Finite groups whose character degrees are products of at most three prime numbers

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Abstract. Huppert and Manz determined the structure of nonsolvable groups whose character degrees are products of at most two primes. In this paper, we classify nonsolvable groups satisfying that their character degrees are products of at most three primes.

Keywords: simple group, nonsolvable group, character degree.

1. Introduction

Suppose that all groups considered are finite. Let $\mathrm{Irr}(G)$ be the set of all complex irreducible characters of G. Set $\mathrm{cd}(G) = \{\chi(1) : \chi \in \mathrm{Irr}(G)\}$, the set of degrees of all irreducible characters of G. It is well known that if G is a finite group, and p is a prime, then $p \nmid \chi(1)$ for all $\chi \in \mathrm{Irr}(G)$, if and only if G has an abelian normal Sylow p-subgroup; see Theorem 12.34 of [14]. Gow-Huphreys proved that if G is a finite group, p is a prime, and P is a Sylow p-subgroup of G, then $p\big|\chi(1)$ for all nonlinear character $\chi \in \mathrm{Irr}(G)$ if and only if G has a normal p-complement K and $C_{K'}(P) = 1$; see Theorem 7.6 of [14]. Recently, some authors considered the relation between character degrees of certain simple groups and the structure of finite groups; see [11, 13, 19] for instance.

Let $\mathbb{N}$ be the set of positive integers. If $n \in \mathbb{N}$ can be written by

$$n = \prod_{i=1}^{k} p_i^{a_i},$$

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where $p_1, p_2, \cdots, p_s$ are different primes and $a_1, a_2, \cdots, a_k$ are positive integers, then define

$$\omega(n) = \sum_{i=1}^{k} a_i$$

and $\omega(G) = \max_{\chi \in \mathrm{Irr}(G)} \omega(\chi(1))$.

If $\omega(G) = 1$, then all non-linear characters have prime-number-degrees and so by [10, pp. 243 Corollary 14.4], G is solvable. Huppert and Manz in [8] proved that the groups whose character degrees are square-free are solvable. Lewis and White in [12] classified the nonsolvable groups whose degrees of all irreducible characters are odd-square-free. Huppert and Manz in [9] have classified nonsolvable groups whose character degrees are products of at most two prime numbers. From Theorem of [9], we can get the following result.

Theorem 1.1. *Let G be a nonabelian simple group with $\omega(G) \le 2$. Then G is isomorphic to A_5 of A_7.*

Huppert in [7] showed that the derived length of a group G is at most 4 if $\omega(G) = 2$ and G is solvable.

Let G be a finite group and put

$$\pi(G) = \{p : p \text{ is a prime divisor of } |G|\}.$$

The other notation is standard; see [4].

We first give the structure of a non-abelian simple group G with $\omega(G) \le 3$.

Theorem 1.2. *Assume that G is a non-abelian simple group with $\omega(G) \le 3$.*

If $\omega(G) = 2$, then G is isomorphic to A_5 or A_7.

If $\omega(G) = 3$, then the following hold.

(1) *If $|\pi(G)| = 3$, then G is isomorphic to $L_2(7)$, $L_2(8)$ or $L_2(9)$;*

(2) *If $|\pi(G)| = 4$, then G is isomorphic to $L_2(11)$, $L_2(19)$ or $L_2(27)$;*

(3) *If $|\pi(G)| = 5$, then G is isomorphic to $L_2(p)$ with either $p - 1 = 2^2 \cdot p_1$, $p + 1 = 2 \cdot 3 \cdot p_2$ or $p - 1 = 2 \cdot 3 \cdot p_1$, $p + 1 = 2^2 \cdot p_2$ where p, p_1, p_2 are different primes and $p_1, p_2 \ge 5$.*

In this paper, we will also give the structure of a nonsolvable group G with $\omega(G) = 3$.

Theorem 1.3. *Assume that G is a nonsolvable group. If $\omega(G) = 3$, then G has a normal subgroup N such that $G/N \cong L_2(q)$ for some prime power q or A_7. Furthermore, each $\theta \in \mathrm{Irr}(N)$ is invariant in G and one of the following statements holds.*

(1) *G/N is isomorphic to S where $\mathrm{cd}(N) = \{1, p\}$ for some prime p, and $S \in \{A_5, S_5, A_7\}$;*

(2) *G/N is isomorphic to S where N is abelian, $S \in \{L_2(8), S_7, L_2(8).3, L_2(7), L_2(7).2, L_2(9), L_2(11), L_2(11).2, L_2(19), L_2(19).2, L_2(27), L_2(27).2,$*

$L_2(9).2_2$, $L_2(q)$, $L_2(q).2\}$ with q a prime satisfying either $q-1=2^2\cdot p_1, q+1=2\cdot 3\cdot p_2$ or $q-1=2\cdot 3\cdot p_1, q+1=2^2\cdot p_2$ where p_1,p_2 are different primes and $p_1,p_2\geq 5$.

2. Some needed results

Lemma 2.1. *Let N be a subgroup of G. Then $\omega(N)\leq\omega(G)$. If $N\trianglelefteq G$, then $\omega(G/N)\leq\omega(G)$.*

Proof. It is easy to get from the definition of $\omega(G)$. □

Lemma 2.2 ([3])**.** *If x or y is prime, then the equation $x^m-y^n=1$, $x>1$, $y>1$, $m>1$, $n>1$, $x,y,m,n\in\mathbb{N}$, has only solution $(x,y,m,n)=(3,2,2,3)$.*

Recall that a group G is called a ***simple*** K_n***-group*** if $|\pi(G)|=n\geq 3$ and G is simple.

Lemma 2.3 ([5])**.** *If G is a simple K_3-group. Then G is isomorphic to one of the groups: A_5, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$ and $U_4(2)$.*

Lemma 2.4 (Lemma 1 of [18])**.** *Let G be an insolvable group. Then G has a normal series $1\trianglelefteq K\trianglelefteq L\trianglelefteq G$, such that L/K is a direct product of isomorphic non-abelian simple groups and $|G/L|\mid|\mathrm{Out}(L/K)|$.*

Lemma 2.5. *Let L be a sporadic simple group or ${}^2F_4(2)'$ except M_{11}. Then $\omega(L)\geq 5$. In particular, $\omega(M_{11})=4$.*

Proof. By [4], we have Table 1, and so get the desired result. □

Table 1. Sporadic simple groups and a Tits group

L	Character	Degree	L	Character	Degree
M_{11}	χ_6	2^4	M_{12}	χ_{13}	$2^3\cdot 3\cdot 5$
J_1	χ_9	$2^2\cdot 3\cdot 5$	M_{22}	χ_{10}	$2^3\cdot 5\cdot 7$
J_2	χ_{16}	$2^4\cdot 3^2$	M_{23}	χ_{12}	$2^7\cdot 7$
${}^2F_4(2)'$	χ_{22}	2^{11}	HS	χ_{24}	$2^5 7\cdot 5^2$
J_3	χ_{21}	$2\cdot 3^4\cdot 19$	M_{24}	χ_{24}	$2^3\cdot 3^2\cdot 7\cdot 11$
M^cL	χ_{11}	$2^6\cdot 5\cdot 11$	He	χ_{30}	$2^{10}\cdot 3\cdot 7$
Ru	χ_{11}	$2^3\cdot 3^3\cdot 5^3$	Suz	χ_{43}	$2^{10}\cdot 3^5$
ON	χ_{25}	$2^9\cdot 3^4\cdot 5$	Co_3	χ_{39}	$2^7\cdot 5^2\cdot 7\cdot 11$
Co_2	χ_{60}	$3^6\cdot 5^3\cdot 23$	Fi_{22}	χ_{44}	$2^{16}\cdot 3\cdot 13$
HN	χ_{53}	$2^3\cdot 3^6\cdot 5^3\cdot 7$	Ly	χ_{55}	$2^2\cdot 3\cdot 7\cdot 11\cdot 31\cdot 37\cdot 67$
Th	χ_4	$2^3\cdot 3^3\cdot 5^3$	Fi_{23}	χ_3	$2^2\cdot 3\cdot 13\cdot 23$
Co_1	χ_6	$2\cdot 3\cdot 5^2\cdot 23$	J_4	χ_4	$3^2\cdot 29\cdot 31\cdot 37$
Fi_{24}	χ_4	$2\cdot 11\cdot 17\cdot 23\cdot 29$	B	χ_5	$2\cdot 5^4\cdot 7\cdot 23\cdot 47$
M	χ_3	$2^2\cdot 31\cdot 41\cdot 59\cdot 71$			

3. The proof of theorems

The structure of the simple groups whose degrees of the irreducible complex characters are products of at most three primes, are determined first, and then the structure of a nonsolvable group G with $\omega(G)=3$ is given.

3.1 The proof of Theorem 1.2

Proof. If G is a non-abelian simple group, then G is isomorphic to one of the following simple groups: alternating groups A_n with $n \geq 5$, sporadic simple groups or simple groups of Lie type.

If $\omega(G) = 2$, then by [9], we conclude that G is isomorphic to A_5 or A_7.

So, we consider $\omega(G) = 3$ in the following.

(1) Let G be an alternating group A_n of degree $n \geq 5$. If $n = 5$ or $n = 7$, then by [4], $\omega(G) = 2$.

If $n = 6$, then by [4], $\mathrm{cd}(A_6) = \{1, 5, 8, 9, 10\}$ and so $\omega(A_6) = 3$, the desired result.

If $13 \geq n \geq 8$, then $\omega(A_n) \geq 4$.

Some degree for A_n with $13 \geq n \geq 5$[1]

A_n	Character χ_i	Degree $\chi_i(1)$
A_8	χ_{12}	$2^3 \cdot 7$
A_9	χ_{11}	$2^3 \cdot 7$
A_{10}	χ_4	$2^2 \cdot 3^2$
A_{11}	χ_6	$2^3 \cdot 3 \cdot 5$
A_{12}	χ_3	$2 \cdot 3^3$
A_{13}	χ_6	$2^2 \cdot 5 \cdot 11$

Note that the irreducible characters χ^α of A_n are the same as that of the symmetric group S_n if the partitions α of n are non-self-conjugate. If $n \geq 14$, then by hook formula, we have

$$\chi^\alpha(1) = \frac{(n-1)(n-2)(n-3)(n-4)(n-5)}{5!}, \quad \alpha = (n-5, 1, 1, 1, 1, 1),$$

where α is a partition of n for S_n and non-self-conjugate. So, for $n \geq 8$, we get $\omega(A_n) \geq 4$, a contradiction.

(2) Let G be a simple group of Lie type. Let G be a finite group of Lie type in characteristic p, then the Steinberg character is obviously a p-Steinberg character; see [16, pp. 304]. By Corollary 3 of [16], if a finite simple group G of has p-Steinberg character χ with $p \big| |G|$, then G is a finite simple group of Lie type in characteristic p, and χ is then the Steinberg character St of G. Note that if G is a simple group of Lie type of characteristic p, p a prime, then $\mathrm{St}(1) = |G|_p$. Then, we can construct Table 3 from [4].

Table 2. Simple groups of Lie type

L	St(1)	Condition	L	St(1)	Condition
$A_n(q)$	$q^{\frac{n(n+1)}{2}}$	$n \geq 1$	${}^2A_n(q)$	$q^{\frac{n(n+1)}{2}}$	$n \geq 2$
$B_n(q)$	q^{n^2}	$n \geq 2$	$C_n(q)$	q^{n^2}	$n \geq 3$
$D_n(q)$	$q^{n(n-1)}$	$n \geq 4$	${}^2D_n(q)$	$q^{n(n-1)}$	$n \geq 4$
${}^2G_2(q)$	q^6		$F_4(q)$	q^{24}	
$E_6(q)$	q^{36}		${}^2E_2(q)$	q^{36}	
$E_7(q)$	q^{63}		$E_8(q)$	q^{120}	
${}^2B_2(q)$	q^2	$q = 2^{2m+1} \geq 8$	${}^3D_4(q)$	q^{12}	
${}^2G_2(q)$	q^3	$q = 3^{2m+1} \geq 27$	${}^2F_4(q)$	q^{12}	$q = 2^{2m+1} \geq 2^5$

By Table 2, we have that the possible groups G satisfying $\omega(G) \leq 3$ are $A_n(q)$ with $n \geq 1$ and ${}^2A_n(q)$ with $n \geq 2$.

We consider the following cases:

Case 2.1. G is of Type A_l: $L_{l+1}(q)$.

As in this case $l(l+1)/2 \leq 3$, so $l = 1$, or 2. So, two cases are considered: $l = 1$ and $l = 2$.

Let $l = 1$. First, we assume that q is even. Then, $q = 2^n$ for some integer $n \geq 2$. By hypothesis, $2 \leq n \leq 3$ since $L_2(2^n)$ has a Steinberg character of degree 2^n. If $n = 2$, then G is isomorphic to $L_2(4) \cong A_5$ as $\mathrm{cd}(A_5) = \{1, 3, 4, 5\}$. If $n = 3$, then $G \cong L_2(8)$ as $\mathrm{cd}(L_2(8)) = \{1, 7, 8, 9\}$.

Now, we suppose that q is odd and $q \geq 7$ since if $q = 3$, $L_2(3)$ is solvable and if $q = 5$, then $L_2(5) \cong L_2(4)$ is considered. Note that

$$\mathrm{cd}(L_2(q)) = \{1, q-1, q, q+1, \frac{q + (-1)^{(q-1)/2}}{2}\}.$$

Since $q \geq 7$ is odd, then one of the numbers $q+1$ and $q-1$ has the form $4r$ for some integer $r \geq 2$. Thus, we only need to consider q. Since $\omega(G) = 3$, then $q | p^3$ for some odd prime p. So, we consider the following three cases: $q = p$, $q = p^2$ and $q = p^3$.

If $q = p$, then since for some $k \in \mathbb{N}$, $p = 4k+1$ or $4k-1$, one has that

$$p + 1 = 4k + 2 = 2 \cdot 3 \cdot p_1 \ \text{ and } \ p - 1 = 4k = 2^2 \cdot p_2$$

or

$$p + 1 = 2^2 \cdot p_1 \ \text{ and } \ p - 1 = 4k - 2 = 2 \cdot 3 \cdot p_2.$$

It follows that G is a simple K_n-group for $n \leq 5$.

If G is a simple K_3-group, then by Lemma 2.3, G is isomorphic to $L_2(7)$.

If G is a simple K_4-group, then by Theorem 2 of [2], G is isomorphic to $L_2(11)$ or $L_2(19)$.

If G is a K_5-group, then G is isomorphic to $L_2(p)$ satisfying one of the two conditions:

$$\begin{cases} p - 1 = 2^2 \cdot p_1 \\ p + 1 = 2 \cdot 3 \cdot p_2 \end{cases} \quad \text{with } p, p_1, p_2 \text{ different primes and } p_1, p_2 \geq 5$$

and

$$\begin{cases} p - 1 = 2 \cdot 3 \cdot p_1 \\ p + 1 = 2^2 \cdot p_2 \end{cases} \quad \text{with } p, p_1, p_2 \text{ different primes and } p_1, p_2 \geq 5.$$

If $q = p^2$, then $p^2 - 1 = (p-1)(p+1) = 8 \cdot m$ for some positive integer m and so $m = 1$, $q = 9$. Thus, G is isomorphic to $L_2(9)$.

If $q = p^3$, then we get that $p^3 + 1 = (p+1)(p^2 - p + 1) = 2m(p^2 - p + 1)$ and $p^3 - 1 = (p-1)(p^2 + p + 1) = 4n(p^2 + p + 1)$ for some $m, n \in \mathbb{N}$ or $p^3 + 1 = (p+1)(p^2 - p + 1) = 4m(p^2 - p + 1)$ and $p^3 - 1 = (p-1)(p^2 + p + 1) = 2n(p^2 + p + 1)$ for some $m, n \in \mathbb{N}$.

If the former, then $\omega(G) = 3$ implies $n = 1$. It follows that $q^2 + q + 1$ is a prime, so $p^3 - 1 = 4(p^2 + p + 1)$. Thus, $p = 5$ and $q + 1 = p^3 + 1 = 125 + 1 = 126 = 2 \cdot 3^2 \cdot 7$. Therefore $\omega(G) \geq 4$, a contradiction to the hypothesis.

If the latter, then $m = 1$, $p^2 - p + 1$ is a prime and so $p^3 + 1 = 4(p^2 - p + 1)$. Thus, $p = 3$. It follows that $p^3 - 1 = 3^3 - 1 = 26 = 2 \cdot 13$ and so $G \cong L_2(27)$.

Let $l = 2$. In this case, the Steinberg character St of $L_3(q)$ has the degree q^3, so by hypothesis, q is a prime.

If $q = 2$, then $\omega(L_3(2)) = 3$ as $L_3(2) \cong L_2(7)$ and $\mathrm{cd}(L_2(7)) = \{1, 3, 6, 7, 8\}$.

If q is an odd prime, then by [15] there is a character $\chi \in \mathrm{Irr}(L_3(q))$ such that $\chi(1) = (q-1)^2(q+1)$ is divisible by 16 since $(q-1)(q+1)$ is divisible by 8. By Lemma 2.1, $\omega(G) \geq 4$, a contradiction.

Case 2.2. G is of Type ${}^2A_l$: $U_{l+1}(q)$ $(l \geq 2)$.

In this case, $l(l+1)/2 \leq 3$, so $l = 2$ or $l = 1$. If $l = 1$, then $U_2(q) \cong L_2(q)$ is considered in Case 2.1. Now let $l = 2$. If $q = 2$, then $U_3(2)$ is solvable, so $q \geq 3$. Then by [15], there is a character $\chi \in \mathrm{Irr}(U_3(q^2))$ such that if q is odd, then $\chi(1) = (q-1)(q+1)^2 = 16k$ for some positive integer k and that if q is even, then $q \geq 4$ and $q^3 \in \mathrm{cd}(G)$. It follows that $\omega(G) \geq 4$ in both cases, a contradiction to the hypothesis $\omega(G) = 3$.

(3) Let G be a sporadic simple group or a Tits group. By Lemma 2.5, we rule out this case.

This completes the proof. □

3.2 The Proof of Theorem 1.3

Lemma 3.1. *If G is a nonsolvable group with $\omega(G) = 3$, then there is a normal series $1 \leq K \leq L \leq G$ such that L/K is isomorphic to one of the groups as listed in Main Theorem 1.2 and $|G/L| \mid |\mathrm{Out}(L/K)|$.*

Proof. Since G is nonsolvable, by Lemma 2.4 there is a normal series $1 \leq K \leq L \leq G$ such that L/K is a direct product of isomorphic non-abelian simple groups and $|G/L|\big||\mathrm{Out}(L/K)|$. Since $\omega(G) = 3$ and G is nonsolvable, then L/K is isomorphic to $B_1 \times B_2 \times \cdots \times B_n$, where B_i is the group listed in Main Theorem 1.2. If $n \geq 2$, then $\omega(B_1 \times B_2 \times \cdots \times B_n) \geq 4 > \omega(G) = 3$, a contradiction to Lemma 2.1. Therefore $n = 1$. □

Lemma 3.2. *Let N be a normal subgroup of a group G such that G/N is isomorphic to $L_2(q)$ with $q \geq 7$ an odd prime power. If $\omega(G) = 3$, then for each $\theta \in \mathrm{Irr}(N)$, θ is G-invariant.*

Proof. Let $\theta \in \mathrm{Irr}(N)$, and let $I := I_G(\theta)$, be the inertia group of θ in G. Note that $N \trianglelefteq I \leq G$.

If $I = N$, then Problem 6.1 of [10] gives $\theta^G \in \mathrm{Irr}(G)$ for $\theta \in \mathrm{Irr}(N)$, so $\theta^G(1) = |L_2(q)|\theta(1)$ is the product of at least four primes, a contradiction.

If $N < I < G$, then by [6, Chap III, Theorem 8.27], I is isomorphic to $N : F$ where F is a Frobenius group of order $q(q-1)/2$, $N : D_{(q+1)}$, or $N : D_{(q-1)}$.

If $I = N : F$, then $|G : I| = q + 1$. Let $\eta \in \mathrm{Irr}(I/N)$ with $\eta(1) = \frac{q-1}{2}$ as I/N is a Frobenius group and $\frac{q-1}{2} \in \mathrm{cd}(I/N)$. Now by Corollary 6.17 of [10], $\eta\chi \in \mathrm{Irr}(I)$ for each irreducible constituent χ of θ^G, and $(\eta\chi)_N = \theta$. So, by Theorem 6.11 of [10], $(\eta\chi)^G \in \mathrm{Irr}(G)$, so $(\eta\chi)^G(1) = (q+1)(q-1)/2 = 8r$ or $4rq$ for some numbers $r \geq 2$ and $q \geq 2$, a contradiction.

If $I = N : D_{q-1}$, then let $\eta \in \mathrm{Irr}(I)$ with $\eta(1) = 2$ since $I/N \cong D_{q-1}$ is a dihedral group, one has $\eta^G(1) = |G : I|\eta(1) = 2q(q+1)$, also a contradiction. Similarly we may exclude $I = D_{q-1}$.

Thus, $I = G$, θ is G-invariant. □

Now, we can prove Main theorem 1.3

Proof of Main Theorem 1.3

Proof. Let $H \trianglelefteq G$. By Lemma 2.1, $\omega(H) \leq \omega(G)$ and $\omega(G/H) \leq \omega(G) = 3$. By hypothesis, G is nonsolvable.

By Lemma 3.1 we can assume that G has a normal series $1 \trianglelefteq K \trianglelefteq L \trianglelefteq G$ such that L/K is isomorphic to a group listed in Main Theorem 1.2, and that p is a prime divisor of $|G|$.

We consider the groups listed in Main Theorem 1.2.

Case 1. $L/K \cong A_5$. By Lemma 3.1, $A_5 \leq G/K \leq S_5$.

Let $G/K \cong A_5$.

Let $\theta \in \mathrm{Irr}(K)$, and let $I = I_G(\theta)$ be the inertia group of θ in G. Hypothesis and the fact $\mathrm{cd}(A_5) = \{1, 3, 4, 5\}$ show that $\mathrm{cd}(K) = \{1, p\}$, for some prime p.

Let $G/K \cong S_5$.

Let $\theta \in \mathrm{Irr}(K)$, and let $I = I_G(\theta)$, be the inertia of θ in G. Then by Lemma 3.2, $I = G$, so by Corollary 6.17 of [10], for each $\eta \in \mathrm{Irr}(G/K)$, $\chi\eta \in \mathrm{Irr}(G)$ with $\chi_K = \theta$. Hypothesis and $\mathrm{cd}(S_5) = \{1, 4, 5, 6\}$ show that $\mathrm{cd}(K) = \{1, p\}$, for some prime p.

Case 2. $L/K \cong A_7$. We know that $\mathrm{cd}(A_7) = \{1,\ 2 \cdot 3,\ 2 \cdot 5,\ 2 \cdot 7,\ 3 \cdot 5,\ 3 \cdot 7,\ 5 \cdot 7\}$ and that $\mathrm{cd}(S_7) = \{1,\ 2 \cdot 3,\ 2 \cdot 7,\ 3 \cdot 5,\ 2^2 \cdot 5,\ 3 \cdot 7,\ 5 \cdot 7\}$, so $\omega(A_7) = 2$ and $\omega(S_7) = 3$.

By Lemma 3.1, $A_7 \leq G/K \leq S_7$.

Let $G/K \cong A_7$.

Similarly as Case 1, we can get that $G/K \cong A_7$ such that for every $\theta \in \mathrm{Irr}(K)$, θ is G-invariant and $\mathrm{cd}(K) = \{1, p\}$ for some prime p.

Let $G/K \cong S_7$.

Now $\omega(S_7) = 3$. Also we can get that $G/K \cong S_7$ such that for every $\theta \in \mathrm{Irr}(K)$, θ is G-invariant and K is abelian.

Case 3. $L/K \cong L_2(7)$. Note that $\mathrm{cd}(L_2(7)) = \{1, 3, 2 \cdot 3, 7, 2^3\}$ and that $\mathrm{cd}(L_2(7).2) = \{1, 3, 2^2, 3 \cdot 3, 7, 2^3\}$, so $\omega(L_2(7)) = \omega(L_2(7).2) = 3$.

By Lemma 3.1, $L_2(7) \leq G/K \leq L_2(7).2$.

Similarly as Case 1, we can get that G/K is isomorphic to $L_2(7)$ or $L_2(7).2$ where K is abelian and for every $\theta \in \mathrm{Irr}(K)$, θ is G-invariant.

Case 4. $L/K \cong L_2(8)$. Note that $\omega(L_2(8)) = \omega(L_2(8).3) = 3$. We can also get that G/K is isomorphic to $L_2(8)$ or $L_2(8).3$, where K is abelian and for every $\theta \in \mathrm{Irr}(K)$, θ is G-invariant.
Case 5. $L/K \cong L_2(9)$. By [1],

$$\begin{aligned}
\mathrm{cd}(A_6) &= \{1,5,8,9,10\},\\
\mathrm{cd}(A_6.2_1) &= \{1,5,9,10,16\},\\
\mathrm{cd}(A_6.2_2) &= \{1,8,9,10\},\\
\mathrm{cd}(A_6.2_3) &= \{1,9,10,16\},\\
\mathrm{cd}(A_6.2^2) &= \{1,9,10,16,20\}.
\end{aligned}$$

By Lemma 3.1, $L_2(9) \leq G/K \leq \mathrm{Aut}(L_2(9))$, so hypothesis gives that G/K is isomorphic to A_6, or $A_6.2_2$ and that K is abelian and, for every $\theta \in \mathrm{Irr}(K)$, θ is G-invariant.
Case 6. $L/K \cong L_2(p)$ where p is a prime with either $p-1 = 2^2\cdot p_1, p+1 = 2\cdot 3\cdot p_2$ or $p-1 = 2\cdot 3\cdot p_1, p+1 = 2^2\cdot p_2$ where p, p_1, p_2 are different primes and $p_1, p_2 \geq 5$.

Now, we have that $\omega(L_2(p)) = 3$, and $|\mathrm{Out}(L_2(p))| = 2$.

We can get from [17, pp. 192], that $\omega(L_2(p)) = \omega(L_2(p).2) = 3$. By Lemma 3.1, $L_2(p) \leq G/K \leq L_2(p).2$, so G/K is isomorphic to $L_2(p)$ or $L_2(p).2$ where K is abelian and for every $\theta \in \mathrm{Irr}(K)$, θ is G-invariant.
Case 7. $L/K \cong L_2(11)$ or $L_2(19)$.

Note that $\mathrm{cd}(L_2(11)) = \{1,\ 5,\ 2\cdot 5,\ 11,\ 2^2\cdot 3\}$ and $\mathrm{cd}(L_2(11).2) = \{1,\ 2\cdot 5,\ 11,\ 2^2\cdot 3\}$. Then $G/K \cong L_2(11)$ or $L_2(11).2$. Corollary 6.17 of [10] means that K is abelian and $\theta \in \mathrm{Irr}(K)$ is G-invariant by Lemma 3.2. Similarly we can get that $G/K \cong L_2(19)$ or $L_2(19).2$ with K abelian and $\theta \in \mathrm{Irr}(K)$ G-invariant.
Case 8. $L/K \cong L_2(27)$.

We know from [4] that

$$\begin{aligned}
\mathrm{cd}(L_2(27)) &= \{1,13,2\cdot 13,3^3,2^2\cdot 7\},\\
\mathrm{cd}(L_2(27).2) &= \{1,2\cdot 13,3^3,2^2\cdot 7\},\\
\mathrm{cd}(L_2(27).3) &= \{1,13,2\cdot 3\cdot 13,2^2\cdot 3\cdot 7\},\\
\mathrm{cd}(L_2(27).6) &= \{1,2\cdot 13,2\cdot 3\cdot 13,3^3,2^2\cdot 3\cdot 7\}.
\end{aligned}$$

So, as the proof of Case 5, we get similarly that $G/K \cong L_2(27)$, or $L_2(27).2$ where K is abelian by Corollary 6.17 of [10], and so by Lemma 3.2, $\theta \in \mathrm{Irr}(K)$ is G-invariant.

This completes the proof. □

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