $x, y \in H$. Then any subset I of H containing 0 is an implicative hyper GR-ideal of H.*

Definition 2.5 ([11]). *Suppose that H and K are hyper GR-algebras. Define a hyperoperation*

$$\circledast : (H \times K) \times (H \times K) \to P^*(H \times K) \text{ by}$$
$$(a, b) \circledast (c, d) = (a \circledast_H c) \times (b \circledast_K d)$$

and $(a, b) \ll (c, d)$ if and only if $a \ll_H c$ and $b \ll_K d$, *for all $a, c \in H$ and $b, d \in K$ with $\ll_H$ and $\ll_K$ as hyper orders of $(H, \circledast_H, 0_H)$ and $(K, \circledast, 0_k)$, respectively. If A, B are nonempty subsets of H, then we define $(A, B) = A \times B$.*

Lemma 2.1 ([11]). *Let H and K be hyper GR-algebras, $A, C \subseteq H$, anf $B, D \subseteq K$. Then $(A, B) \circledast (C, D) = (A \circledast_H C, B \circledast_K D)$.*

Definition 2.6 ([15]). *A fuzzy set μ of a nonempty set M is a function $\mu : M \to [0, 1]$.*

Definition 2.7 ([20]). *Let μ be a fuzzy set of M. For a fixed $t \in [0, 1]$, the set $\mu_t = \{x \in M | \mu(x) \geq t\}$ is called a level subset of μ.*

Definition 2.8 ([5]). *A fuzzy set μ in a hyper GR-algebra H is a fuzzy hyper GR-ideal of type 1 if for all $x, y \in H$,*

(F1) $\mu(0) \geq \mu(x) \geq \min\{\inf_{u \in x \circledast y} \mu(u), \mu(y)\}$.

Definition 2.9 ([17]). *A fuzzy relation on any set H is a fuzzy set*

$$\lambda : H \times H \to [0, 1].$$

Definition 2.10 ([17]). *If λ is a fuzzy relation on a set H and μ is a fuzzy set in H, then λ is a fuzzy relation on μ if $\lambda(x, y) \leq \min\{\mu(x), \mu(y)\}$* for all $x, y \in H$.

Definition 2.11 ([17]). *If μ is a fuzzy set in a set H, then a strongest fuzzy relation on H, denoted by λ_μ, is a fuzzy relation on μ given for all $x, y \in H$ by $\lambda_\mu(x, y) = \min\{\mu(x), \mu(y)\}$.*

Definition 2.12 ([17]). *Let μ_1 and μ_2 be fuzzy sets in a set H. The Cartesian product of μ_1 and μ_2 is defined for all $x, y \in H$ by*

$$(\mu_1 \times \mu_2)(x, y) = \min\{\mu_1(x), \mu_2(y)\}.$$

Definition 2.13 ([21]). *For a family $\{\mu_i : i \in I\}$ of fuzzy sets in H, define the join $\vee_{i \in I} \mu_i$ and the meet $\wedge_{i \in I} \mu_i$ as follows:*

$$(\vee_{i \in I} \mu_i)(x) = \sup_{i \in I} \mu_i(x) \text{ and } (\wedge_{i \in I} \mu_i)(x) = \inf_{i \in I} \mu_i(x),$$

for all $x \in H$, where I is any indexing set.

Definition 2.14 ([9]). *A partially ordered set P is a set in which a binary relation $x \leq y$ is defined, which satisfies for all $x, y, z \in P$ the following conditions:*

(P1) *For all* x, $x \geq x$.

(P2) *If* $x \geq y$ *and* $y \geq x$, *then* $x = y$.

(P3) *If* $x \geq y$ *and* $y \geq z$, *then* $x \geq z$.

Definition 2.15 ([9]). *A partially ordered set* P *is finite if the number of its elements is finite.*

Definition 2.16 ([9]). *A totally ordered set* P *is a partially ordered set which satisfies the following condition:*

(P4) *Given* x *and* y *in* P, *either* $x \geq y$ *or* $y \geq x$.

A totally ordered set is also called a chain.

Theorem 2.2 ([9]). *Any subset of a chain is a chain.*

Definition 2.17 ([9]). *Let* X *be a subset of a partially ordered set* P. *An upperbound of* X *is an element* $u \in P$ *such that* $u \geq x$ *for all* $x \in X$. *The least upperbound or supremum of* X, *denoted by* $\sup X$, *is an upperbound* u_0 *such that* $u_0 \leq u$ *for any upperbound* u *of* X. *A lowerbound of* X *is an element* $l \in P$ *such that* $l \leq x$ *for all* $x \in X$. *The greatest lowerbound or infimum of* X, *denoted by* $\inf X$, *is a lowerbound* l_0 *such that* $l_0 \geq l$ *for any lowerbound* l *of* X.

Definition 2.18 ([9]). *The length of a finite chain is defined to be* $n - 1$ *where* n *is the number of elements of the chain. The length* $l(P)$ *of a partially ordered set* P *is defined as the least upper bound of the lengths of the chains in* P. *When* $l(P)$ *is finite,* P *is said to be of finite length.*

Definition 2.19 ([9]). *A lattice* L *is a partially ordered set where any two of whose elements* x *and* y *have an infimum (or "meet") denoted by* $x \wedge y$, *and a supremum (or "join") denoted by* $x \vee y$. *A lattice* L *is complete when each of its subsets has an infimum and supremum in* L.

Remark 2.1 ([9]). Any finite lattice or lattice of finite length is complete.

Definition 2.20 ([9]). *A sublattice of a lattice* L *is a subset* X *of* L *such that* $a \in X$, $b \in X$ *imply* $a \wedge b \in X$ *and* $a \vee b \in X$.

Definition 2.21 ([9]). *A lattice* L *is distributive if either* $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ *or* $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ *holds for all* $x, y, z \in L$.

Lemma 2.2 ([9]). *Any totally ordered lattice is a distributive lattice.*

3. Distributive hyper GR-ideal

Definition 3.1. *A nonempty subset A of a hyper GR-algebra H is called a distributive hyper GR-ideal if it satisfies the following:*

(DH1) $0 \in A$, *and*

(DH2) *for all* $x, y, z \in H$, $[(x \circledast z) \circledast z] \circledast (y \circledast z) \subseteq A$ *and* $y \in A$ *imply* $x \in A$.

Example 3.1. Let $H = \{0, 1, 2, 3\}$ with the Cayley table below.

$\circledast$	0	1	2	3
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{0,2\}$	$\{0,2\}$	$\{0,1,2\}$	$\{0,1,2\}$
3	$\{0,3\}$	$\{0,3\}$	$\{0,3\}$	$\{0,1,3\}$

It can be verified that H is a hyper GR-algebra. By routine calculations, we see that $\{0,1\}, \{0,1,2\}$ and $\{0,1,3\}$ are both hyper GR-ideals and distributive hyper GR-ideals of H.

In Example 3.1, $[(x \circledast z) \circledast z] \circledast (y \circledast z) \subseteq \{0,1,2\}$ for any $x \neq 3$, $z \neq 0$ (or 1 or 2 or 3) and $y \neq 0$ (or 1 or 2 or 3). Will $\{0,1,2\}$ remain a distributive hyper GR-ideal of H if $[(x \circledast z) \circledast z] \circledast (y \circledast z) \subseteq \{0,1,2\}$, for all $x, y, z \in H$? The next theorem will answer this question.

Theorem 3.1. *Let A be a proper subset of a hyper GR-algebra H. If $[(x \circledast z) \circledast z] \circledast (y \circledast z) \subseteq A$, for all $x, y, z \in H$, then A is not a distributive hyper GR-ideal of H.*

Proof of Theorem 3.1. Since A is a proper subset of H, there exists $w \in H$ such that $w \notin A$. Suppose A is a distributive hyper GR-ideal of H. Let $v \in A$. By the hypothesis, $[(w \circledast z) \circledast z] \circledast (v \circledast z) \subseteq A$. Then $w \in A$. This is a contradiction. Hence, A is not a distributive hyper GR-ideal of H.

Example 3.2. Consider the set $H = \{0, 1, 2\}$ with the Cayley table below.

$\circledast$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{0,1\}$	$\{0,2\}$	$\{0\}$
2	$\{0,2\}$	$\{2\}$	$\{0\}$

It can be verified that H is a hyper GR-algebra and $\{0\}$ is a hyper GR-ideal of H. Observe that $\{0\}$ and $\{0,1\}$ are hyper GR-ideals of H. Since $[(2 \circledast 2) \circledast 2] \circledast (0 \circledast 2) = \{0\}$ and $0 \in \{0\}$ but $2 \notin \{0\}$, $\{0\}$ is not a distributive hyper GR-ideal of H. Moreover, since $[(2 \circledast 2) \circledast 2] \circledast (1 \circledast 2) = \{0\} \subset \{0,1\}$ and $1 \in \{0,1\}$ but $2 \notin \{0,1\}$, $\{0,1\}$ is also not a distributive hyper GR-ideal of H.

Remark 3.1. Example 3.2 shows that not every hyper GR-ideal of a hyper GR-algebra H is a distributive hyper GR-ideal.

In Example 3.2, we can see that $2 \in 2 \circledast 2$ and $\{0\}$ is not a distributive hyper GR-ideal of H.

Example 3.3. Consider the hyper GR-algebra $H = \mathbb{Z}$ such that $x \circledast y = \{0, x, y\}$ for $x, y \in H$. Clearly, $x \in x \circledast y$, for all $x, y \in H$. Let $a, b, c \in H$ such that $[(a \circledast c) \circledast c] \circledast (b \circledast c) \subseteq \{0, x, y\}$ and $b \in \{0, x, y\}$. Since $a \in [(a \circledast c) \circledast c] \circledast (b \circledast c)$, $a \in \{0, x, y\}$. Hence, $\{0, x, y\}$ is a distributive hyper GR-ideal of H.

The next theorem gives a generalization when $x \in x \circledast y$, for all $x, y \in H$.

Theorem 3.2. *Let H be a hyper GR-algebra, $A \subset H$ and $0 \in A$. If $x \in x \circledast y$, for all $x, y \in H$, then A is both hyper GR-ideal and distributive hyper GR-ideal of H.*

Proof of Theorem 3.2. Let $x, y \in H$ such that $x \circledast y \subseteq A$ and $y \in A$. By the hypothesis, $x \in x \circledast y \subseteq A$. Hence, A is a hyper GR-ideal of H. Moreover, let $x, y, z \in H$ such that $[(x \circledast z) \circledast z] \circledast (y \circledast z) \subseteq A$ and $y \in A$. By the hypothesis, $x \in x \circledast z$ and so $x \in (x \circledast z) \circledast z$. Since$[(x \circledast y) \circledast z] \circledast (y \circledast z) \subseteq A$, $x \circledast v \subseteq A$ for $v \in y \circledast z$. By the hypothesis $x \in x \circledast v$ and so $x \in A$. Hence, A is a distributive hyper GR-ideal of H.

By Theorems 2.1 and 3.2, we obtained a theorem that makes a hyper GR-ideal both a distributive hyper GR-ideal and an implicative hyper GR-ideal of a hyper GR-algebra.

Theorem 3.3. *If $x \in x \circledast y$ for any x and y in a hyper GR-algebra H, then a subset A containing 0 of H is a hyper GR-ideal, distributive hyper GR-ideal and implicative hyper GR-ideal of H.*

Example 3.4. Consider the hyper GR-algebra $H = \{0, 1, 2, 3\}$ with the Cayley table below.

$\circledast$	0	1	2	3
0	$\{0,1,3\}$	$\{0,1,3\}$	$\{0,1,3\}$	$\{0,1,3\}$
1	$\{0,1,3\}$	$\{0,1,3\}$	$\{0,1,3\}$	$\{0,1,3\}$
2	$\{0,1,3\}$	$\{0,1\}$	$\{0,1\}$	$\{3\}$
3	$\{0,1,3\}$	$\{0,1,3\}$	$\{0,1,3\}$	$\{0\}$

We can see that H, $\{0\}, \{0, 1, 2\}$ and $\{0, 2\}$ are the only hyper GR-ideals of H. Note that $[(x \circledast z) \circledast z] \circledast (y \circledast z) = \{0, 1, 3\}$, for all $x, y, z \in H$. This implies that all subsets of H containing 0 which are not equal to $\{0, 1, 3\}$ are distributive hyper GR-ideals of H. Particularly, $\{0, 1\}$ is a distributive hyper GR-ideal of H but not a hyper GR-ideal of H because $2 \circledast 1 = \{0, 1\}$, $1 \in \{0, 1\}$ and $2 \notin \{0, 1\}$.

Remark 3.2. Example 3.4 shows that not every distributive hyper GR-ideal is a hyper GR-ideal.

Theorem 3.4. *A distributive hyper GR-ideal A such that $A \circledast A \subseteq A$ is a hyper GR-ideal.*

Proof of Theorem 3.4. Let $x \circledast y \subseteq A$ where $y \in A$. Since $A \circledast A \subseteq A, (x \circledast y) \circledast y \subseteq A$ and $y \circledast y \subseteq A$. It implies that $[(x \circledast y) \circledast y)] \circledast (y \circledast y) \subseteq A$. Since A is a distributive hyper GR-ideal, $x \in A$. Thus, A is a hyper GR-ideal.

Theorem 3.5. *Let H be a hyper GR-algebra such that $[(x \circledast 0) \circledast (y \circledast 0)] \circledast 0 = x \circledast y$, for all $x, y \in H$. Then a distributive hyper GR-ideal A is a hyper GR-ideal.*

Proof of Theorem 3.5. Let A be a distributive hyper GR-ideal of H. Let $a, b \in H$ such that $a \circledast b \subseteq A$ and $b \in A$. Then, by HGR2 and the hypothesis, we have

$$[(a \circledast 0) \circledast 0] \circledast (b \circledast 0) = [(a \circledast 0) \circledast (b \circledast 0)] \circledast 0 = a \circledast b \subseteq A.$$

Since A is a distributive hyper GR-ideal of H, we have $a \in A$. Hence, A is a hyper GR-ideal of H.

4. Fuzzy hyper GR-ideal of type 2

Definition 4.1. *A fuzzy set μ in a hyper GR-algebra H is a fuzzy hyper GR-ideal of type 2 if for all $x, y \in H$*

- **(FF1)** *$x \ll y$ implies $\mu(y) \leq \mu(x)$; and*
- **(FF2)** $\mu(x) \geq \min\{\inf_{u \in x \circledast y} \mu(u), \mu(y)\}$.

Example 4.1. Consider the hyper GR-algebra H in Example 2.1. Define a fuzzy set μ in H by

$$\mu(a) = \begin{cases} l, & \text{if } a = 0 \\ k, & \text{if } a \neq 0 \end{cases}$$

where $k, l \in [0, 1]$ and $k < l$. By routine calculations we see that μ is a fuzzy hyper GR-ideal of type 2 in H.

Remark 4.1. By FF1, $x \ll y$ and $y \ll x$ imply $\mu(x) = \mu(y)$.

Example 4.2. Let $H = \{0, 1, 2, 3\}$ with the Cayley table below.

$\circledast$	0	1	2	3
0	{0,1,2}	{0,1,2}	{0,1,2}	{0,1,2}
1	{0,1}	{0,1,2}	{0,1,2}	{0,1,2}
2	{0,2}	{0,1,2}	{0,1,2}	{0,1,2}
3	{0,1,3}	{1,2,3}	{1,2,3}	{0,1,2}

It can be shown that H is a hyper GR-algebra. Define a fuzzy set μ in H by $\mu(1) = 0.75$, $\mu(2) = 0.6$, $\mu(3) = 0.5$, and $\mu(0) = 0.8$. By routine calculations, we can see that μ is a fuzzy hyper GR-ideal of type 1 in H. On the contrary, μ is not a fuzzy hyper GR-ideal of type 2 in H since $2 \ll 1$ but $\mu(2) = 0.6 < 0.75 = \mu(1)$.

Remark 4.2. Example 4.2 shows that not all fuzzy hyper GR-ideal of type 1 are fuzzy hyper GR-ideal of type 2.

In Example 3.1, we see that $2 \ll x$ and $x \ll 2$, for all $x \in H$. So, if a fuzzy set μ on H satisfies FF1, then $\mu(x) = \mu(2)$, for all $x \in H$. Hence, μ is constant. The next theorem is a generalization for this case.

Theorem 4.1. *Let H be a hyper GR-algebra and μ a fuzzy set such that $x \ll y$ implies $\mu(y) \leq \mu(x)$. If there exists $\hat{x} \in H$ such that $\hat{x} \ll x$ and $x \ll \hat{x}$, for all $x \in H$, then the fuzzy set μ is constant.*

Proof of Theorem 4.1. Let $x \in H$ and μ a fuzzy set in H such that $x \ll y$ implies $\mu(y) \leq \mu(x)$. Then by the hypothesis, $\mu(x) \leq \mu(\hat{x})$ and $\mu(\hat{x}) \leq \mu(x)$. It follows that $\mu(x) = \mu(\hat{x})$. Since x is an arbitrary element in H, μ is constant.

Theorem 4.2. *Let H be a hyper GR-algebra such that $0 \ll x$, for all $x \in H$. Then a fuzzy set μ is a fuzzy hyper GR-ideal of type 2 in H if and only if μ_d is a hyper GR-ideal of H whenever $\mu_d \neq \emptyset$, $d = \min\{\inf_{u \in x \circledast y} \mu(u), \mu(y)\}$ for $x, y \in H$ and for $x \neq y$ such that $x \ll y$ implies $d = \mu(y)$.*

Proof of Theorem 4.2. Let μ be a fuzzy hyper GR-ideal of type 2 in H and $d = \min\{\inf_{u \in x \circledast y} \mu(u), \mu(y)\}$ for $x, y \in H$. Since $0 \ll x$, for all $x \in H$, $\mu(0) \geq \mu(x)$, for all $x \in H$. This implies that $\mu(0) \geq \mu(u)$, for all $u \in x \circledast y$. Then $\mu(0) \geq \inf_{u \in 0 \circledast y} \mu(u) \geq \min\{\inf_{u \in x \circledast y} \mu(u), \mu(y)\} = d$. It follows that $0 \in \mu_d$. Let $x', y' \in H$ such that $x' \circledast y' \subseteq \mu_d$ and $y' \in \mu_d$. Then, for any $v \in x' \circledast y'$, $d \leq \mu(v)$. Since μ is a fuzzy hyper GR-ideal of type 2, $\mu(x') \geq \min\{\inf_{v \in x' \circledast y'} \mu(v), \mu(y')\}$. Note that $\mu(v) \geq d$ and $\mu(y') \geq d$. This implies that $\mu(x') \geq \min\{\inf_{v \in x' \circledast y'} \mu(v), \mu(y')\} \geq d$. Thus, $x' \in \mu_d$ and so μ_d is a hyper GR-ideal of H.

Conversely, let $x, y \in H$ and $d = \min\{\inf_{u \in x \circledast y} \mu(u), \mu(y)\}$. Since $\mu(y) \geq \min\{\inf_{u \in x \circledast y} \mu(u), \mu(y)\} = d$, $y \in \mu_d$. Let $w \in x \circledast y$. Then we have $\mu(w) \geq \inf_{u \in x \circledast y} \mu(u) \geq \min\{\inf_{u \in x \circledast y} \mu(u), \mu(y)\} = d$. It implies that $w \in \mu_d$ and so $x \circledast y \subseteq \mu_d$. Since μ_d is a hyper GR-ideal, $x \in \mu_d$. Then, $\mu(x) \geq d = \min\{\inf_{u \in x \circledast y} \mu(u), \mu(y)\}$. Moreover, suppose that $x \ll y$ where $x \neq y$. By hypothesis, $d = \mu(y)$. Since $x \in \mu_d$, $\mu(x) \geq d = \mu(y)$. Suppose $x = y$. Clearly, $\mu(x) = \mu(y)$. Therefore, μ is a fuzzy hyper GR-ideal of type 2 in H.

Theorem 4.3. *Let μ be a fuzzy hyper GR-ideal of type 2 in H. If $x \circledast y \ll z$, then $\mu(x) \geq \min\{\mu(z), \mu(y)\}$.*

Proof of Theorem 4.3. Let $x, y, z \in H$ such that $x \circledast y \ll z$. Then by FF1, $\mu(z) \leq \mu(u)$, for all $u \in x \circledast y$. Since μ is a fuzzy hyper GR-ideal of type 2 in H,

$$\mu(x) \geq \min\left\{\inf_{u \in x \circledast y} \mu(u), \mu(y)\right\} \geq \min\{\mu(z), \mu(y)\}.$$

5. Fuzzy distributive hyper GR-ideal of type 2

Definition 5.1. *A fuzzy set μ in a hyper GR-algebra H is a fuzzy distributive hyper GR-ideal of type 2 if it satisfies FF1 and for all $x, y, z \in H$,*

$$\text{(FFD) } \mu(x) \geq \min\left\{\inf_{v\in[(x\circledast z)\circledast z]\circledast(y\circledast z)} \mu(v), \mu(y)\right\}.$$

Example 5.1. The fuzzy set μ in Example 2.1 is also a fuzzy distributive hyper GR-ideal of type 2.

Example 5.2. Let $H = \langle a_n \rangle$ be a sequence of numbers where $n \in \mathbb{N} \cup \{0\}$ and $a_0 = 0$ such that

$$a_n \circledast a_m = \begin{cases} \{0, a_1\}, & \text{if } m \neq 0 \text{ or } n = 0 = m, \\ \{a_n\}, & \text{if } n \neq 0 \text{ and } m = 0. \end{cases}$$

It can be verified that H is a hyper GR-algebra. Define a fuzzy set μ in H by $\mu(0) = l$ and $\mu(a_n) = k$ $(n \neq 0)$ where $k, l \in [0, 1]$ and $k < l$. By routine calculations, μ is both a fuzzy hyper GR-ideal of type 2 and a fuzzy distributive hyper GR-ideal of type 2.

Theorem 5.1. *Let H be a hyper GR-algebra such that $[(x \circledast 0) \circledast (y \circledast 0)] \circledast 0 \subseteq x \circledast y$, for all $x, y \in H$. If μ is a fuzzy distributive hyper GR-ideal of type 2, then μ is a fuzzy hyper GR-ideal of type 2.*

Proof of Theorem 5.1. Let $x \in H$. By Definition 5.1, HGR2 and infimum property,

$$\begin{aligned} \mu(x) &\geq \min\left\{\inf_{u\in[(x\circledast 0)\circledast 0]\circledast(y\circledast 0)} \mu(u), \mu(y)\right\} \\ &= \min\left\{\inf_{u\in[(x\circledast 0)\circledast(y\circledast 0)]\circledast 0} \mu(u), \mu(y)\right\} \\ &\geq \min\left\{\inf_{u\in x\circledast y} \mu(u), \mu(y)\right\}. \end{aligned}$$

Hence, μ is a fuzzy hyper GR-ideal of type 2.

Theorem 5.2. *Let H be a hyper GR-algebra such that $0 \ll x$, for all $x \in H$ and $[(x \circledast 0) \circledast (y \circledast 0)] \circledast 0 = x \circledast y$, for all $x, y \in H$. Then a fuzzy set μ of a hyper GR-algebra H is a fuzzy distributive hyper GR-ideal of type 2 in H if and only if for all $x, y \in H$, μ_d is a distributive hyper GR-ideal of H whenever $\mu_d \neq \emptyset$, $d = \min\{\inf_{u\in x\circledast y} \mu(u), \mu(y)\}$ and for $x \neq y$ such that $x \ll y$ implies $d = \mu(y)$.*

Proof of Theorem 5.2. Let $d = \min\{\inf_{u\in x\circledast y} \mu(u), \mu(y)\}$ for $x, y \in H$. Suppose μ is a fuzzy distributive hyper GR-ideal of type 2 in H, $\mu_d \neq \emptyset$ and $[(x\circledast 0)\circledast(y\circledast 0)]\circledast 0 = x\circledast y$. By Theorem 5.1, μ is a fuzzy hyper GR-ideal of type 2. It follows from Theorem 4.2 that μ_d is a hyper GR-ideal of H. Hence, $0 \in \mu_d$.

Moreover, let $x, y, z \in H$ such that $[(x \circledast z) \circledast z] \circledast (y \circledast z) \subseteq \mu_d$ and $y \in \mu_d$. For any $u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)$, $u \in \mu_d$. Thus, $d \leq \mu(u)$. It implies that d is a lower-bound for $\{\mu(u) : u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)\}$ and hence $\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} \mu(u) \geq d$. Since μ is a fuzzy distributive hyper GR-ideal of type 2 in H, for all $x, y, z \in H$ we have $\mu(x) \geq \min\left\{\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} \mu(u), \mu(y)\right\} \geq \min\{d, d\} = d$. Thus, $x \in \mu_d$ and so μ_d is a distributive hyper GR-ideal of H.

Conversely, for all $x, y \in H$ such that $d = \min\{\inf_{u \in x \circledast y} \mu(u), \mu(y)\}$, μ_d is a distributive hyper GR-ideal of H, whenever $\mu_d \neq \emptyset$ and $[(x \circledast 0) \circledast (y \circledast 0)] \circledast 0 = x \circledast y$. By Theorem 3.5, μ_d is a hyper GR-ideal of H. Hence, from Theorem 4.2, μ is a fuzzy hyper GR-ideal of type 2 in H. Hence, FF1 holds. Let $x, y, z \in H$ and let $l \in [0, 1]$ such that $l = \min\left\{\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} \mu(u), \mu(y)\right\}$. Let $v \in [(x \circledast z) \circledast z] \circledast (y \circledast z)$. Then, $\mu(v) \geq \inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} \mu(u) \geq \min\{\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} \mu(u), \mu(y)\} = l$. It implies that $v \in \mu_l$ and hence $[(x \circledast z) \circledast z] \circledast (y \circledast z) \subseteq \mu_l$. Note that $\mu(y) \geq \min\left\{\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} \mu(u), \mu(y)\right\} = l$. Then, $y \in \mu_l$. Since μ_l is a distributive hyper GR-ideal of H, $x \in \mu_l$. It follows that $\mu(x) \geq l = \min\left\{\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} \mu(u), \mu(y)\right\}$. Therefore, μ is a fuzzy distributive hyper GR-ideal of type 2.

Corollary 5.1. *Let H be a hyper GR-algebra such that $0 \ll x$, for all $x \in H$ and $[(x \circledast 0) \circledast (y \circledast 0)] \circledast 0 = x \circledast y$, for all $x, y \in H$. For any subset A of H, let μ_A be a fuzzy set in H defined by*

$$\mu_A(x) = \begin{cases} d, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases},$$

for all $x \in H$ where $d = \min\{\inf_{u \in x \circledast y} \mu(u), \mu(y)\}$ and $x \ll y$ implies $d = \mu(y)$. Then A is a distributive hyper GR-ideal if and only if μ_A is a fuzzy distributive hyper GR-ideal of type 2 in H.

Proof of Corollary 5.1. Let $d = \min\{\inf_{u \in x \circledast y} \mu(u), \mu(y)\}$ for $x \in H$. Note that

$$(\mu_A)_t = \begin{cases} \emptyset, & \text{if } d < t \leq 1, \\ A, & \text{if } 0 < t \leq d \\ H, & \text{if } t \geq 0 \end{cases} \tag{1}$$

are all possible level subsets of μ_A. Let $t = d$. It follows from Equation 1 that $(\mu_A)_d = A$. By Theorem 5.2, A is a distributive hyper GR-ideal if and only if μ_A is a fuzzy distributive hyper GR-ideal of type 2 in H.

Lemma 5.1. *Let H be a hyper GR-algebra, $I \subseteq \mathbb{N}$, and for $x, y, z \in H$,*

$$A = \sup_{i \in I} \left\{\min\left\{\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} \mu_i(u), \mu_i(y)\right\}\right\},$$

$$B = \min\left\{\sup_{i \in I} \left(\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} \mu_i(u)\right), \sup_{i \in I} \mu_i(y)\right\},$$

$$C = \min\left\{\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} (\sup_{i \in I} \mu_i(u)), \sup_{i \in I} \mu_i(y)\right\},$$

$D = \inf_{i\in I}\left\{\min\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u),\mu_i(y)\right\}\right\}$,

$E = \min\left\{\inf_{i\in I}\left(\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u)\right),\inf_{i\in I}\mu_i(y)\right\}$, *and*

$F = \min\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}(\inf_{i\in I}\mu_i(u)),\inf_{i\in I}\mu_i(y)\right\}$

where μ_i *is a fuzzy distributive hyper GR-ideal of type* 2*, for all* $i\in I$*. Then, for* $x,y,z\in H$*,* (i) $A\geq B$*;* (ii) $B\geq C$*;* (iii) $D\geq E$*;* (iv) $E\geq F$*; and* (v) $\mu_i(y)\leq\mu_i(x)$ *for* $i\in I$ *and* $x,y\in H$ *imply* $\sup_{i\in I}\mu_i(y)\leq\sup_{i\in I}\mu_i(x)$ *and* $\inf_{i\in I}\mu_i(y)\leq\inf_{i\in I}\mu_i(x)$

Proof of Lemma 5.1. (i) Case 1: Suppose
$\min\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u),\mu_i(y)\right\}=\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u)$. Then,

$$\begin{aligned}\sup_{i\in I}\left\{\min\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u),\mu_i(y)\right\}\right\}&=\sup_{i\in I}\left(\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u)\right)\\&\geq\min\left\{\sup_{i\in I}\left(\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u)\right),\sup_{i\in I}\mu_i(y)\right\}.\end{aligned}$$

Case 2: Suppose $\min\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u),\mu_i(y)\right\}=\mu_i(y)$. Then,

$$\begin{aligned}\sup_{i\in I}\left\{\min\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u),\mu_i(y)\right\}\right\}&=\sup_{i\in I}\mu_i(y)\\&\geq\min\left\{\sup_{i\in I}\left(\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u)\right),\sup_{i\in I}\mu_i(y)\right\}.\end{aligned}$$

So, $A\geq B$.
(ii) For $u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)$, let $\mu_1(u),\mu_2(u)\in\{\mu_i(u):i\in I\}$. Then

$$\begin{aligned}\mu_1(u)\wedge\mu_2(u)&=\inf\{\mu_1(u),\mu_2(u)\}\\&=\min\{\mu_1(u),\mu_2(u)\}\in\{\mu_i(u):i\in I\}\end{aligned}$$

and

$$\begin{aligned}\mu_1(u)\vee\mu_2(u)&=\sup\{\mu_1(u),\mu_2(u)\}\\&=\max\{\mu_1(u),\mu_2(u)\}\in\{\mu_i(u):i\in I\}.\end{aligned}$$

Hence, $\{\mu_i(u):i\in I\}$ for $u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)$ is a sublattice of [0,1]. Since $\{\mu_i(u):i\in I\}\subseteq[0,1]$, $\{\mu_i(u):i\in I\}$ is of finite length. By Remark 2.1, $\{\mu_i(u):i\in I\}$ is complete and so $\sup_{i\in I}\mu_i(u)$ and $\inf_{i\in I}\mu_i(u)$ are in $\{\mu_i(u):i\in I\}$. This implies that $\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}(\sup_{i\in I}\mu_i(u))\in\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u):i\in I\right\}$. Then $\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}(\sup_{i\in I}\mu_i(u))\leq\sup_{i\in I}\left(\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u)\right)$. Hence, $B\geq C$.

(iii) Case 1: Suppose
$\min\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u),\mu_i(y)\right\}=\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u)$. Then,

$$\inf_{i\in I}\left\{\min\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u),\mu_i(y)\right\}\right\}=\inf_{i\in I}\left(\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u)\right)$$

$$\geq \min\left\{\inf_{i\in I}\left(\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u)\right), \inf_{i\in I}\mu_i(y)\right\}.$$

Case 2: Suppose $\min\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u), \mu_i(y)\right\} = \mu_i(y)$. Then,

$$\inf_{i\in I}\left\{\min\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u), \mu_i(y)\right\}\right\} = \inf_{i\in I}\mu_i(y)$$
$$\geq \min\left\{\inf_{i\in I}\left(\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u)\right), \inf_{i\in I}\mu_i(y)\right\}.$$

Thus, $D \geq E$.
(iv) Since $\mu_i(u) \geq \inf_{i\in I}\mu_i(u)$ for each $i \in I$,

$$\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u) \geq \inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\inf_{i\in I}\mu_i(u).$$

Hence, $E \geq F$.
(v) Since $\mu_i(y) \leq \mu_i(x)$ for $i \in I$ and $x, y \in H$,

$$\begin{aligned}
sup_{i\in I}\mu_i(y) &= \sup\{\mu_1(y), \mu_2(y), ..., \mu_n(y), ...\}\\
&\leq \sup\{\mu_1(x), \mu_2(y), ..., \mu_n(y), ...\}\\
&\leq \sup\{\mu_1(x), \mu_2(x), ..., \mu_n(y), ...\}\\
&\vdots \qquad\qquad \vdots\\
&\leq \sup\{\mu_1(x), \mu_2(x), ..., \mu_n(x), ...\} = \sup_{i\in I}\mu_i(x) \text{ and}\\
\inf_{i\in I}\mu_i(y) &= \inf\{\mu_1(y), \mu_2(y), ..., \mu_n(y), ...\}\\
&\leq \inf\{\mu_1(x), \mu_2(y), ..., \mu_n(y), ...\}\\
&\leq \inf\{\mu_1(x), \mu_2(x), ..., \mu_n(y), ...\}\\
&\vdots \qquad\qquad \vdots\\
&\leq \inf\{\mu_1(x), \mu_2(x), ..., \mu_n(x), ...\} = \inf_{i\in I}\mu_i(x).
\end{aligned}$$

Theorem 5.3. *The family M of fuzzy distributive hyper GR-ideals of type 2 of a hyper GR-algebra H is a complete distributive lattice with respect to join and meet.*

Proof of Theorem 5.3. Let M be a family of fuzzy distributive hyper GR-ideals of type 2 in H and $\mu_1, \mu_2, \mu_3 \in M$. Let a binary relation $\leq$ be defined by $\mu \leq \mu'$ if and only if $\mu(x) \leq \mu'(x)$, for all $x \in H$ and $\mu, \mu' \in M$. Clearly, $\mu_1 = \mu_1$. Suppose $\mu_1 \leq \mu_2$ and $\mu_2 \leq \mu_1$. Then $\mu_1(x) \leq \mu_2(x)$ and $\mu_2(x) \leq \mu_1(x)$, for all $x \in H$. Since $\mu_1(x), \mu_2(x) \in [0,1]$ for all $x \in H$, $\mu_1(x) = \mu_2(x)$, for all $x \in H$ and hence $\mu_1 = \mu_2$. Let $\mu_1 \leq \mu_2$ and $\mu_2 \leq \mu_3$. Then $\mu_1(x) \leq \mu_2(x)$ and $\mu_2(x) \leq \mu_3(x)$, for all $x \in H$. Since $\mu_1(x), \mu_2(x), \mu_3(x) \in [0,1]$, for all $x \in H$, $\mu_1(x) \leq \mu_3(x)$, for all $x \in H$ and so $\mu_1 \leq \mu_3$. Since $\mu_1(x), \mu_2(x) \in [0,1]$, for all

$x \in H$, either $\mu_1(x) \leq \mu_2(x)$ or $\mu_2(x) \leq \mu_1(x)$, for all $x \in H$. That is, either $\mu_1 \leq \mu_2$ or $\mu_2 \leq \mu_1$. Thus, M is a totally ordered set.

Moreover, let $\mu_1, \mu_2 \in M$ and let $x, y \in H$ such that $x \ll y$. Note that $(\mu_1 \vee \mu_2)(x) = sup\{\mu_1(x), \mu_2(x)\} = \max\{\mu_1(x), \mu_2(x)\}$ and $(\mu_1 \wedge \mu_2)(x) = inf\{\mu_1(x), \mu_2(x)\} = \min\{\mu_1(x), \mu_2(x)\}$ for any $x \in H$. Since μ_1 and μ_2 are fuzzy distributive hyper GR-ideals of type 2 in H, $\mu_1(y) \leq \mu_1(x)$ and $\mu_2(y) \leq \mu_2(x)$. Then $\mu_1(y) \leq \mu_1(x) \leq \max\{\mu_1(x), \mu_2(x)\}$, $\mu_2(y) \leq \mu_2(x) \leq \max\{\mu_1(x), \mu_2(x)\}$, $\mu_1(x) \geq \mu_1(y) \geq \min\{\mu_1(y), \mu_2(y)\}$, and $\mu_2(x) \geq \mu_2(y) \geq \min\{\mu_1(y), \mu_2(y)\}$. It follows that

$$(\mu_1 \vee \mu_2)(y) = \max\{\mu_1(y), \mu_2(y)\} \leq \max\{\mu_1(x), \mu_2(x)\} = (\mu_1 \vee \mu_2)(x) \text{ and}$$
$$(\mu_1 \wedge \mu_2)(y) = \min\{\mu_1(y), \mu_2(y)\} \leq \min\{\mu_1(x), \mu_2(x)\} = (\mu_1 \wedge \mu_2)(x).$$

Let $x, y, z \in H$ and $I = \{1, 2\}$. Since μ_1, μ_2 are fuzzy distributive hyper GR-ideals of type 2 in H,

$$\mu_1(x) \geq \min\left\{\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} \mu_1(u), \mu_1(y)\right\} \text{ and}$$
$$\mu_2(x) \geq \min\left\{\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} \mu_2(u), \mu_2(y)\right\}.$$

Thus

$$(\mu_1 \vee \mu_2)(x) = \max_{i \in I} \mu_i(x) \geq \max_{i \in I}\left\{\min\left\{\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} \mu_i(u), \mu_i(y)\right\}\right\}.$$

By Lemma 5.1(i) and (ii), we have

$$\begin{aligned}(\mu_1 \vee \mu_2)(x) = \max_{i \in I} \mu_i(x) &\geq \max_{i \in I}\left\{\min\left\{\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} \mu_i(u), \mu_i(y)\right\}\right\} \\ &\geq \min\left\{\max_{i \in I}\left(\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} \mu_i(u)\right), \max_{i \in I} \mu_i(y)\right\} \\ &\geq \min\left\{\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} (\max_{i \in I} \mu_i(u)), \max_{i \in I} \mu_i(y)\right\} \\ &= \min\left\{\left\{\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} (\mu_1 \vee \mu_2)(u), (\mu_1 \vee \mu_2)(y)\right\}\right\}.\end{aligned}$$

Also, by Lemma 5.1(iii) and (iv), we have

$$\begin{aligned}(\mu_1 \wedge \mu_2)(x) = \min_{i \in I} \mu_i(x) &\geq \min_{i \in I}\left\{\min\left\{\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} \mu_i(u), \mu_i(y)\right\}\right\} \\ &\geq \min\left\{\min_{i \in I}\left(\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} \mu_i(u)\right), \min_{i \in I} \mu_i(y)\right\} \\ &\geq \min\left\{\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} \left(\min_{i \in I} \mu_i(u)\right), \min_{i \in I} \mu_i(y)\right\} \\ &= \min\left\{\left\{\inf_{u \in [(x \circledast z) \circledast z] \circledast (y \circledast z)} (\mu_1 \wedge \mu_2)(u), (\mu_1 \wedge \mu_2)(y)\right\}\right\}.\end{aligned}$$

It implies that $\mu_1 \vee \mu_2$ and $\mu_1 \wedge \mu_2$ are fuzzy distributive hyper GR-ideals of type 2 in H. Hence, $\mu_1 \vee \mu_2$ and $\mu_1 \wedge \mu_2$ exist in M and so M is a lattice. Since M is totally ordered set, M is distributive lattice by Lemma 2.2.

Furthermore, let $R = \{\mu_i : i \in I\}$ be a subset of M and let $x, y \in H$ such that $x \ll y$. Since μ_i is a fuzzy distributive hyper GR-ideal of type 2 in H, for all $i \in I$, $\mu_i(y) \leq \mu_i(x)$. By Lemma 5.1(v),

$$\begin{aligned}(\vee_{i\in I}\mu_i)(y) &= \sup_{i\in I}\mu_i(y) \leq \sup_{i\in I}\mu_i(x) = (\vee_{i\in I}\mu_i)(x) \text{ and}\\ (\wedge_{i\in I}\mu_i)(y) &= \inf_{i\in I}\mu_i(y) \leq \inf_{i\in I}\mu_i(x) = (\wedge_{i\in I}\mu_i)(x).\end{aligned}$$

Let $x, y, z \in H$. Since μ_i is a fuzzy distributive hyper GR-ideal of type 2 in H, for all $i \in I$, $\mu_i(x) \geq \min\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)} \mu_i(u), \mu_i(y)\right\}$.

By Lemma 5.1(i) and (ii), we get

$$\begin{aligned}(\vee_{i\in I}\mu_i)(x) = \sup_{i\in I}\mu_i(x) &\geq \sup_{i\in I}\left\{\min\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u), \mu_i(y)\right\}\right\}\\ &\geq \min\left\{\sup_{i\in I}\left(\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u)\right), \sup_{i\in I}\mu_i(y)\right\}\\ &\geq \min\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\left(\sup_{i\in I}\mu_i(u)\right), \sup_{i\in I}\mu_i(y)\right\}\\ &= \min\left\{\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}(\vee_{i\in I}\mu_i)(u), (\vee_{i\in I}\mu_i)(y)\right\}\right\}.\end{aligned}$$

Also, by Lemma 5.1(iii) and (iv), we have

$$\begin{aligned}(\wedge_{i\in I}\mu_i)(x) = \inf_{i\in I}\mu_i(x) &\geq \inf_{i\in I}\left\{\min\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u), \mu_i(y)\right\}\right\}\\ &\geq \min\left\{\inf_{i\in I}\left(\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\mu_i(u)\right), \inf_{i\in I}\mu_i(y)\right\}\\ &\geq \min\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}\left(\inf_{i\in I}\mu_i(u)\right), \inf_{i\in I}\mu_i(y)\right\}\\ &= \min\left\{\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)}(\wedge_{i\in I}\mu_i)(u), (\wedge_{i\in I}\mu_i)(y)\right\}\right\}.\end{aligned}$$

Hence, $\vee_{i\in I}\mu_i$ and $\wedge_{i\in I}\mu_i$ are fuzzy distributive hyper GR-ideals of type 2 in H. Thus, $\vee_{i\in I}\mu_i$ and $\wedge_{i\in I}\mu_i$ exist in M. Therefore, M is a complete distributive lattice.

Theorem 5.4. *Let μ_1 and μ_2 be fuzzy distributive hyper GR-ideals of type 2 in H. Then $\mu_1 \times \mu_2$ is a fuzzy distributive hyper GR-ideal of type 2 in $H \times H$.*

Proof of Theorem 5.4. Let $(x_1, y_1), (x_2, y_2) \in H \times H$ such that $(x_1, y_1) \ll (x_2, y_2)$. Then $x_1 \ll x_2$ and $y_1 \ll y_2$. Since μ_1 and μ_2 are fuzzy distributive hyper GR-ideals of type 2 in H, $\mu_1(x_2) \ll \mu_1(x_1)$ and $\mu_2(y_2) \ll \mu_2(y_1)$. Now,

$$\begin{aligned}(\mu_1 \times \mu_2)(x_2, y_2) &= \min\{\mu_1(x_2), \mu_2(y_2)\} \\ &\leq \min\{\mu_1(x_1), \mu_2(y_1)\} = (\mu_1 \times \mu_2)(x_1, y_1).\end{aligned}$$

Moreover, let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in H \times H$. Since μ_1 and μ_2 are fuzzy distributive hyper GR-ideals of type 2 in H,
$\mu_1(x_1) \geq \min\left\{\inf_{u\in[(x_1\circledast x_3)\circledast x_3]\circledast(x_2\circledast x_3)} \mu_1(u), \mu_1(x_2)\right\}$
and $\mu_2(y_1) \geq \min\left\{\inf_{v\in[(y_1\circledast y_3)\circledast y_3]\circledast(y_2\circledast y_3)} \mu_2(v), \mu_2(y_2)\right\}$. Then

$$\begin{aligned}(\mu_1 \times \mu_2)(x_1,y_1) &= \min\{\mu_1(x_1), \mu_2(y_1)\} \\ &\geq \min\left\{\min\left\{\inf_{u\in[(x_1\circledast x_3)\circledast x_3]\circledast(x_2\circledast x_3)} \mu_1(u), \mu_1(x_2)\right\},\right. \\ &\qquad \left.\min\left\{\inf_{v\in[(y_1\circledast y_3)\circledast y_3]\circledast(y_2\circledast y_3)} \mu_2(v), \mu_2(y_2)\right\}\right\} \\ &= \min\left\{\min\left\{\inf_{u\in[(x_1\circledast x_3)\circledast x_3]\circledast(x_2\circledast x_3)} \mu_1(u), \inf_{v\in[(y_1\circledast y_3)\circledast y_3]\circledast(y_2\circledast y_3)} \mu_2(v)\right\},\right. \\ &\qquad \left.\min\{\mu_1(x_2), \mu_2(y_2)\}\right\}.\end{aligned}$$

It follows that

$$\begin{aligned}(\mu_1 \times \mu_2)(x_1, y_1) &= \min\{\min\{\inf\{\mu_1(u), \mu_2(v)\}\}, (\mu_1 \times \mu_2)(x_2, y_2)\} \\ &= \min\{\inf\{\min\{\mu_1(u), \mu_2(v)\}\}, (\mu_1 \times \mu_2)(x_2, y_2)\} \\ &= \min\{\inf\{(\mu_1 \times \mu_2)(u, v)\}, (\mu_1 \times \mu_2)(x_2, y_2)\},\end{aligned}$$

for $u \in [(x_1 \circledast x_3) \circledast x_3] \circledast (x_2 \circledast x_3)$ and $v \in [(y_1 \circledast y_3) \circledast y_3] \circledast (y_2 \circledast y_3)$. By Definition 2.5 and Lemma 2.1,

$$\begin{aligned}(u, v) &\in \{[(x_1 \circledast x_3) \circledast x_3] \circledast (x_2 \circledast x_3)\} \times \{[(y_1 \circledast y_3) \circledast y_3] \circledast (y_2 \circledast y_3)\} \\ &= ((x_1 \circledast x_3) \circledast x_3, (y_1 \circledast y_3) \circledast y_3) \circledast (x_2 \circledast x_3, y_2 \circledast y_3) \\ &= [(x_1 \circledast x_3, y_1 \circledast y_3) \circledast (x_3, y_3)] \circledast [(x_2, y_2) \circledast (x_3, y_3)] \\ &= \{[(x_1, y_1) \circledast (x_3, y_3)] \circledast (x_3, y_3)\} \circledast [(x_2, y_2) \circledast (x_3, y_3)].\end{aligned}$$

Hence, $(\mu_1\times\mu_2)(x_1, y_1) \geq \min\left\{\inf_{(u,v)\in\Omega}(\mu_1 \times \mu_2)(u, v), (\mu_1 \times \mu_2)(x_2, y_2)\right\}$ where $\Omega = \{[(x_1, y_1) \circledast (x_3, y_3)] \circledast (x_3, y_3)\} \circledast [(x_2, y_2) \circledast (x_3, y_3)]$. Therefore, $\mu_1 \times \mu_2$ is a fuzzy distributive hyper GR-ideal of type 2 in $H \times H$.

Theorem 5.5. *Let μ be a fuzzy set in a hyper GR-algebra H and λ_μ the strongest fuzzy relation on H. Then μ is a fuzzy distributive hyper GR-ideal of type 2 in H if and only if λ_μ is a fuzzy distributive hyper GR-ideal of type 2 in $H \times H$.*

Proof of Theorem 5.5. Let μ be a fuzzy distributive hyper GR-ideal of type 2 in H. Suppose $(x_1, x_2) \ll (y_1, y_2)$. Then $x_1 \ll y_1$ and $x_2 \ll y_2$. Since μ is a fuzzy distributive hyper GR-ideal of type 2 in H, $\mu(y_1) \leq \mu(x_1)$ and $\mu(y_2) \leq \mu(x_2)$. Then $\lambda_\mu(y_1, y_2) = \min\{\mu(y_1), \mu(y_2)\} \leq \min\{\mu(x_1), \mu(x_2)\} = \lambda_\mu(x_1, x_2)$. Let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in H \times H$. Then,

$$\begin{aligned}\lambda_\mu(x_1, x_2) &= \min\{\mu(x_1), \mu(x_2)\} \\ &\geq \min\left\{ \min\left\{ \inf_{u\in[(x_1\circledast z_1)\circledast z_1]\circledast(y_1\circledast z_1)} \mu(u), \mu(y_1)\right\},\right. \\ &\qquad \left. \min\left\{ \inf_{v\in[(x_2\circledast z_2)\circledast z_2]\circledast(y_2\circledast z_2)} \mu(v), \mu(y_2)\right\} \right\} \\ &= \min\left\{ \min\left\{ \inf_{u\in[(x_1\circledast z_1)\circledast z_1]\circledast(y_1\circledast z_1)} \mu(u), \inf_{v\in[(x_2\circledast z_2)\circledast z_2]\circledast(y_2\circledast z_2)} \mu(v)\right\},\right. \\ &\qquad \left. \min\{\mu(y_1), \mu(y_2)\} \right\}.\end{aligned}$$

It follows that

$$\begin{aligned}\lambda_\mu(x_1, x_2) &= \min\{\min\{\inf\{\mu(u), \mu(v)\}\}, \lambda_\mu(y_1, y_2)\} \\ &= \min\{\inf\{\min\{\mu(u), \mu(v)\}\}, \lambda_\mu(y_1, y_2)\} \\ &= \min\{\inf \lambda_\mu(u, v), \lambda_\mu(y_1, y_2)\}\end{aligned}$$

for $u \in [(x_1 \circledast z_1) \circledast z_1] \circledast (y_1 \circledast z_1)$, and $v \in [(x_2 \circledast z_2) \circledast z_2] \circledast (y_2 \circledast z_2)$. By Definition 2.5 and Lemma 2.1,

$$\begin{aligned}(u, v) \in&\{[(x_1 \circledast z_1) \circledast z_1] \circledast (y_1 \circledast z_1)\} \times \{[(x_2 \circledast z_2) \circledast z_2] \circledast (y_2 \circledast z_2)\} \\ =&([(x_1 \circledast z_1) \circledast z_1], [(x_2 \circledast z_2) \circledast z_2]) \circledast (y_1 \circledast z_1, y_2 \circledast z_2) \\ =&[(x_1 \circledast z_1, x_2 \circledast z_2) \circledast (z_1, z_2)] \circledast [(y_1, y_2) \circledast (z_1, z_2)] \\ =&\{[(x_1, x_2) \circledast (z_1, z_2)] \circledast (z_1, z_2)\} \circledast ((y_1, y_2) \circledast (z_1, z_2)).\end{aligned}$$

Thus, $\lambda_\mu(x_1, x_2) \geq \min\left\{\inf_{(u,v)\in\omega} \lambda_\mu(u, v), \lambda_\mu(y_1, y_2)\right\}$ where $\omega = \{[(x_1, x_2) \circledast (z_1, z_2)] \circledast (z_1, z_2)\} \circledast ((y_1, y_2) \circledast (z_1, z_2))$. Therefore, λ_μ is a fuzzy distributive hyper GR-ideal of type 2 in $H \times H$.

Conversely, let λ_μ be a fuzzy distributive hyper GR-ideal of type 2 in $H \times H$. Let $x, y \in H$ such that $x \ll y$. Then $(x, x) \ll (y, y)$. It implies that

$$\mu(x) = \min\{\mu(x), \mu(x)\} = \lambda_\mu(x, x) \geq \lambda_\mu(y, y) = \min\{\mu(y), \mu(y)\} = \mu(y).$$

Moreover, let $x, y, z \in H$. Then

$$\begin{aligned}\min\{\mu(x), \mu(0)\} &= \lambda_\mu(x, 0) \\ &\geq \min\left\{ \inf_{(u,v)\in[((x,0)\circledast(z,0))\circledast(z,0)]\circledast((y,0)\circledast(z,0))} \lambda_\mu(u, v), \lambda_\mu(y, 0)\right\} \\ &= \min\left\{ \inf_{(u,v)\in([(x\circledast z)\circledast z]\circledast(y\circledast z),[(0\circledast 0)\circledast 0]\circledast(0\circledast 0))} \lambda_\mu(u, v), \lambda_\mu(y, 0)\right\}\end{aligned}$$

$$
\begin{aligned}
&= \min\{\inf \lambda_\mu(u,v), \lambda_\mu(y,0)\} \\
&= \min\{\inf\{\min\{\mu(u),\mu(v)\}\}, \min\{\mu(y),\mu(0)\}\} \\
&= \min\{\min\{\inf\{\mu(u),\mu(v)\}\}, \min\{\mu(y),\mu(0)\}\} \\
&= \min\{\min\{\inf \mu(u), inf \mu(v)\}, \min\{\mu(y),\mu(0)\}\} \\
&= \min\left\{ \min\left\{ \inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)} \mu(u), \mu(y)\right\}, \right. \\
&\qquad \left. \min\left\{ \inf_{v\in[(0\circledast 0)\circledast 0]\circledast(0\circledast 0))} \mu(v), \mu(0)\right\} \right\}.
\end{aligned}
$$

Thus, $\mu(x) \geq \min\left\{\inf_{u\in[(x\circledast z)\circledast z]\circledast(y\circledast z)} \mu(u), \mu(y)\right\}$ and so μ is a fuzzy distributive hyper GR-ideal of type 2 in H.

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