Isomorphic properties of neighborly irregular cubic graph

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Abstract. Cubic graph can manage the uncertainty relevant to the inconsistent and indeterminate information of all real-world problems, in which fuzzy graphs possibly will not succeed into bringing about satisfactory results. Hence, in this research, we describe weak isomorphism, co-weak isomorphism and isomorphism of neighborly irregular cubic graphs. Some results on order, size and degree of nodes in isomorphic neighborly irregular and isomorphic highly irregular cubic graphs are discussed. Isomorphism between neighborly irregular and highly irregular cubic graphs are proved to be an equivalence relation. Likewise, density and balanced irregular cubic graphs are introduced.
Keywords: cubic graph, weak isomorphism, neighborly irregular, highly irregular.

1. Introduction

Fuzzy sets theory and its related fuzzy logic has been proposed by Zadeh [37]. A fuzzy set of a universe X is a function from X into the unit closed interval $[0, 1]$ of real number. In [38] Zadeh made an extension of the concept of a fuzzy set by an interval-valued fuzzy set, i.e., a fuzzy set with an interval-valued membership function. The first definition of fuzzy graphs was proposed by Kaufmann [12] in 1993. Interval-valued fuzzy sets have been actively used in real-life applications. For example, Sambuc [27] in medical diagnosis in thyroiditis pathology, Kohout

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[14] also in medicine, Turksen in preferences modeling [36], etc. These works and others show the importance of these sets. Jun et al. [11] introduced cubic sets. The fuzzy graph theory as a generalization of Euler's graph theory was first introduced by Rosenfeld [26] in 1975. Later, Bhattacharya [2] gave some remarks on fuzzy graphs and some operations on fuzzy graphs were introduced by Mordeson and Peng [15]. The complement of a fuzzy graph was dened by Mordeson [16] and further studied by Sunitha and Vijayakumar [28]. Hongmei and Lianhua gave the definition of interval-valued fuzzy graphs [10]. Akram and Dudek defined some operations on interval-valued fuzzy graphs [1]. Rashmanlou et al. [17, 18, 19] introduced some properties of highly irregular interval-valued fuzzy graphs, and new concepts of bipolar fuzzy graphs. Karunambigai et al. [13] introduced edge regular intuitionistic fuzzy graph. Samanta and Pal [29, 30, 31, 32, 33] defined fuzzy tolerance graph, fuzzy threshold graph, fuzzy k-competition graph and p-competition fuzzy graph and new concepts of fuzzy planar graph. Shao et al. [34, 35] investigated new concepts in intuitionistic fuzzy graphs and vague graphs. Borzooei et al. [3, 4, 5, 6] described several domination sets in vague graphs. Recently, some research works have been done by the authors in continuation of previous works related to interval valued intuitionistic fuzzy graphs, vague graphs, bipolar fuzzy graphs, and intuitionistic fuzzy graphs which are mentioned in [7, 8, 9, 21, 22, 23, 24, 25].

In this paper some properties of an edge regular cubic graph are given. Particularly, strongly regular, edge regular and biregular cubic graphs are defined and the necessary and sufficient condition for a cubic graph to be strongly regular is studied. Also, we have introduced a partially edge regular cubic graph and fully edge regular cubic graph with suitable illustrations.

2. Preliminaries

A graph is an ordered pair $G = (V, E)$, where V is the set of vertices of G and E is the set of edges of G. A subgraph of a graph $G = (V, E)$ is a graph $H = (W, F)$, where $W \subseteq V$ and $F \subseteq E$. A fuzzy graph $G = (\sigma, \mu)$ is a pair of functions $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$ with $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$, for all $u, v \in V$, where V is a finite non-empty set and $\wedge$ denote minimum. We introduce below necessary notions and present a few auxiliary results that will be used throughout the paper.

A map $\lambda : X \to [0, 1]$ is called a fuzzy subset of X. For any two fuzzy subsets λ and μ of X, $\lambda \leq \mu$ means that, for all $x \in X, \lambda(x) \leq \mu(x)$. The symbol $\lambda \wedge \mu$ and $\lambda \vee \mu$ will mean the following fuzzy subsets of X.

$(\lambda \wedge \mu)(x) = \lambda(x) \wedge \mu(x)$ and $(\lambda \vee \mu)(x) = \lambda(x) \vee \mu(x)$, for all $x \in X$.

Let X be a non-empty set. A function $A : X \to [I]$ is called an interval-valued fuzzy set(shortly, an IVF set) in X. Let $[I]^X$ stands for the set of all IVF sets in X. For every $A \in [I]^X$ and $x \in X$, $A(x) = [A^-(x), A^+(x)]$ is called the degree of membership of an element x to A, where $A^- : X \to I$ and $A^+ : X \to I$ are fuzzy sets in X which are called a lower fuzzy set and an upper fuzzy set in

X, respectively. For simplicity, we denote $A = [A^-, A^+]$. For every $A, B \in [I]^X$, we define $A \subseteq B$ if and only if $A(x) \leq B(x)$, for all $x \in X$.

Definition 2.1 ([10]). *Let $A = [A^-, A^+]$, and $B = [B^-, B^+]$ be two interval-valued fuzzy set in X. Then, we define* $r\min\{A(x), B(x)\} = [\min\{A^-(x), B^-(x)\}, \min\{A^+(x), B^+(x)\}], r\max\{A(x), B(x)\} = [\max\{A^-(x), B^-(x)\}, \max\{A^+(x), B^+(x)\}]$.

Definition 2.2 ([11]). *Let X be a non-empty set. By a cubic set in X, we mean a structure $A = \{< x, A(x), \lambda(x) : x \in X\}$ in which A is an interval-valued fuzzy sets in X and λ is a fuzzy set in X. A cubic set $A = \{< x, A(x), \lambda(x) : x \in X >\}$ is simply denoted by $A =< A, \lambda >$. The collection of all cubic sets in X is denoted by $CP(X)$.*

Definition 2.3 ([20]). *A cubic graph is a triple $G = (G^*, P, Q)$ where $G^* = (V, E)$ is a graph, $P = (\widetilde{\mu_P}, \lambda_P)$ is a cubic set on V and $Q = (\widetilde{\mu_Q}, \widetilde{\lambda_Q})$ is a cubic set on $V \times V$ such that $\widetilde{\mu_Q}(xy) \leq r\min\{\widetilde{\mu_P}(x), \widetilde{\mu_P}(y)\}$ and $\widetilde{\lambda_Q}(xy) \geq$* $\max\{\widetilde{\lambda_P}(x), \widetilde{\lambda_P}(y)\}$.

The underlying crisp graph of a cubic graph $G = (A, B)$, is the graph $G = (V, E)$, where $V = \{v : \widetilde{\mu_P}>0$ and $\lambda_P>0\}$ and $E = \big\{\{u, v\} : \widetilde{\mu_Q}(\{u, v\})>0, \widetilde{\mu_Q}(\{u, v\})>0\big\}$. V is called the vertex set and E is called the edge set. A cubic graph maybe also denoted as $G = (V, E)$.

Definition 2.4 ([20]). *A cubic graph $G = (G^*, P, Q)$ is called complete if $\widetilde{\mu_Q}(xy) = r\min\{\mu_P(x), \mu_P(y)\}$ and $\widetilde{\lambda_Q}(xy) = \max\{\widetilde{\lambda_P}(x), \widetilde{\lambda_P}(y)\}$ for all $x, y \in V$.*

Definition 2.5 ([20]). *A cubic graph $G = (G^*, P, Q)$ is called strong if $\widetilde{\mu_Q}(xy) = r\min\{\mu_P(x), \mu_P(y)\}$ and $\widetilde{\lambda_Q}(xy) = \max\{\widetilde{\lambda_P}(x), \widetilde{\lambda_P}(y)\}$, for all $xy \in E$.*

Definition 2.6 ([20]). *The complement of a cubic graph $G = (A, B)$ is a cubic graph $\overline{G} = (\overline{G^*}, \overline{P}, \overline{Q})$, where $\overline{P} = (\overline{\widetilde{\mu_P}}, \overline{\lambda_P})$ and $\overline{Q} = (\overline{\widetilde{\mu_Q}}, \overline{\widetilde{\lambda_Q}})$ is defined by: $\overline{\widetilde{\mu_Q}}(xy) = r\min\{\mu_P(x), \mu_P(y)\} - \widetilde{\mu_Q}(xy)$ and $\overline{\widetilde{\lambda_Q}}(xy) = \widetilde{\lambda_Q}(xy) -$* $\max\{\widetilde{\lambda_P}(x), \widetilde{\lambda_P}(y)\}$.

Definition 2.7. *Let $G = (V, E)$ be a cubic graph.*

(i) The neighborhood degree of a vertex v is defined as $d_N(v) = (d_{N_{\widetilde{\mu_P}}(v)}, d_{N_{\lambda_Q}}(v))$, where $d_{N_{\mu_P}}(v) = (\sum_{w \in N(\widetilde{\mu_P^-})} \widetilde{\mu_P^-}(w), \sum_{w \in N(\mu_P^+)} \mu_P^+(w))$ and $d_{N_{\lambda_Q}}(v) = \sum_{w \in N(\lambda_Q)} \lambda_Q(w)$.

(ii) The degree of a vertex v_i is defined by $d_G(v_i) = (d_{\widetilde{\mu_P}}(v_i), d_{\lambda_Q}(v_i)) = (k_1, k_2)$, where $k_1 = d_{\widetilde{\mu_P}}(v_i) = (\sum_{v_i \neq v_j} \widetilde{\mu_Q^-}(v_i v_j), \sum_{v_i \neq v_j} \widetilde{\mu_Q^+}(v_i v_j))$ and $k_2 = d_{\lambda_Q}(v_i) = \sum_{v_i \neq v_j} \lambda_Q(v_i v_j)$.

Definition 2.8. *A cubic graph $G = (V, E)$ is said to be*

(i) $(k_1, k_2)-$regular if $d_G(v_i) = (k_1, k_2)$, for all $v_i \in V$ and also G is said to be a regular cubic graph of degree (k_1, k_2).

(ii) bipartite if the vertex set V can be partitioned into two non-empty sets V_1 and V_2 such that:

(a) $\widetilde{\mu_Q}(v_iv_j) = 0$ and $\lambda_Q(v_iv_j) = 0$, if $(v_i, v_j) \in V_1$ or $(v_i, v_j) \in V_2$;

(b) $\widetilde{\mu_Q}(v_iv_j) = 0$, $\lambda_Q(v_iv_j) > 0$, if $v_i \in V_1$ or $v_j \in V_2$;

(c) $\widetilde{\mu_Q}(v_iv_j) > 0$, $\lambda_Q(v_iv_j) = 0$, if $v_i \in V_1$ or $v_j \in V_2$, for some i and j.

Definition 2.9. *Let $G^* = (V, E)$ be a crisp graph and let $e = v_iv_j$ be an edge in G^*. Then, the degree of an edge $e = v_iv_j \in E$ is defined as $d_{G^*}(v_iv_j) = d_{G^*}(v_i) + d_{G^*}(v_j) - 2$.*

Definition 2.10. *(i) The order of G is defined to be $O(G) = (O_{\widetilde{\mu_P}}, O_{\lambda_P})$, where $O_{\widetilde{\mu_P}} = \sum_{u \in V} \widetilde{\mu_P(u)}$ and $O_{\lambda_P} = \sum_{u \in V} \lambda_P(u)$.*

(ii) The size of G is defined to be $S(G) = (S_{\widetilde{\mu_Q}}(G), S_{\lambda_Q}(G))$, where $S_{\widetilde{\mu_Q}}(G) = \sum_{u \neq v} \widetilde{\mu_Q(uv)}$ and $S_{\lambda_Q}(G) = \sum_{u \neq v} \lambda_Q(uv)$.

3. Isomorphic properties of neighborly irregular and highly irregular cubic graphs

In this section, we describe weak isomorphism, co-weak isomorphism, and isomorphism of neighborly irregular cubic graphs and prove that isomorphisms between neighborly irregular and highly irregular cubic graphs are equivalence relations.

Definition 3.1. *Let $G = (A, B)$ be a connected cubic graph. G is said to be a neighborly irregular cubic graph if every two adjacent vertices of G have distinct degree.*

Definition 3.2. *Let $G = (A, B)$ be a connected cubic graph. G is said to be a highly irregular cubic graph if every vertex of G is adjacent to vertices with distinct degrees.*

Definition 3.3. *A homomorphism h of neighborly irregular cubic graphs (highly irregular cubic graphs) G_1 and G_2 is a mapping $h : V_1 \to V_2$ which satisfies the following conditions:*

(a) $\widetilde{\mu}_{A_1}(u_1) \leq \widetilde{\mu}_{A_2}(h(u_1))$, $\lambda_{A_1}(u_1) \geq \lambda_{A_2}(h(u_1))$, for all $u_1 \in V_1$,

(b) $\widetilde{\mu}_{B_1}(u_1v_1) \leq \widetilde{\mu}_{B_2}(h(u_1)h(v_1))$, $\lambda_{B_1}(u_1v_1) \geq \lambda_{B_2}(h(u_1)h(v_1))$, for all $u_1v_1 \in E_1$.

Example 3.1. Let $V_1 = \{a, b, c, d\}$ and $V_2 = \{u, v, x, w\}$. Consider two neighborly irregular cubic graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$. The below graph in Figure 1 shows the homomorphism of neighborly irregular cubic graphs G_1 and G_2. There is a homomorphism $h : V_1 \to V_2$ such that $h(a) = u$, $h(b) = v$, $h(c) = x$, $h(d) = w$.

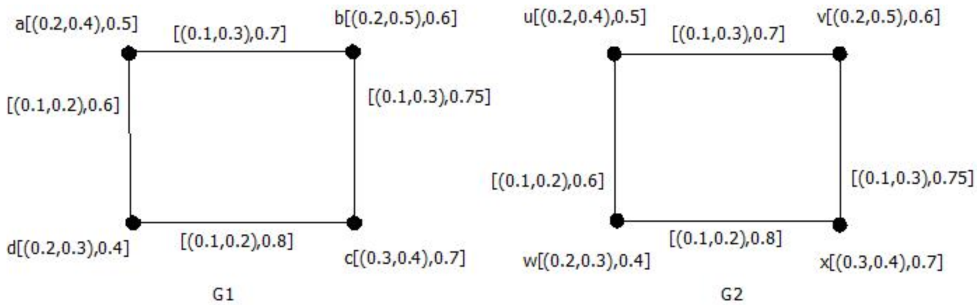

Figure 1: Homomorphism of neighborly irregular cubic graphs G_1 and G_2

Definition 3.4. *A weak isomorphism h of neighborly irregular cubic graphs (highly irregular cubic graphs) G_1 and G_2 is a bijective mapping $h : V_1 \to V_2$ which satisfies the following conditions:*

(c) *h is homomorphism,*

(d) $\widetilde{\mu}_{A_1}(u_1) = \widetilde{\mu}_{A_2}(h(u_1)),\ \lambda_{A_1}(u_1) = \lambda_{A_2}(h(u_1))$, *for all* $u_1 \in V_1$.

Example 3.2. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two highly irregular cubic graphs defined as follows.

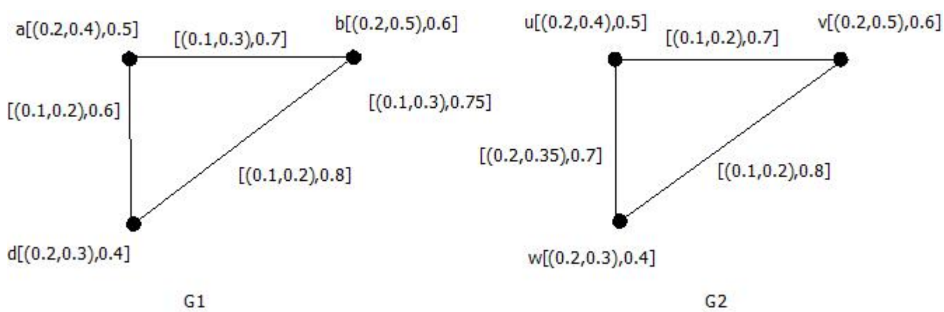

Figure 2: Weak isomorphism of highly irregular cubic graphs G_1 and G_2

There is a weak isomorphism $h : V_1 \to V_2$ such that $h(a) = u$, $h(b) = v$, $h(c) = w$.

Definition 3.5. *A co-weak isomorphism h of neighborly irregular cubic graphs (highly irregular cubic graphs) G_1 and G_2 is a bijective mapping $h : V_1 \to V_2$ which satisfies the following conditions:*

(e) *h is homomorphism,*

(f) $\widetilde{\mu}_{B_1}(u_1v_1) = \widetilde{\mu}_{B_2}(h(u_1)h(v_1)), \lambda_{B_1}(u_1v_1) = \lambda_{B_2}(h(u_1)h(v_1))$, *for all* $u_1v_1 \in E_1$.

Example 3.3. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two neighborly irregular cubic graphs given in Figure 3 as follows.

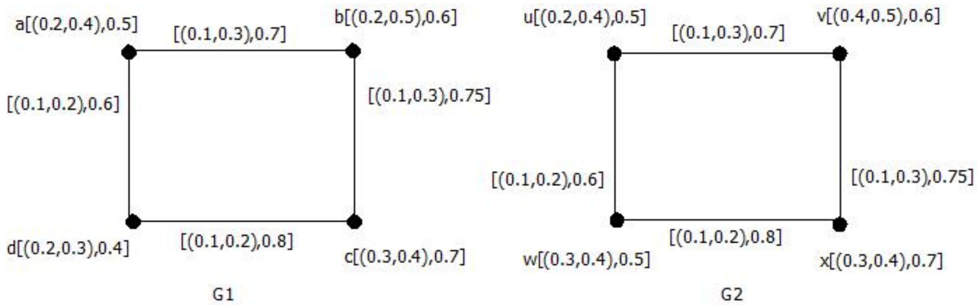

Figure 3: Co-weak isomorphism of highly irregular cubic graphs G_1 and G_2

There is a co-weak isomorphism $h : V_1 \to V_2$ such that $h(a) = u, h(b) = v, h(c) = w, h(d) = x$.

Definition 3.6. *An isomorphism h of neighborly irregular cubic graphs (highly irregular cubic graphs) G_1 and G_2 is a bijective mapping $h : V_1 \to V_2$ which satisfies the following conditions:*

(g) $\widetilde{\mu}_{A_1}(u_1) = \widetilde{\mu}_{A_2}(h(u_1)),\ \lambda_{A_1}(u_1) = \lambda_{A_2}(h(u_1)),$

(h) $\widetilde{\mu}_{B_1}(u_1v_1) = \widetilde{\mu}_{B_2}(h(u_1)h(v_1)),\ \lambda_{B_1}(u_1v_1) = \lambda_{B_2}(h(u_1)h(v_1))$, *for all* $u_1 \in V_1$ *and* $u_1v_1 \in E_1$.

Theorem 3.1. *For any two isomorphic neighborly irregular cubic graphs, their order and size are the same.*

Proof. Let h from G_1 to G_2 be an isomorphism between the neighborly irregular cubic graphs G_1 and G_2 with the underlying sets V_1 and V_2 respectively. Then $\widetilde{\mu}_{A_1}(u) = \widetilde{\mu}_{A_2}(h(u)),\ \lambda_{A_1}(u) = \lambda_{A_2}(h(u))$, for all $u \in V_1$, $\widetilde{\mu}_{B_1}(uv) = \widetilde{\mu}_{B_2}(h(u)h(v)),\ \lambda_{B_1}(uv) = \lambda_{B_2}(h(u)h(v))$, for all $u, v \in V$. So, we have

$$\begin{aligned}
O(G_1) &= (\sum_{u_1\in V_1} \widetilde{\mu}_{A_1}(u_1), \sum_{u_1\in V_1} \lambda_{A_1}(u_1)) \\
&= (\sum_{u_1\in V_1} \widetilde{\mu}_{A_2}(h(u_1)), \sum_{u_1\in V_1} \lambda_{A_2}(h(u_1))) \\
&= (\sum_{u_2\in V_2} \widetilde{\mu}_{A_2}(u_2), \sum_{u_2\in V_2} \lambda_{A_2}(u_2)) = O(G_2)
\end{aligned}$$

$$
\begin{aligned}
S(G_1) &= (\sum_{u_1v_1\in E_1}\widetilde{\mu}_{B_1}(u_1v_1), \sum_{u_1v_1\in E_1}\lambda_{B_1}(u_1v_1)) \\
&= (\sum_{u_1,v_1\in V_1}\widetilde{\mu}_{B_2}(h(u_1)h(v_1)), \\
&\qquad \sum_{u_1,v_1\in V_1}\lambda_{B_2}(h(u_1)h(v_1))) \\
&= (\sum_{u_2v_2\in E_2}\widetilde{\mu}_{B_2}(u_2v_2), \sum_{u_2v_2\in E_2}\lambda_{B_2}(u_2v_2)) \\
&= S(G_2). \quad \square
\end{aligned}
$$

Remark 3.1. The above theorem is also true for highly irregular cubic graphs.

Corollary 3.1. *Converse of Theorem 3.10 need not be true for both neighborly irregular and highly irregular cubic graphs. It can be seen from the following example.*

Example 3.4. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two highly irregular cubic graphs given in Fig. 4.

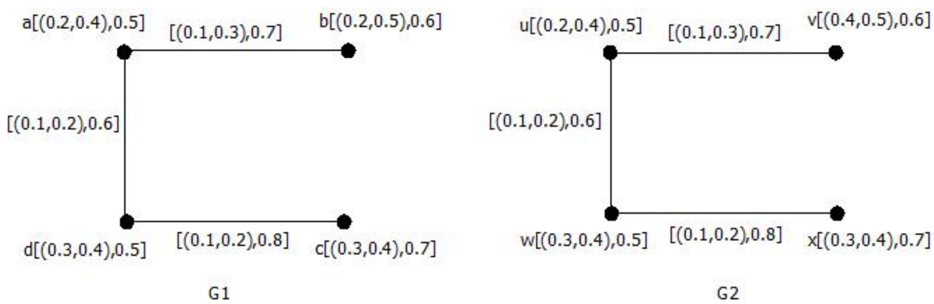

Figure 4: Highly irregular cubic graphs G_1 and G_2

In both the graphs, $O(G_1) = O(G_2) = [(1.0, 1.8), 2.3]$ and $S(G_1) = S(G_2) = [(0.3, 0.7), 2.1]$. But G_1 is not isomorphic to G_2

Proposition 3.1. *If two neighborly irregular cubic graphs (highly irregular cubic graphs) are weak isomorphic, then their orders are same. But neighborly irregular cubic graphs (highly irregular cubic graphs) of same order need not be weak isomorphic.*

Example 3.5. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two neighborly irregular cubic graphs defined as follows.

In both graphs, $O(G_1) = O(G_2) = [(1.0, 1.7), 2.3]$, but they are not weak isomorphic.

Proposition 3.2. *If two neighborly irregular cubic graphs (highly irregular cubic graphs) are co-weak isomorphic, their sizes are the same. But neighborly*

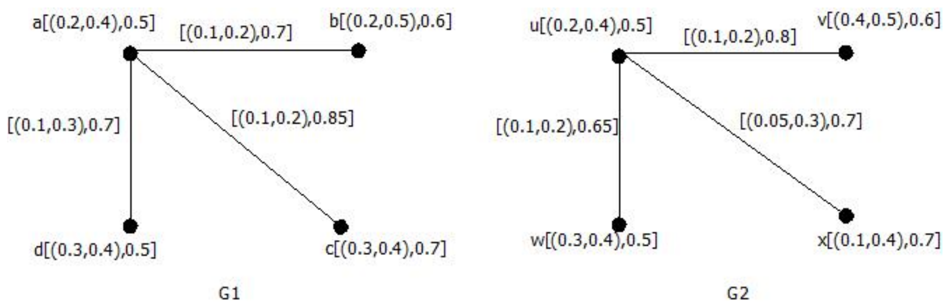

Figure 5: Neighborly irregular cubic graphs G_1 and G_2

irregular cubic graphs (highly irregular cubic graphs) of same size need not be co-weak isomorphic.

Example 3.6. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two highly irregular cubic graphs given as follows.

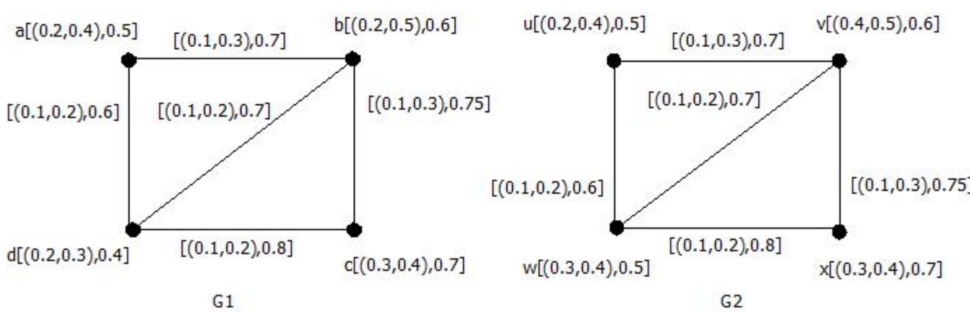

Figure 6: Highly irregular cubic graphs G_1 and G_2

The size of the above two highly irregular cubic graphs are same ($S(G_1) = S(G_2) = [(0.5, 1.2), 3.55]$. But they are not co-weak isomorphic.

Theorem 3.2. *If G_1 and G_2 are isomorphic neighborly irregular cubic graphs, then the degrees of the corresponding vertices u and $h(u)$ are preserved.*

Proof. Let $h : G_1 \to G_2$ be an isomorphism between the neighborly irregular cubic graphs G_1 and G_2 with underlying sets V_1 and V_2 respectively. Then $\widetilde{\mu}_{B_1}(u_1v_1) = \widetilde{\mu}_{B_2}(h(u_1)h(v_1)),\ \lambda_{B_1}(u_1v_1) = \lambda_{B_2}(h(u_1)h(v_1))$, for all $u_1, v_1 \in V_1$.

Therefore,

$$\begin{aligned} d_{\widetilde{\mu}}(u_1) &= \sum_{u_1,v_1\in V_1} \widetilde{\mu}_{B_1}(u_1v_1) \\ &= \sum_{u_1,v_1\in V_1} \widetilde{\mu}_{B_2}(h(u_1)h(v_1)) = d_{\widetilde{\mu}}(h(u_1)), \\ d_{\lambda(u_1)} &= \sum_{u_1,v_1\in V_1} \lambda_{B_1}(u_1v_1) \\ &= \sum_{u_1,v_1\in V_1} \lambda_{B_2}(h(u_1)h(v_1)) = d_{\lambda(h(u_1))}, \end{aligned}$$

for all $u_1 \in V_1$. That is, the degrees of the corresponding vertices of G_1 and G_2 are the same. □

Remark 3.2. The above theorem is true for highly irregular cubic graphs also.

Corollary 3.2. *Converse of the Theorem 3.18 and Remark 3.19 need not be true. This can be seen from the following example.*

Example 3.7. Consider two neighborly irregular cubic graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ given in Figure 7.

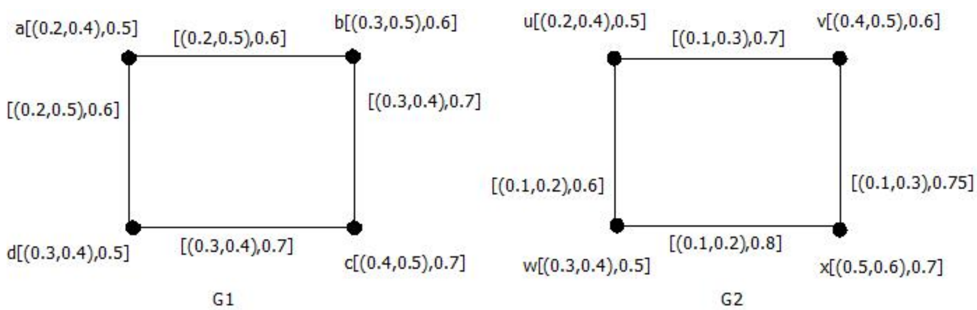

Figure 7: Neighborly irregular cubic graphs G_1 and G_2 with same degrees

$d(a) = d(u) = [(0.4, 1), 1.2]$, $d(b) = d(v) = [(0.5, 0.9), 1.3]$, $d(c) = d(w) = [(0.6, 1), 1.4]$, $d(d) = d(x) = [(0.5, 0.9), 1.3]$. In both the graphs the degrees of the corresponding vertices are the same, but G_1 and G_2 are only co-weak isomorphic but not isomorphic.

Remark 3.3. Isomorphism between neighborly irregular cubic graphs is an equivalence relation.

Theorem 3.3. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two highly irregular cubic graphs. G_1 and G_2 are isomorphic if and only if their complements are isomorphic. But the complement need not be highly irregular.*

Proof. Assume that G_1 and G_2 are isomorphic, with their isomorphism map $h : V_1 \to V_2$. By the definition of complement we have,

$$\begin{aligned}
\overline{\widetilde{\mu}_{B_1}}(xy) &= \min(\widetilde{\mu}_{A_1}(x), t_{A_1}(y)) - \widetilde{\mu}_{B_1}(xy) \\
&= \min(\widetilde{\mu}_{A_2}(h(x)), \widetilde{\mu}_{A_2}(h(y))) \\
&- t_{B_2}(h(x)h(y)) \\
\overline{\lambda_{B_1}}(xy) &= \lambda_{B_1}(xy) - \max(\lambda_{A_1}(x), \lambda_{A_1}(y)) \\
&= \lambda_{B_2}(h(x)h(y)) \\
&- \max(\lambda_{A_2}(h(x)), \lambda_{A_2}(h(y))),
\end{aligned}$$

for all $xy \in E_1$. Hence $\overline{G}_1 \cong \overline{G}_2$. The proof of the converse part is straight forward. □

Theorem 3.4. *Let G_1 and G_2 be two highly irregular cubic graphs. If G_1 is weak isomorphic with G_2, then $\overline{G}_1$ is weak isomorphic with $\overline{G}_2$.*

Proof. Assume that G_1 and G_2 are weak isomorphic, with their weak isomorphism map $h : V_1 \to V_2$. As $h^{-1} : V_2 \to V_1$ is bijective also, for every $x_2 \in V_2$ there is an $x_1 \in V_1$ such that $h^{-1}(x_2) = x_1$. By the definition of complement we have,

$$\begin{aligned}
\overline{\widetilde{\mu}_{B_1}}(x_1y_1) &= \min(\widetilde{\mu}_{A_1}(x_1), \widetilde{\mu}_{A_1}(y_1)) \\
&- \widetilde{\mu}_{B_1}(x_1y_1), \ for \ all \ x_1, y_1 \in V_1.
\end{aligned}$$

$$\begin{aligned}
\overline{\widetilde{\mu}_{B_1}}(h^{-1}(x_2)h^{-1}(y_2)) &\geq \min(\widetilde{\mu}_{A_2}(h(x_1)), \\
&\widetilde{\mu}_{A_2}(h(y_1))) - \widetilde{\mu}_{B_2}(h(x_1)h(y_1)) \\
&= \min(\widetilde{\mu}_{A_2}(x_2), \widetilde{\mu}_{A_2}(y_2)) - \widetilde{\mu}_{B_2}(x_2y_2) \\
&= \overline{\widetilde{\mu}_{B_2}}(x_2y_2), \text{for all } x_2, y_2 \in V_2.
\end{aligned}$$

i.e. $\overline{\widetilde{\mu}_{B_2}}(x_2y_2) \leq \overline{\widetilde{\mu}_{B_1}}(h^{-1}(x_2)h^{-1}(y_2))$, for all $x_2, y_2 \in V_2$. Also, $\overline{\lambda_{B_1}}(x_1y_1) = \lambda_{B_1}(x_1y_1) - \max(\lambda_{A_1}(x_1), \lambda_{A_1}(y_1))$, for all $x_1, y_1 \in V_1$. So,

$$\begin{aligned}
\overline{\lambda_{B_1}}(h^{-1}(x_2)h^{-1}(y_2)) &\leq \lambda_{B_2}(h(x_1)h(y_1)) \\
&- \max(\lambda_{A_2}(h(x_1)), \lambda_{A_2}(h(y_1))) \\
&= \lambda_{B_2}(x_2y_2) - \max(\lambda_{A_2}(x_2), \lambda_{A_2}(y_2)) \\
&= \overline{\lambda_{B_2}}(x_2y_2), \quad \text{for all } x_2, y_2 \in V_2
\end{aligned}$$

i.e. $\overline{\lambda_{B_2}}(x_2y_2) \geq \overline{\lambda_{B_1}}(h^{-1}(x_2)h^{-1}(y_2))$, for all $x_2, y_2 \in V_2$. Therefore, $h^{-1} : V_2 \to V_1$ is a weak isomorphism between $\overline{G}_1$ and $\overline{G}_2$. □

Remark 3.4. The above theorem is true for neighborly irregular cubic graphs.

Theorem 3.5. *Let G_1 and G_2 be two highly irregular cubic graphs (neighborly irregular cubic graphs). If G_1 is a co-weak isomorphic with G_2, then there exists a homomorphism between $\overline{G}_1$ and $\overline{G}_2$.*

Definition 3.7. *A cubic graph G is said to be a self weak complementary if G is weak isomorphic with $\overline{G}$.*

Theorem 3.6. *Let G be a self weak complementary highly irregular cubic graph, then*

$$\sum_{x\neq y}\widetilde{\mu}_B(xy) \leq \frac{1}{2}\sum_{x\neq y}\min(\widetilde{\mu}_A(x),\widetilde{\mu}_A(y)) \quad and$$
$$\sum_{x\neq y}\lambda_B(xy) \geq \frac{1}{2}\sum_{x\neq y}\max(\widetilde{\mu}_A(x),\widetilde{\mu}_A(y)).$$

Proof. Let $G=(A,B)$ be a self weak complementary highly irregular cubic graph of $G^*=(V,E)$. Using the definition of complement for all $x,y\in V$ we have,

$$\begin{aligned}
\widetilde{\mu}_B(xy) &\leq \overline{\widetilde{\mu}_B}(h(x)h(y))\\
&= \min(\widetilde{\mu}_A(h(x)),\widetilde{\mu}_A(h(y)))-\widetilde{\mu}_B(h(x)h(y)),\\
f_B(xy) &\geq \overline{\lambda_B}(h(x)h(y))=\lambda_B(h(x)h(y))\\
&- \max(\lambda_A(h(x)),\lambda_A(h(y))),\\
\widetilde{\mu}_B(xy) &+ \widetilde{\mu}_B(h(x)h(y))\leq \min(\widetilde{\mu}_A(h(x)),\widetilde{\mu}_A(h(y)))\\
\lambda_B(xy) &+ \max(\lambda_A(h(x)),\lambda_A(h(y)))\geq \lambda_B(h(x)h(y)).
\end{aligned}$$

So,

$$\begin{aligned}
\sum_{x\neq y}\widetilde{\mu}_B(xy) &+ \sum_{x\neq y}\widetilde{\mu}_B(h(x)h(y))\\
&\leq \sum_{x\neq y}\min(\widetilde{\mu}_A(h(x)),\widetilde{\mu}_A(h(y)))
\end{aligned}$$

and

$$\begin{aligned}
\sum_{x\neq y}\lambda_B(xy) &+ \sum_{x\neq y}\lambda_B(h(x)h(y))\\
&\geq \sum_{x\neq y}\max(\lambda_A(h(x)),\lambda_A(h(y))).
\end{aligned}$$

Hence, $2\sum_{x\neq y}\widetilde{\mu}_B(xy) \leq \sum_{x\neq y}\min(\widetilde{\mu}_A(x),\widetilde{\mu}_A(y))$ and $2\sum_{x\neq y}\lambda_B(xy) \geq \sum_{x\neq y}\max(\lambda_A(x),\lambda_A(y))$. Therefore, $\sum_{x\neq y}\widetilde{\mu}_B(xy) \leq \frac{1}{2}\sum_{x\neq y}\min(\widetilde{\mu}_A(x), \widetilde{\mu}_A(y))$ and $\sum_{x\neq y}\lambda_B(xy)\geq\frac{1}{2}\sum_{x\neq y}\max(\lambda_A(x),\lambda_A(y))$. □

Definition 3.8. *Let $G=(A,B)$ be an irregular cubic graph then, the irregularity of G is defined as $Irreg(G)=(Irreg_{\widetilde{\mu}}(G), Irreg_{\lambda(G)})$ where $Irreg_{\widetilde{\mu}}(G)=$*

$\sum_{xy\in E}|d_{\widetilde{\mu}}(x)-d_{\widetilde{\mu}}(y)|$ *and* $Irreg_{\lambda}(G)=\sum_{xy\in E}|d_{\lambda}(x)-d_{\lambda}(y)|$, *for all* $x,y\in V$. *The total irregularity of the irregular cubic graph is defined as*

$$Irreg_{total}(G)=\frac{1}{2}\sum_{x,y\in V}|d(x)-d(y)|.$$

Example 3.8. Consider an irregular cubic graph G as in Figure 8.

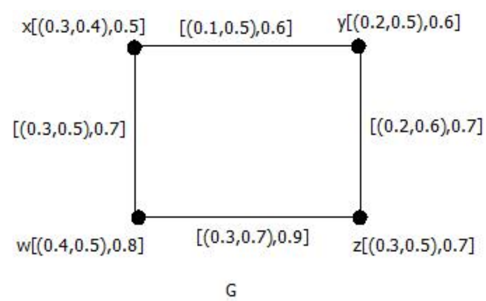

Figure 8: Irregular cubic graph G

Here, $d(x)$=$[(0.4,1),1.2]$, $d(y)$=$[(0.3,1.1),1.3]$, $d(z)$=$[(0.5,1.3),1.5]$, $d(w)=[(0.6,1.2),1.5]$. It is easy to show that $Irreg(G)=[(0.1,0.1),0.1]+[(0.2,0.2),0.2]+[(0.1,0.1),0.1]+[(0.2,0.2),0.4]=[(0.6,0.6),0.8]$. But the total irregularity is $Irreg_{total}(G)=\frac{1}{2}\{[(0.1,0.1),0.1]+[(0.1,0.3),0.3]+[(0.2,0.2),0.4]+[(0.2,0.2),0.2]+[(0.3,0.1),0.3]+[(0.1,0.1),0.3]\}=[(0.5,0.5),0.8]$.

Remark 3.5. (i) For any irregular cubic graph G, $Irreg_{total}(G)\leq Irreg(G)$.

(ii) Let $G=(A,B)$ be the irregular cubic graph which is complete then, $Irreg(G)=2[Irreg_{total}(G)]$.

Definition 3.9. *Let* $G=(A,B)$ *be an irregular cubic graph. The density of* G *is defined as* $D(G)=(D_{\widetilde{\mu}}(G),D_{\lambda}(G))$ *where* $D_{\widetilde{\mu}}(G)=\frac{2\sum_{x,y\in V}\widetilde{\mu}_B(xy)}{\sum_{xy\in E}\widetilde{\mu}_A(x)\wedge\widetilde{\mu}_A(y)}$, *for all* $x,y\in V$ *and* $D_{\lambda}(G)=\frac{2\sum_{x,y\in V}\lambda_B(xy)}{\sum_{xy\in E}\lambda_A(x)\vee\lambda_A(y)}$, *for all* $x,y\in V$.

Example 3.9. In Example 3.30 we have:

$$\begin{aligned}&D(G)=(D_{\widetilde{\mu}}(G),D_{\lambda}(G))\\&=\left(\frac{2[0.1+0.2+0.3+0.3]}{0.2+0.2+0.3+0.3},\frac{2[0.5+0.6+0.7+0.5]}{0.5+0.5+0.5+0.5},\frac{2[0.6+0.7+0.7+0.7]}{0.6+0.7+0.8+0.8}\right)\\&=[(1.8,2.42),1.86].\end{aligned}$$

Theorem 3.7. *Let* $G=(A,B)$ *be the irregular cubic graph with* $\widetilde{\mu}_B(xy)=\frac{1}{2}(\widetilde{\mu}_A(x)\wedge\widetilde{\mu}_A(y))$ *and* $\lambda_B(xy)=\frac{1}{2}(\lambda_A(x)\vee\lambda_A(y))$ *then,* $D(G)=(1,1)$.

Proof.

$$\begin{aligned} D(G) &= \Big(\frac{2\sum_{x,y\in V}\widetilde{\mu}_B(xy)}{\sum_{xy\in E}(\widetilde{\mu}_A(x)\wedge\widetilde{\mu}_A(y))}, \\ &= \Big(\frac{2\sum_{x,y\in V}\frac{1}{2}(\widetilde{\mu}_A(x)\wedge\widetilde{\mu}_A(y))}{\sum_{xy\in E}(\widetilde{\mu}_A(x)\wedge\widetilde{\mu}_A(y))}, \\ &\quad \frac{2\sum_{x,y\in V}\frac{1}{2}(\lambda_A(x)\vee\lambda_A(y))}{\sum_{xy\in E}(\lambda_A(x)\vee\lambda_A(y))}\Big) \\ &= \Big(\frac{2(\frac{1}{2})\sum_{x,y\in V}(\widetilde{\mu}_A(x)\wedge\widetilde{\mu}_A(y))}{\sum_{xy\in E}(\widetilde{\mu}_A(x)\wedge\widetilde{\mu}_A(y))}, \\ &\quad \frac{2(\frac{1}{2})\sum_{x,y\in V}(\lambda_A(x)\vee\lambda_A(y))}{\sum_{xy\in E}(\lambda_A(x)\vee\lambda_A(y))}\Big) = (1,1). \quad \square \end{aligned}$$

Theorem 3.8. *Let G be an irregular and complete cubic graph then, $D(G) = (2,2)$.*

Proof. Since G is complete and irregular we have $\widetilde{\mu}_B(xy) = \widetilde{\mu}_A(x)\wedge\widetilde{\mu}_A(y)$ and $\lambda_B(xy) = \lambda_A(x)\vee\lambda_A(y)$, for all $x,y\in V$. Now, $\sum_{x,y\in V}\widetilde{\mu}_B(xy) = \sum_{xy\in E}(\widetilde{\mu}_A(x)\wedge \widetilde{\mu}_A(y))$, $\sum_{x,y\in V}\lambda_B(xy) = \sum_{xy\in E}(\lambda_A(x)\vee\lambda_A(y))$. Therefore,

$$\begin{aligned} D(G) &= \Big(\frac{2\sum_{x,y\in V}\widetilde{\mu}_B(xy)}{\sum_{xy\in E}(\widetilde{\mu}_A(x)\wedge\widetilde{\mu}_A(y))}, \\ &= \Big(\frac{2\sum_{xy\in E}(\widetilde{\mu}_A(x)\wedge\widetilde{\mu}_A(y))}{\sum_{xy\in E}(\widetilde{\mu}_A(x)\wedge\widetilde{\mu}_A(y))}, \\ &\quad \frac{2\sum_{xy\in E}(\lambda_A(x)\vee\lambda_A(y))}{\sum_{xy\in E}(\lambda_A(x)\vee\lambda_A(y))}\Big) = (2,2). \quad \square \end{aligned}$$

Definition 3.10. *An irregular cubic graph G is said to be balanced if $D_{\widetilde{\mu}}(H) \leq D_{\widetilde{\mu}}(G)$ and $D_{\lambda}(H) \leq D_{\lambda}(G)$, for all subgraph H of G.*

Example 3.10. Consider an irregular cubic graph G as in Figure 9.

Here, $D(G) = (D_{\widetilde{\mu}}(G), D_{\lambda}(G)) = [(1.3, 1.5), 2]$, and subgraph of G are $H_1 = \{x,y\}$, $H_2 = \{x,z\}$, $H_3 = \{x,w\}$, $H_4 = \{y,z\}$, $H_5 = \{y,w\}$, $H_6 = \{z,w\}$, $H_7 = \{x,y,z\}$, $H_8 = \{x,z,w\}, H_9 = \{x,y,w\}$, $H_{10} = \{y,z,w\}$, $H_{11} = \{x,y,z,w\}$.

Now, density $(D_{\widetilde{\mu}}(H), D_{\lambda}(H))$ is $D(H_1) = [(1.3,1.5),2]$, $D(H_2) = [(0,0),0]$, $D(H_3) = [(1.3,1.5),2]$, $D(H_4) = [(1.25,1.5),2]$, $D(H_5) = [(0,0),0]$, $D(H_6) = [(1.25,1.5),2]$, $D(H_7) = [(1.3,1.5),2]$, $D(H_8) = [(0.92,1.5),2]$, $D(H_9) = [(1.3,1.5),2]$, $D(H_{10}) = [(1.25,1.5),2]$, $D(H_{11}) = [(1.3,1.5),2]$. So, $D(H) \leq D(G)$, for all subgraphs H of G. Hence, G is balanced irregular cubic graph.

Theorem 3.9. *Let $G = (A,B)$ be an irregular cubic graph and all the edges of G are strong then, G is balanced.*

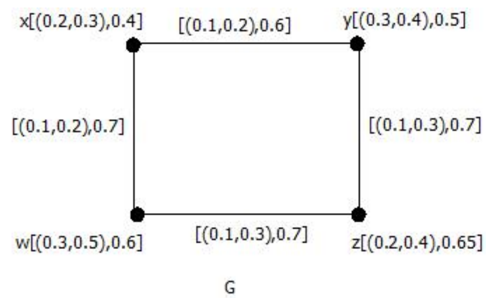

Figure 9: Balanced irregular cubic graph G

Proof. Since all the edges are strong then $\widetilde{\mu}_B(xy) = \widetilde{\mu}_A(x) \wedge \widetilde{\mu}_A(y)$ and $f_B(xy) = f_A(x) \vee f_A(y)$. Now, by Theorem 3.8, $D(G) = (2, 2)$. For the subgraphs H, $(D_{\widetilde{\mu}}(H), D_\lambda(H)) = (2, 2)$ if the vertices of H having edges, otherwise $(D_{\widetilde{\mu}}(H), D_\lambda(H)) = (0, 0)$ i.e. $D_{\widetilde{\mu}}(H) \leq D_{\widetilde{\mu}}(G)$ and $D_\lambda(H) \leq D_\lambda(G)$ which implies G is balanced. □

4. Conclusion

Fuzzy graph theory has numerous applications in modern science and technology, especially in the fields of operations research, neural networks, and decision making. Since the cubic models give more precision, flexibility and compatibility to the system as compared to the classical and fuzzy models, hence, in this paper, the definitions of partial edge regular and fully edge regular cubic graph are given and several properties of edge regular cubic graph are studied. Also, we have introduced the condition under which edge regular cubic graph and totally edge regular cubic graph are equivalent. In our future work, we will introduce cubic incidence graphs and study the concepts of connected perfect dominating set, regular perfect dominating set, inverse perfect dominating set, and independent perfect dominating set on cubic incidence graph.

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