Fixed point theorem for non-self mappings satisfying generalized contraction condition of integral type in metrically convex spaces

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Abstract. A common fixed point theorem for single valued non-self mapping from nonempty closed subset K of a metrically convex metric spaces X is proved satisfying the contraction condition of integral type on the subset K. Result generalizing the previous results due to Banach [3], Branciari [4], Ćirić [6], Rhoades [11], Khan [9] and others.
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1. Introduction

In this paper we consider generalized contraction condition of integral type in metrically convex spaces and prove a fixed point theorem for nonself single valued maps. Here, let us point out that there do exist many situations when the mapping on a metric space or Banach space is not always a self map. Therefore, it is of great interest to find the sufficient conditions on non-self mapping which will ensure the existence of a fixed point.

In 1972, Assad and Kirk [1] introduced the definition of convexity in metric space and established a wonderful result. Since then there have been many theorems dealing with non-self mappings satisfying various types of contractive inequalities. The recent literature witness various extensions and generalizations of this theorem which includes Assad[2], Imdad et al. [7], Khan and Imdad [10], Khan [8] and others.

Here, let us recall the following results for the convenience of the reader.

Theorem 1.1 ([4]). *Let (X,d) be a complete metric space, $c \in [0,1), f : X \to X$ be a mapping such that, for each $x, y \in X$,*

$$\int_0^{d(fx,fy)} \phi(t)dt \leq \ c\int_0^{d(x,y)} \phi(t)dt,$$

where $\phi : R^+ \to R^+$ is a Lebesgue-integrable mapping which is summable,non-negative and such that, for each $\epsilon > 0$, $\int_0^{\epsilon} \phi(t)dt > 0$. Then f has a unique fixed point $z \in X$ such that, for each $x \in X, \lim_{n\to\infty} f^n x = z$.

Theorem 1.2 ([11])**.** *Let (X,d) be a complete metric space, $k \in [0,1), f : X \to X$ be a mapping such that, for each $x, y \in X$,*

$$\int_0^{d(fx,fy)} \phi(t)dt \leq \ k \int_0^{m(x,y)} \phi(t)dt,$$

where $m(x,y) = \max\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}\}$ and $\phi : R^+ \to R^+$ is a Lebesgue-integrable mapping which is summable,non-negative and such that, for each $\epsilon > 0$, $\int_0^\epsilon \phi(t)dt > 0$. Then f has a unique fixed point $z \in X$ such that, for each $x \in X, \lim_{n\to\infty} f^n x = z$.

In the present paper we prove a fixed point theorem for a single valued non-self mapping which generalize the earlier results due to Banach [3], Branciari [4], Ćirić [6], Rhoades [11], Khan [9] and others.

Before proving the results, we collect the following definitions for future discussion.

Definition 1.3. Let (X,d) be a metric space and K be a nonempty subset of a metric space X. Let a mapping $T : K \to X$ is said to be generalized contraction condition on K, if for each $x, y \in K$,

$$\int_0^{d(Tx,Ty)} \phi(t)dt \leq \ c \int_0^{m(x,y)} \phi(t)dt, c \in [0,1), \tag{1.1}$$

where $m(x,y) = h \max\left\{\frac{1}{2}d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{q}\right\}, 0 < h(1+h) \leq k < 1$ and any real number q satisfying $q \geq 1+2k$, and $\phi : R^+ \to R^+$ is a Lebesgue-integrable mapping which is summable, non-negative and such that

$$\int_0^\epsilon \phi(t)dt > 0 \text{ for each } \epsilon > 0. \tag{1.2}$$

Definition 1.4 ([1])**.** A metric space (X,d) is said to be metrically convex if for any $x, y \in X$ with $x \neq y$ there exists a point $z \in X, x \neq z \neq y$ such that

$$d(x,z) + d(z,y) = d(x,y).$$

2. Results

The result of this paper runs as follows.

Theorem 2.1. *Let (X,d) be a complete metrically convex metric space and K be a nonempty closed subset of X. Let $T : K \to X$ be a mapping satisfying generalized contraction condtion and for each $x \in \delta K, Tx \in K$. Then T has a unique fixed point $x \in K$ such that, for each $x \in K, \lim_{n\to\infty} T^n x = x$.*

Proof. Firstly, we proceed to construct two sequences $\{x_n\}$ and $\{y_n\}$ in the following way. Let $x_0 \in K$. Define $y_1 = Tx_0$. If $y_1 \in K$ set $y_1 = x_1$. If $y_1 \notin K$, then choose $x_1 \in \delta K$ so that

$$d(x_0, x_1) + d(x_1, y_1) = d(x_0, y_1).$$

If $y_2 \in K$ then set $y_2 = x_2$. If $y_2 \notin K$, then choose $x_2 \in \delta K$ so that

$$d(x_1, x_2) + d(x_2, y_2) = d(x_1, y_2).$$

Thus, repeating the foregoing arguments, one obtains two sequences $\{x_n\}$ and $\{y_n\}$ such that

(i) $y_{n+1} = Tx_n$,

(ii) $y_n = x_n$ if $y_n \in K$,

(iii) if $x_n \in \delta K$ then $d(x_{n-1}, x_n) + d(x_n, y_n) = d(x_{n-1}, y_n)$, where $y_n \notin K$.

Here, one obtains two types of sets we denote as follows:

$$P = \{x_i \in \{x_n\} : x_i = y_i\} \text{ and } Q = \{x_i \in \{x_n\} : x_i \neq y_i\}.$$

One can note that if $x_n \in Q$ then x_{n-1} and $x_{n+1} \in P$. We, wish to estimate $d(x_n, x_{n+1})$. Now, we distinguish the following three cases.

Case 1. If x_n and $x_{n+1} \in P$, then

$$\int_0^{d(x_n, x_{n+1})} \phi(t)dt = \int_0^{d(Tx_{n-1}, Tx_n)} \phi(t)dt \leq c \int_0^{m(x_{n-1}, x_n)} \phi(t)dt.$$

Since

$$\begin{aligned} &d(Tx_{n-1}, Tx_n) \\ &\leq h \max\left\{\frac{d(x_{n-1}, x_n)}{2}, d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n),\right. \\ &\qquad \left.\frac{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})}{q}\right\}, \end{aligned}$$

$$d(Tx_{n-1}, Tx_n) \leq h \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{q}\right\}. \tag{1.3}$$

Here, if $d(x_{n-1}, x_n)$ is maximum, then we have

$$d(Tx_{n-1}, Tx_n) \leq h\ d(x_{n-1}, x_n) < d(x_{n-1}, x_n).$$

Otherwise, if we suppose that $d(x_{n-1}, x_n) < d(x_n, x_{n+1})$, then we obtain

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq d(x_n, x_{n+1}),$$

which is a contradiction.

Now, if $\frac{d(x_{n-1},x_{n+1})}{q}$ is maximum then from equation (1.3) we have

$$\begin{aligned} d(x_n,x_{n+1}) = d(Tx_{n-1},Tx_n) &\leq \frac{h}{q}\{d(x_{n-1},x_n)+d(x_n,x_{n+1})\} \\ &\leq \frac{h}{q-h}d(x_{n-1},x_n) < hd(x_{n-1},x_n) < kd(x_{n-1},x_n) < d(x_{n-1},x_n). \end{aligned}$$

Therefore,

$$d(x_n,x_{n+1}) = d(Tx_{n-1},Tx_n) \leq d(x_{n-1},x_n).$$

Hence

$$\int_0^{d(x_n,x_{n+1})} \phi(t)dt \leq \ c\int_0^{m(x_{n-1},x_n)} \phi(t)dt.$$

Case 2. If $x_n \in P$ and $x_{n+1} \in Q$, then

$$d(x_n,x_{n+1}) + d(x_{n+1},y_{n+1}) = d(x_n,y_{n+1})$$

which in turn yields

$$d(x_n,x_{n+1}) \leq d(x_n,y_{n+1}).$$

Now, proceeding as in case 1, we have

$$\int_0^{d(x_n,x_{n+1})} \phi(t)dt \leq \ c\int_0^{m(x_{n-1},x_n)} \phi(t)dt.$$

Case 3. If $x_n \in Q$ and $x_{n+1} \in P$. Since $x_n \in Q$ and is a convex linear combination of x_{n-1} and y_n, it follows that

$$d(x_n,x_{n+1}) \leq \max\{d(x_{n-1},x_{n+1}),d(y_n,x_{n+1})\}.$$

If $d(x_{n-1},x_{n+1}) \leq d(x_{n+1},y_n)$, then proceeding as in case 1, we have

$$\int_0^{d(x_n,x_{n+1})} \phi(t)dt \leq \ c\int_0^{m(x_{n-1},x_n)} \phi(t)dt.$$

Otherwise, if $d(x_{n+1},y_n) \leq d(x_{n-1},x_{n+1})$ then, we have

$$\begin{aligned} &\int_0^{d(x_n,x_{n+1})} \phi(t)dt \leq \int_0^{d(x_{n-1},x_{n+1})} \phi(t)dt \\ &= \int_0^{d(Tx_{n-2},Tx_n)} \phi(t)dt \leq \ c\int_0^{m(x_{n-2},x_n)} \phi(t)dt. \end{aligned}$$

Here

$$\begin{aligned}
&d(x_n, x_{n+1}) \leq \\
&h \max \left\{ \frac{d(x_{n-2}, x_n)}{2}, d(x_{n-2}, Tx_{n-2}), d(x_n, Tx_n), \right. \\
&\qquad \left. \frac{d(x_{n-2}, Tx_n) + d(x_n, Tx_{n-2})}{q} \right\} \\
&\leq h \max \left\{ \frac{d(x_{n-2}, x_n)}{2}, d(x_{n-2}, x_{n-1}), d(x_n, x_{n+1}), \right. \\
&\qquad \left. \frac{d(x_{n-2}, x_{n+1}) + d(x_n, x_{n-1})}{q} \right\}.
\end{aligned}$$

Notice that

$$\begin{aligned}
&\frac{d(x_{n-2}, x_n)}{2} \leq \frac{d(x_{n-2}, x_{n-1}) \ + \ d(x_{n-1}, x_n)}{2} \\
&\leq \max \{d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n)\}.
\end{aligned}$$

Here, if

$$d(x_{n-2}, x_{n-1}) \leq d(x_{n-1}, x_n), \text{then } d(x_{n-2}, x_n) \leq d(x_{n-1}, x_n).$$

Otherwise, if

$$d(x_{n-1}, x_n) \leq d(x_{n-2}, x_{n-1}), \ \text{ then } d(x_{n-2}, x_n) \leq d(x_{n-2}, x_{n-1}).$$

Therefore, we obtain

$$\begin{aligned}
d(x_n, x_{n+1}) \leq h \max &\left\{ d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \right. \\
&\left. \frac{d(x_{n-2}, x_{n+1}) + d(x_n, x_{n-1})}{q} \right\}
\end{aligned}$$

which in turn yields

$$d(x_n, x_{n+1}) \leq \begin{cases} c \ d(x_{n-1}, x_n) & \text{if} \ \ d(x_{n-1}, x_n) \geq d(x_{n-2}, x_{n-1}) \text{ or} \\ c \ d(x_{n-2}, x_{n-1}) & \text{if} \ \ d(x_{n-1}, x_n) \leq d(x_{n-2}, x_{n-1}). \end{cases}$$

Thus, in all the cases, we have

$$d(x_n, x_{n+1}) \leq c \max\{d(x_{n-1}, x_n), d(x_{n-2}, x_{n-1})\}.$$

It can be easily shown by induction that for $n > 1$, we have

$$d(x_n, x_{n+1}) \leq c \max\{d(x_0, x_1), d(x_1, x_2)\}.$$

Thus,

$$\int_0^{d(x_n,x_{n+1})} \phi(t)dt \le c \int_0^{\max\{d(x_0,x_1),d(x_1,x_2)\}} \phi(t)dt$$

which implies that

$$\int_0^{d(x_n,x_{n+1})} \phi(t)dt \le c \max\left\{ \int_0^{d(x_0,x_1)} \phi(t)dt, \int_0^{d(x_1,x_2)} \phi(t)dt \right\}.$$

It follows that the sequence $\{d(x_n, x_{n+1})\}$ is monotonically decreasing. Hence $\int_0^{d(x_n,x_{n+1})} \phi(t)dt \to 0$ as $n \to \infty$. From equation (1.2) it implies that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Now, we prove that the sequence $\{x_n\}$ is a Cauchy sequence. Let on contrary that the sequence $\{x_n\}$ is not Cauchy. Then, there exists $\epsilon > 0$ for which we can find subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ such that $d(x_{m_k}, x_{n_k}) \ge \epsilon$.
Here, we proceed on the lines of Rhoades [11] it can be shown that the sequence $\{x_n\}$ is Cauchy and converges to a point say x.
From equation (1.1) we have

$$\begin{aligned} \int_0^{d(Tx,x_{n+1})} \phi(t)dt &\le c \int_0^{m(x,x_n)} \phi(t)dt \\ &\le c \max\Bigg\{ \int_0^{d(x,x_n)} \phi(t)dt, \int_0^{d(x,Tx)} \phi(t)dt, \int_0^{d(x_n,x_{n+1})} \phi(t)dt, \\ &\qquad \int_0^{d(x,x_{n+1})} \phi(t)dt, \int_0^{d(x_n,Tx)} \phi(t)dt, \Bigg\}. \end{aligned} \tag{1.4}$$

On letting $k \to \infty$, in equation (1.4) then, we have

$$\int_0^{d(Tx,x)} \phi(t)dt \le c \int_0^{d(Tx,x)} \phi(t)dt$$

which implies that

$$\int_0^{d(Tx,x)} \phi(t)dt = 0, \tag{1.5}$$

which from equation (1.5), implies that $d(Tx, x) = 0 \Rightarrow Tx = x$. This shows that x is a fixed point of T.

To prove that the uniqueness of fixed points. Let us suppose that x_1 and x_2 are two fixed points of T. Then

$$\begin{aligned} \int_0^{d(x_1,x_2)} \phi(t)dt &= \int_0^{d(Tx_1,Tx_2)} \phi(t)dt \le c \int_0^{m(x_1,x_2)} \phi(t)dt \\ &= c \max\left\{ \int_0^{d(x_1,x_2)} \phi(t)dt, 0 \right\} = c \int_0^{d(x_1,x_2)} \phi(t)dt, \end{aligned}$$

which implies that $\int_0^{d(x_1,x_2)} \phi(t)dt = 0$. Also imply that $d(x_1, x_2) = 0$ or $x_1 = x_2$. This shows the uniqueness of fixed point. This completes the proof. □

Remark 2.2. By setting $K = X$ and $\phi(t) = 1$ for each $t \geq 0$ in the Theorem 2.1, then we deduce a partial generalization of the result due to Banach [3].

Remark 2.3. By setting $K = X$ in the Theorem 2.1, then we deduce a sharpened version of the result due to Branciari [4].

Remark 2.4. By setting $K = X$ in the Theorem 2.1, then we deduce a partial generalization of the result due to Rhoades [11].

Remark 2.5. By setting some minor changes in the Theorem 2.1, then we deduce the result due to Khan [9].

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