Some characterization of rarely ω-continuous functions

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Abstract. The notion of rare continuity was introduced by Popa [15]. In this present paper, we introduce a new class of functions, called rarely ω-continuous function via ω-open sets and we investigate several properties of this function.

Keywords: rare set, ω-open, rarely continuous, rarely ω-continuous.

1. Introduction

Throughout the present paper, (X, τ) or X will denote a topological space and $Cl(A)$ (resp. $Int(A)$) will denote the closure (resp. interior) of a subset A of X. A subset A of X is said to be regular open (resp. regular closed) if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$). We denote by $Ro(X)$ (resp. $Rc(X)$) the family of all regular open (resp. regular closed) sets in a space (X, τ). Let (X, τ) be a space and A a subset of X. A point $x \in X$ is called a condensation point of A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A set A is said to be ω-closed [9] if it contains all its condensation points [8]. The complement of an ω-closed set is said to be ω-open. It is well known that a subset W of a space (X, τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U - W$ is countable. The family of all ω-open sets of a space (X, τ), denoted by τ_ω or $\omega O(X)$, forms a topology on X finer than τ. The ω-closure and ω-interior, that can be defined in the same way as $Cl(A)$ and $Int(A)$, respectively, will be denoted by $Cl_\omega(A)$ and $Int_\omega(A)$, respectively. Recently papers [1–7] have introduced some new classes of ω-open set. A subset A of X is a called a dense set if and only if the intersection of every non-empty

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open set with A is non-empty, that is, $A \cap U \neq \emptyset$ for all non-empty open $U \in \tau$. A subset A of X is called a rare set in X if and only if $Int(A) = \emptyset$ that is, if $X - A$ is a dense set.

Example 1.1. Let $(\mathbb{R}, \tau)$ be the real numbers with the left-ray topology, i.e. $\tau = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$. Let $A = [0, 1]$, since $Int(A) = \emptyset$, then A is a rare set in $\mathbb{R}$.

For a point $x \in X$, the family $\{U : x \in U \text{ and } U \in \tau\}$ is denoted by $\tau(x)$. For a point $x \in X$, the family $\{U : x \in U \text{ and } U \in \tau_\omega\}$ is denoted by $\tau_\omega(x)$. For a point $x \in X$, the family $\{U : x \in U \text{ and } U \in Ro(X)\}$ is denoted by $Ro(x)$.

Popa [15] introduced the notion of rare continuity. Recently papers [4–6] have introduced several properties of this function.

Definition 1.1. *A function $f : (X, \tau) \to (Y, \sigma)$ is called rarely continuous if for each $x \in X$ and each open set W in Y containing $f(x)$, there exist a rare set R_W with $W \cap Cl(R_W) = \emptyset$ and an open set U in X containing x such that $f(U) \subset W \cup R_W$.*

Definition 1.2. *A subset $\mathcal{N}$ of a topological space (X, τ) is called an ω-neighbourhood of a point $x \in X$ if there exists an ω-open set B such that $x \in B \subseteq \mathcal{N}$.*

Definition 1.3 ([10])**.** *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be:*

1. *ω-continuous if $f^{-1}(V) \in \tau_\omega$ for every $V \in \sigma$.*
2. *ω-open if $f(U) \in \sigma_\omega$ for every $U \in \tau_\omega$.*
3. *ω-closed if $f(U) \in (\sigma_\omega)^c$ for every $U \in (\tau_\omega)^c$.*

2. Rarely ω-continuous functions

Definition 2.1. *A function $f : (X, \tau) \to (Y, \sigma)$: is called rarely ω-continuous if for each $x \in X$ and each $W \in \sigma(f(x))$, there exist a rare set R_w with $W \cap Cl(R_w) = \emptyset$ and an ω-open set U containing x such that $f(U) \subset W \cup R_w$.*

Example 2.1. Let $\mathbb{R}$ be the set of real numbers, τ_u be the usual topology and τ_l be the left-ray topology, i.e. $\tau_l = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$. We define $f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, \tau_l)$ be the identity function $f(x) = x$. Let $a \in \mathbb{R}$ then for each open set $W = (-\infty, a + \delta) \in \tau_l$ containing $f(a) = a$ there exists $U = I \cup \{a\} \in \tau_\omega(a)$ where I is the set of irrational numbers. Then, $Int[f(U) \cap (Y - W)] = Int[(I \cup \{a\}) \cap [a + \delta, \infty)] = \emptyset$. Hence, if $a \in \mathbb{R}$ then for each open set W containing $f(a) = a$ there exists $U \in \tau_\omega(a)$ such that $Int[f(U) \cap (Y - W)] = \emptyset$. Hence, f is rarely continuous function and rarely ω-continuous function.

Theorem 2.1. *Let (X, τ) and (Y, σ) be topological spaces. Then, the following statements are equivalent for a function $f : (X, \tau) \to (Y, \sigma)$:*

1. *f is rarely ω-continuous at $x \in X$.*
2. *For each $W \in \sigma(f(x))$, there exists $U \in \tau_\omega(x)$ such that $Int[f(U) \cap (Y - W)] = \emptyset$.*
3. *For each $W \in \sigma(f(x))$, there exists $U \in \tau_\omega(x)$ such that $Int[f(U)] \subset Cl(W)$.*
4. *For each $W \in \sigma(f(x)))$, there exists a rare set R_w with $W \cap Cl(R_w) = \emptyset$ such that $x \in Int_w(f^{-1}(W \cup R_w))$.*
5. *For each $W \in \sigma(f(x))$, there exists a rare set R_w with $Cl(W) \cap R_w = \emptyset$ such that $x \in Int_w(f^{-1}(Cl(W) \cup R_w))$.*
6. *For each $W \in Ro(f(x))$, there exists a rare set R_w with $W \cap Cl(R_w) = \emptyset$ such that $x \in Int_w(f^{-1}(W \cup R_w))$.*

Proof. (1) $\Rightarrow$ (2): Let $W \in \sigma(f(x))$. By $f(x) \in W \subset Int(Cl(W))$ and the fact that $Int(Cl(W)) \in \sigma(f(x))$, then there exist a rare set R_W with $Int(Cl(W)) \cap Cl(R_W) = \emptyset$ and an ω-open set $U \subset X$ containing x such that $f(U) \subset Int(Cl(W)) \cup R_W$. We have $Int[f(U) \cap (Y - W)] = Int[f(U)] \cap Int(Y - W) \subset Int[Int(Cl(W)) \cup R_W] \cap (Y - Cl(W)) \subset (Cl(W) \cup Int(R_W)) \cap (Y - Cl(W)) = \emptyset$. Hence, $Int[f(U) \cap (Y - W)] = \emptyset$.

(2) $\Rightarrow$ (3): Let $W \in \sigma(f(x))$, then by (2), there exists $U \in \tau_\omega(x)$ such that $Int[f(U) \cap (Y - W)] = \emptyset$. We have $Int[f(U) \cap (Y - W)] = Int(f(U)) \cap Int((Y - W)) = Int(f(U)) \cap (Y - Cl(W)) = \emptyset$. Then $Int[f(U)] \subset Cl(W)$.

(3) $\Rightarrow$ (1): Let $W \in \sigma(f(x))$. Then, by (3), there exists $U \in \tau_\omega(x)$ such that $Int[f(U)] \subset Cl(W)$. We have $f(U) = [f(U) - Int(f(U))] \cup Int(f(U)) \subset [f(U) - Int(f(U))] \cup Cl(W) = [f(U) - Int(f(U))] \cup W \cup (Cl(W) - W) = [f(U) - Int(f(U))] \cap (Y - W) \cup W \cup (Cl(W) - W)$. Set $R_1 = [f(U) - Int(f(U))] \cap (Y - W)$ and $R_2 = (Cl(W) - W)$. Then R_1 and R_2 are rare sets. More $R_w = R_1 \cup R_2$ is a rare set such that $Cl(R_w) \cap W = \emptyset$ and $f(U) \subset W \cup R_w$. This shows that f is rarely w-continuous function.

(1) $\Rightarrow$ (4): Suppose that $W \in \sigma(f(x))$. Then, there exists a rare set R_w with $W \cap cl(R_w) = \emptyset$ and a ω-open set U in X containing x, such that $f(U) \subset W \cup R_w$. It follows that $x \in U \subset f^{-1}(W \cup R_w)$. This implies that $x \in Int_\omega(f^{-1}(W \cup R_w))$.

(4) $\Rightarrow$ (5): Suppose that $W \in \sigma(f(x))$. Then, there exists a rare set R_w with $W \cap Cl(R_w) = \emptyset$ such that $x \in Int_w(f^{-1}(W \cup R_w))$. Since $W \cap Cl(R_w) = \emptyset$, $R_w \subset Y - W$, where $Y - W = (Y - Cl(W)) \cup (Cl(W) - W)$. Now, we have $R_w \subset (R_w \cup (Y - cl(W)) \cup (Cl(W) - W))$. Set $R_1 = R_w \cap (Y - Cl(W))$. It follows that R_1 is a rare set with $cl(W) \cap R_1 = \emptyset$. Therefore, $x \in Int_w[f^{-1}(W \cup R_w)] \subset Int_w[f^{-1}(Cl(W) \cup R_1)]$.

(5) $\Rightarrow$ (6): Assume that $W \in Ro(f(x))$. Then, there exists a rare set R_w with $Cl(W) \cap R_w = \emptyset$ such that $x \in Int_w[f^{-1}(Cl(W) \cup R_w)]$. Set $R_1 = R_w \cup (Cl(W) - W)$. It follows that R_1 is a rare set and $W \cap Cl(R_1) = \emptyset$.

Hence, $x \in Int_w[f^{-1}(cl(W) \cup R_w)] = Int_w[f^{-1}(W \cup (Cl(W) - W) \cup R_w)] = Int_w[f^{-1}(W \cup R_1)]$.

(6) $\Rightarrow$ (2): Let $W \in \sigma(f(x))$. By $f(x) \in W \subset Int(Cl(W))$ and the fact that $Int(Cl(W)) \in Ro(f(x))$, there exists a rare set R_w with $Int(Cl(W)) \cap Cl(R_w) = \emptyset$ such that $x \in Int_w[f^{-1}(Int(Cl(W)) \cup R_w)]$. Let

$U = Int_w[f^{-1}(Int(Cl(W)) \cup R_w)]$. Hence, $x \in U \in \tau_\omega(x)$ and, therefore $f(U) \subset Int(Cl(W)) \cup R_w$. We have $Int[f(U) \cap (Y - W)] = Int[f(U)] \cap Int(Y - W) \subset Int[Int(Cl(W)) \cup R_W] \cap (Y - Cl(W)) \subset (Cl(W) \cup Int(R_W)) \cap (Y - Cl(W)) = \emptyset$. □

Theorem 2.2. *Let $f : (X, \tau) \to (Y, \sigma)$. Then, every rarely continuous function is rarely ω-continuous.*

Proof. Let $x \in X$ and W be an open set in Y containing $f(x)$. Since f is rarely continuous, by there exists an open set U in X containing x such that $Int(f(U)) \subset Cl(W)$. Since $\tau \subseteq \tau_\omega$, U is an ω-open set containing x. It then f is rarely ω-continuous. □

The converse of Theorem 2.2 is not true in general as shown in the following example.

Example 2.2. Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, b\}, \{c\}\}$ and $\sigma = \{\emptyset, Y, \{a, b\}, \{a\}, \{b\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Since $\tau_\omega = P(X)$, then f is rarely ω-continuous but not rarely continuous function because take $a \in Y$ and $W = \{a\}$, the for any open set $U = \{a, b\}$ or $U = X$ in X containing a we have $Int[f(U) \cap (Y - W)] = Int[U \cap \{b, c\}] \neq \emptyset$.

Theorem 2.3. *A function $f : (X, \tau) \to (Y, \sigma)$ is rarely ω-continuous if and only if for each an open set $W \subset Y$, there exists a rare set R_w with $W \cap Cl(R_w) = \emptyset$ such that $f^{-1}(W) \subset Int_w[f^{-1}(W \cup R_w)]$.*

Proof. Let $W \in \sigma(f(x))$, then there exists a rare set R_w with $W \cap cl(R_w) = \emptyset$ and an ω-open set U in X containing x, such that $f(U) \subset W \cup R_w$. Let $y \in W$ Since $W \subset W \cup R_w$, then $y \in W \cup R_w$. It follows that $x \in f^{-1}(W) \subset f^{-1}(W \cup R_w)$. This implies that $x \in Int_w(f^{-1}(W \cup R_w))$.

Conversely, let $W \in \sigma(f(x))$, then there exists a rare set R_w with $W \cap Cl(R_w) = \emptyset$ such that $f^{-1}(W) \subset Int_w[f^{-1}(W \cup R_w)]$. Let $x \in f^{-1}(W) \subset f^{-1}(W \cup R_w)$. This implies that $x \in Int_w(f^{-1}(W \cup R_w))$. Then, by Theorem 2.1 f is rarely ω-continuous function. □

Definition 2.2. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be ω^*-continuous at $x \in X$ if for each set $W \in \sigma(f(x))$, there exists an ω-open set U in X containing x such that $Int[f(U)] \subset W$. If f has this property at each point $x \in X$, then we say that f is ω^*-continuous on X.*

Theorem 2.4. *Let Y be a regular space a function $f : (X, \tau) \to (Y, \sigma)$ is ω^*-continuous on X if and only if f is rarely ω-continuous on X.*

Proof. Let f be rarely ω^*-continuous on X and $x \in X$. Suppose that $W \in \sigma(f(x))$. Then, there exists a ω-open set U in X containing x such that $Int[f(U)] \subset W$, then $Int[f(U)] \subset Cl(W)$. We have $f(U) = [f(U) - Int(f(U))] \cup Int(f(U)) \subset [f(U) - Int(f(U))] \cup Cl(W) = [f(U) - Int(f(U))] \cup W \cup (Cl(W) - W) = [f(U) - Int(f(U))] \cap (Y - W) \cup W \cup (Cl(W) - W)$.

Set $R_1 = [f(U) - Int(f(U))] \cap (Y - W)$ and $R_2 = (Cl(W) - W)$.

Then R_1 and R_2 are rare sets. Also $R_w = R_1 \cup R_2$ is a rare set such that $Cl(R_w) \cap W = \emptyset$ and $f(U) \subset W \cup R_w$. This shows that f is rarely ω-continuous function.

Conversely, let f be rarely ω-continuous on X and $x \in X$. Suppose that $f(x) \in W$, where W is an open set in Y. By the regularity of Y, there exists an open set $W_1 \in \sigma(f(x))$ such that $Cl(W_1) \subset W$. Since f is rarely ω-continuous, there exists an ω-open set U in X containing x such that $Int[f(U)] \subset Cl(W_1)$. This implies that $Int[f(U)] \subset W$ and so f is ω^*-continuous function on X. □

Definition 2.3. *A function $f : (X, \tau) \to (Y, \sigma)$ is called to be almost weakly ω-continuous if for each open set W in Y containing $f(x)$ there exists an ω-open set U in X containing x such that $f(U) \subset Cl(W)$.*

Theorem 2.5. *Let $f : (X, \tau) \to (Y, \sigma)$ be an ω-open and rarely ω-continuous function, then f is almost weakly ω-continuous.*

Proof. Let $x \in X$ and $W \in \sigma(f(x))$. Since f is rarely ω-continuous, there exists an ω-open set U in X such that $Int(f(U)) \subset Cl(W)$. Since f is ω-open $f(U)$ is ω-open and hence $f(U) = Int_\omega(f(U)) \subseteq Int(f(U)) \subset Cl(W)$. This shows that f is almost weakly ω-continuous. □

Theorem 2.6. *Let $f : X \to Y$ be a rarely ω-continuous function, then the graph function $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$ for every x in X is rarely ω-continuous.*

Proof. Let $x \in X$ and A is any open set containing $g(x)$. It follows that there exist open sets U and W in X and Y, respectively, such that $(x, f(x)) \in U \times W \subseteq A$. Since f is rarely ω-continuous, then there exists $G \in \tau_\omega(x)$ such that $Int[f(G)] \subset Cl(W)$. Let $B = U \cap G$. It follows that $B \in \tau_\omega(x)$ and we have $Int[g(B)] \subseteq Int(U \times f(G)) \subseteq U \times Cl(W) \subseteq Cl(A)$. Hence, g is a rarely ω-continuous function. □

Definition 2.4. *Let $A = \{W_i\}$ be a class of subsets of a topological space (X, τ). The rarely union sets [14] of A we mean $\{W_i \cup R_{W_i}\}$, where each R_{W_i} is a rare set such that each of $\{W_i \cap Cl(R_{W_i})\}$ is empty.*

Definition 2.5. *A subset K of a topological space (X, τ) is said to be*

1. *ω -compact relative to X if every cover of K by ω-open sets in X has a finite subcover. A space X is said to be ω-compact if X is ω-compact relative to X.*

2. *rarely almost compact relative to* X *if for every cover of* K *by open sets of* X*, there exists a finite subfamily whose rarely union sets cover* K*. A topological space* X *is said to be rarely almost compact if the set* X *is rarely almost compact relative to* X*.*

Theorem 2.7. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be rarely* ω*-continuous and* K *be* ω*-compact relative to* X*. Then* $f(K)$ *is rarely almost compact relative to* Y*.*

Proof. Let $\Omega = \{W_k : k \in K\}$ be an open cover of $f(K)$. Let $\mathcal{A}$ be the set of all W in Ω such that $W \cap f(K) \neq \emptyset$. Then $\mathcal{A}$ is an open cover of $f(K)$. Hence, for each $k \in K$, there is some $W_k \in \mathcal{A}$ such that $f(k) \in W_k$. Since f is rarely ω-continuous, there exists a rare set R_{W_k} with $W_k \cap Cl(R_{V_k}) = \emptyset$ and an ω-open set U_k containing k such that $f(U_k) \subset W_k \cup R_{W_k}$. Hence, there is a finite subfamily $\{U_k : k \in \Delta\}$ which covers K, where Δ is a finite subset of K. The subfamily $\{W_k \cup R_{W_k} : k \in \Delta\}$ also covers $f(K)$. □

Lemma 2.1. *If* $f : (X, \tau) \to (Y, \sigma)$ *is continuous and one-to-one, then* f *preserves rare sets.*

Theorem 2.8. *If* $f : (X, \tau) \to (Y, \sigma)$ *is rarely* ω*-continuous and* $g : (Y, \sigma) \to (Z, \rho)$ *is a continuous injection, then* $g \circ f : (X, \tau) \to (Z, \rho)$ *is rarely* ω*-continuous.*

Proof. Let $x \in X$ and $(g \circ f)(x) \in V$, where V is an open set in Z. By hypothesis, g is continuous, therefore $W = g^{-1}(V)$ is an open set in Y containing $f(x)$ such that $g(W) \subset V$. Since f is rarely w-continuous, there exists a rare set R_W with $W \cap Cl(R_W) = \emptyset$ and an ω-open set U containing x such that $f(U) \subset W \cup R_W$. It follows from Lemma 2.1 that $g(R_W)$ is a rare set in Z. Since R_W is a subset of $Y - W$ and g is injective, we have $Cl(g(R_W)) \cap V = \emptyset$. This implies that $(g \circ f)(U) \subset V \cup g(R_W)$. Hence, $g \circ f$ is rarely ω-continuous. □

Definition 2.6. *A topological space* (X, τ) *is called rarely separated [14] if for every pair of distinct points* x *and* y *in* X*, there exist open sets* U_x *and* U_y *containing* x *and* y*, respectively, and rare sets* R_{U_x}*,* R_{U_y} *with* $U_x \cap cl(R_{U_x}) = \emptyset$ *and* $U_y \cap cl(R_{U_y}) = \emptyset$ *such that* $(U_x \cup R_{U_x}) \cap (U_y \cup R_{U_y}) = \emptyset$*.*

Definition 2.7. *A topological space* (X, τ) *is said to be* ω*-*T_2 *[10] if for any distinct pair of points* x *and* y *in* X*, there exist disjoint* ω*-open sets* U *and* V *in* X *containing* x *and* y*, respectively.*

Theorem 2.9. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be rarely* ω*-continuous injection such that* (Y, σ) *is rarely separated, then* (X, τ) *is* ω*-*T_2 *space.*

Proof. Let x and y be any distinct points in X. Then $f(x) \neq f(y)$ (as f is injective). Thus there exist open sets G_x and U_y in Y containing $f(x)$ and $f(y)$, respectively, and rare sets R_{U_x} and R_{U_y} with $U_x \cap Cl(R_{U_x}) = \emptyset$ *and* $U_y \cap Cl(R_{U_y}) = \emptyset$ such that $(U_x \cup R_{U_x}) \cap (U_y \cap R_{U_y}) = \emptyset$. Therefore,

$Int_w[f^{-1}(U_x \cup R_{U_x})] \cap Int_w[f^{-1}(U_y \cup R_{U_y})] = \emptyset$. By Theorem 2.3 we have $x \in f^{-1}(U_x) \subset Int_w[f^{-1}(U_x \cup R_{U_x})]$ and $y \in f^{-1}(U_y) \subset Int_w[f^{-1}(U_y \cup R_{U_y})]$. Since $Int_w[f^{-1}(U_x \cap R_{U_x})]$ and $Int_w[f^{-1}(U_y \cap R_{U_y})]$ are two disjoint ω-open sets, hence (X, τ) is an ω-T_2 space. □

Theorem 2.10. *Let (X, τ) be topological space let $A \subset X$ and $f : (X, \tau) \to (A, \tau_A)$ be a rarely ω-continuous retraction of X onto A. If X is Hausdorff, then A is a closed set.*

Proof. Suppose A is not closed. Then, there exists a point $a \in Cl(A) - A$. Since f is a retraction, $f(a) \neq a$. Moreover, X is Hausdorff, there exist disjoint open sets H and W containing a and $f(a)$, respectively. Now, for the open W, there exists a rare set R_W in the subspace A and an ω-open U containing a such that $Cl(R_W) \cap W = \emptyset$, $U \subset H$ and $f(U) \subset W \cup R_W$. Since $U \cap A$ is ω-open in A, there is a point $z \in U \cap A$ such that $z \notin R_W$. So, $f(z) = z \notin W \cup R_W$, then f is not rarely ω-continuous, a contradiction. Therefore, A is closed. □

Definition 2.8 ([13])**.** *Let (X, τ) be Hausdorff space, then X is H-closed if every open cover $\Omega = \{U_\alpha : \alpha \in \Delta\}$ of X contains a finite subcollection $\Omega_0 = \{Cl(U_{\alpha_i}) : i = 1, 2, 3, ..., n\}$ covers X.*

Theorem 2.11. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function with an ω-closed graph where Y is H-closed. Then f is rarely ω-continuous function.*

Proof. Let $x \in X$ and W be a regular open set containing $f(x)$. For each $y \in Y - W$ there exist an ω-open set $U(x_y)$ containing x and open set $G(y)$ containing y such that $f(U(x_y)) \cap G(y) = \emptyset$ because f has an ω-closed graph. Since Y is Hausdorff, we may also choose $G(y)$ so that $f(x) \notin Cl(G(y))$ for each $y \in Y - W$. Thus the collection $\{G(y) : y \in Y - W\}$ forms an open cover of the regular closed set $Y - W$. Since Y is H-closed spaces, then the regular closed sets are themselves H-closed which implies the open cover $\{G(y) : y \in Y - U\}$ has a finite subcollection $\{G(y_i) : i = 1, 2....n)\}$, such that $Y - W \subset \bigcup_{i=1}^{n}(Cl(G(y_i)) = Cl(\bigcup_{i=1}^{n} G(y_i))$. Now, define $U = \bigcap_{i=1}^{n} U(x_{y_i})$ is an ω-open set. Then, by construction $f(U) \cap (\bigcup_{i=1}^{n} G(y_i)) = \emptyset$. So, if we let $R_W = (Y - W) \cap (Cl(\bigcup_{i=1}^{n} G(y_i)) - \bigcup_{i=1}^{n} G(y_i))$ then R_W is a rare set and $f(U) \subset W \cup R_W$. By Theorem 2.1, f is rarely ω-continuous function. □

Lemma 2.2. *Let $A_1, A_2, \ldots, A_n$ be a finite family of pairwise disjoint sets having the property that when $i \neq j$, either $Cl(A_i) \cap A_j = \emptyset$ or $A_i \cap Cl(A_j) = \emptyset$. If, in addition, $Int(A_i) = \emptyset$ for all $1 < i < n$, then $Int(\bigcup_{i=1}^{n} A_i) = \emptyset$.*

Theorem 2.12. *Let (X, τ) and (Y_a, σ_a) be topological spaces. Let $f_a : (X, \tau) \to (Y_a, \sigma_a)$ be a rarely ω-continuous function for each $a \in \Delta$ and define $f : (X, \tau) \to (\prod_{a \in \Delta} Y_a, \prod_{a \in \Delta} \sigma_a)$ by $f(x) = \{f_a(x)\}$. Then f is rarely ω-continuous.*

Proof. Let $x \in X$ and W be an open set containing $f(x)$. Now, let $H = W_{a_1} \times W_{a_2} \times \ldots.. \times W_{a_n} \times \Pi\{Y_\beta : \beta \neq a_1, a_2, \ldots, a_n\}$ be a basic open set containing $f(x)$ such that $f(x) \in H \subset W$. Then, for each W_{a_1} there exists a rare set $R_{W_{a_i}}$ with $Cl(R_{W_{a_i}}) \cap W_{a_i} = \emptyset$ and some an ω-open U_{a_l} containing x such that $f_{a_i}\ (U_{a_i}) \subset W_{a_i} \cup R_{W_{a_i}}$. Define $U = \bigcap_{i=1}^{n} U_{a_i}$ which is an ω-open. Then $J(U) \subset \Pi\{Y_\beta : \beta \neq a_1, a_2, \ldots, a_n\} \times\ (W_{a1} \cup R_{W_{a1}}) \times\ (W_{a_2} \cup R_{W_{a_2}}) \times\ \ldots \times\ (W_{a_n} \cup R_{W_{a_n}}) = \Pi\{Y_\beta : \beta \neq a_1, a_2, \ldots, a_n\} \times\ W_{a_1} \times\ W_{a_2} \times\ \ldots \times W_{a_n} \cup (\cup\{\Pi\{Y_\beta : \beta \neq a_1, a_2, \ldots, a_n\} \times \prod_{i=1}^{n}\{A_i : A_i = W_{a_i}\ or\ Ai = R_{W_{a_i}}$ and $\prod_{i=1}^{n} A_i \neq \prod_{i=1}^{n} W_{a_i}\})$. Observe that the collection of sets $\{\Pi\{Y_\beta : \beta \neq a_1, a_2, \ldots, a_n\} \times\ \prod_{i=1}^{n}\{A_i : A_i = W_{a_i}\ or\ A_i = R_{W_{a_i}}$ and $\prod_{i=1}^{n} A_i \neq \prod_{i=1}^{n} W_{a_i}\}$ is a finite family of pairwise disjoint sets that satisfies the hypothesis of Lemma 2.2. Thus the interior of the union of this finite family is empty so if we define R_W is rare set, $Cl(R_W) \cap W = \emptyset$ and $f(U) \subset W \cup R_W$. Thus f is rarely ω-continuous function. □

Conclusion

We defined and studied a new class of functions, called rarely ω-continuous function via ω-open sets. In the future, we intend to provide more characterizations and prove soft mapping, as well as define new soft topological concepts using soft ω-open sets.

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