A-contractions relative to a weak distance

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Abstract. Among self maps of a metric space, Akram et. al have considered a class of contractions called A-contractions. On a metric space the concept of a weak distance was defined by Kada et.al.

In this paper we introduce *A-contraction relative to a weak distance* among the self maps of a metric space and prove three fixed point theorems for such maps.
Keywords: weak distance, A-contraction relative to weak distance.

1. Introduction

In [3], Kada, Suzuki and Takahashi have introduced the notion of a weak distance on a metric space as follows

Definition 1.1. Let (X, d) be a metric space. A function $p : X^2 \to [0, \infty)$ is called a *weak distance* on X if it satisfies the following conditions:

(a) $p(x, y) \leqslant p(x, z) + p(z, y)$, for all $x, y, z \in X$;

(b) to each $x \in X$, $p(x, \cdot) : X \to [0, \infty)$ is lower semi continuous. That is, for each sequence $\{y_n\}$ in X with $y_n \to y$ as $n \to \infty$ we have

$$p(x, y) \leq \liminf_{n\to\infty} p(x, y_n);$$

(c) to each $\varepsilon > 0$ there is a $\delta > 0$ such that $p(z, x) < \delta$ and $p(z, y) < \delta$ imply $d(x, y) < \varepsilon$.

Example 1.2. (i) Let (X, d) be a metric space. If a function $p : X^2 \to [0, \infty)$ is defined by $p(x, y) = d(x, y)$, for all $x, y \in X$, then p is a weak distance on X. That is every metric on a set is a weak distance on it.

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(ii) Let $X_0 = [0, 1)$ with $d(x, y) = |x - y|$ for $x, y \in X_0$. Then, (X_0, d) is a metric space. If we define $p : X_0^2 \to [0, \infty)$ by $p(x, y) = y$ for $x, y \in X_0$, it can be verified that p is a weak distance on X_0.

Remark 1.3. If p is a weak distance on a metric space then neither $p(x, y) = 0$ implies $x = y$ nor $x = y$ implies $p(x, y) = 0$. Also, $p(x, y)$ and $p(y, x)$ need not be equal.

Note that $p(x, 0) = 0$ for all $x \in X_0$ in Example 1.2 (ii).

For other examples we refer to [3].

Definition 1.4. Let $\mathbb{R}_+ = [0, \infty)$ and A be the class of all functions $\alpha : \mathbb{R}_+^3 \to \mathbb{R}_+$ such that:

(i) α is continuous on $\mathbb{R}_+^3$ (with respect to Euclidean metric on $\mathbb{R}_+^3$);

(ii) $a \le \alpha(a, b, b)$ or $a \le \alpha(b, a, b)$ or $a \le \alpha(b, b, a)$, for all $a, b \in \mathbb{R}_+$ implies $a \le k.b$ for some $k \in [0, 1)$.

For example, it is proved in [1] (Proof of Theorem 3) that the function α defined by

$$\alpha(u, v, w) = k. \max\{v, w\}, \text{ for } u, v, w \in \mathbb{R}_+, \tag{1}$$

where $k \in [0, 1)$, is in the class A.

Now, we define a new class of contractions on a metric space with a weak distance on it.

Definition 1.5. Suppose (X, d) is a metric space with a weak distance p on it. A self map f of X is said to be a $A -$ *contraction relative to* p if there is an $\alpha \in A$, such that $p(fx, fy) \le \alpha(p(x, y), p(fx, x), p(fy, y))$, for all $x, y \in X$ holds.

Remark 1.6. Any A-contraction relative to d (which is a weak distance, by Example 1.2(i)) on the metric space (X, d) is an A-*contraction* introduced by Akram, Zafar and Siddiqui [1] who observed that contractions due to Kannan [4], Khan [5], Bianchini [2] and Reich [6] are all A-contractions for suitably chosen $\alpha \in A$. Also, they have proved that the class of A-contractions is a wider class which includes these contractions properly. For instance, the A-contraction with $\alpha \in A$ defined in (1) is due to Bianchini [2] which satisfies the condition:

$$d(fx, fy) \le k. \max\{d(fx, x), d(fy, y)\}, \forall x, y \in X \text{ and for some } k \in [0, 1). \tag{2}$$

Now, we give an example of a metric space (X, d) with a weak distance p on it and a A-contraction relative to p but not relative to d.

Example 1.7. Let (X_0, d) be the metric space with weak distance p given in Example 1.2(ii). Define $f : X_0 \to X_0$ by$f(x) = \frac{2}{3}x^2$ for $x \in X_0$. First note that there is a $k \in [0, 1)$ such that

$$(3) \qquad p(fx, fy) \leq k.\max\{p(fx, x), p(fy, y)\}, \quad \text{for all } x, y \in X_0.$$

In fact, (3) holds if and only if $fy \leq k.\max\{x, y\}$ for all $x, y \in X_0$ for some $k \in [0, 1)$ (in view of definition of p here); and this holds with $k = \frac{2}{3}$. Therefore, f is a A-contraction relative to p with $\alpha \in A$ as defined in (1).

Observe that, in (3) if we replace p by d we get (2), the condition for f to be a contraction due to Bianchini [2]. Now, we remark that (2) can not hold for f since $d(f(\frac{3}{4}), f(0))=|f(\frac{3}{4}) - f(0)| = \frac{3}{8}$ and $\max\{d(f(\frac{3}{4}), \frac{3}{4}),\ d(f0, 0)\} = \max\{\frac{3}{8}, 0\} = \frac{3}{8}$ gives that $d(f(\frac{3}{4}), f(0)) \not\leq k.\max\{d(f(\frac{3}{4}), \frac{3}{4}), d(f0, 0)\}$, for any $k \in [0, 1)$. That is (2) fails if $x = \frac{3}{4}$ and $y = 0$.

Thus f is not a Bianchini contraction on X_0 but it is a Bianchini contraction relative to p.

The purpose of this paper is to prove three fixed point theorems for A-contractions relative to a weak distance on a complete metric space. Also, we note that Theorem 5,Theorem 6 and Theorem 7 of [1] are special cases of our results.

2. A fixed point theorem

The following lemma proved in [3] is needed:

Lemma 2.1. *Suppose (X, d) is a complete metric space with a weak distance p on it and suppose $\{x_n\}$ is a sequence in X; $\{a_n\}$ and $\{b_n\}$ are sequences in $\mathbb{R}_+$ such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$ and $x, y, z \in X$, Then*

(i) $p(x_n, x) \leq a_n$ and $p(x_n, y) \leq b_n$ for all $n \geq 1$ imply $x = y$. In particular, $p(z, x)=p(z, y)=0$, for some $z \in X$ implies $x = y$;

(ii) $p(x_m, x_n) \leq a_n$ for all $m > n \geq 1$ implies $\{x_n\}$ is a Cauchy sequence in (X, d).

Theorem 2.2. *Suppose (X, d) is a complete metric space with a weak distance p on it such that $p(x, .)$ is continuous for each $x \in X$. If $f : X \to X$ is a A-contraction relative to p, then:*

(i) for any $x_0 \in X$ the sequence $\{x_n\}$ defined by $x_n = fx_{n-1}$ for $n \geq 1$ converges to some $z \in X$;

(ii) any such $z \in X$ with $p(z, fz) = p(fz, z)$ is the unique fixed point of f if and only if $p(z, z) = 0$.

Proof. Take any $x_0 \in X$ and $x_n = fx_{n-1}$, for $n \geq 1$. Write $P_n = p(x_{n+1}, x_n)$ and $Q_n^r = p(x_{n+r}, x_n)$, for $n \geq 0$, $r \geq 0$. Since f is a A- contraction relative to p there is an $\alpha \in A$ such that

$$p(fx, fy) \leq \alpha(p(x,y), p(fx,x), p(fy,y)), \text{ for all } x, y \in X. \tag{4}$$

Now, for $n \geq 1$ we have by (4), that

$$\begin{aligned} P_n &= p(fx_n, fx_{n-1}) \leq \alpha(p(x_n, x_{n-1}), p(fx_n, x_n), p(fx_{n-1}, x_{n-1})) \\ &= \alpha(p(x_n, x_{n-1}), p(x_{n+1}, x_n), p(x_n, x_{n-1})) \\ &= \alpha(P_{n-1}, P_n, P_{n-1}) \end{aligned}$$

so, that by (ii) of Definition 1.4, we get $P_n \leq kP_{n-1}$ for some $k \in [0,1)$ which on repeated use gives

$$P_n \leq kP_{n-1} \leq k^2 P_{n-2} \leq .... \leq k^n P_0 \tag{5}$$

and therefore

$$\lim_{n\to\infty} P_n = 0. \tag{6}$$

By (a) of Definition 1.1 and (5), we have

$$\begin{aligned} Q_n^r &\leq p(x_{n+r}, x_{n+r-1}) + p(x_{n+r-1}, x_{n+r-2}) + ... + p(x_{n+1}, x_n) \\ &= P_{n+r-1} + P_{n+r-2} + ... + P_n \\ &\leq k^n(1 + k + ... + k^{r-1})P_0 < \frac{k^n P_0}{1-k} = a_n. \end{aligned} \tag{7}$$

That is, if $m = n + r > n \geq 1$ then $p(x_m, x_n) = Q_n^r < a_n$, where $a_n \to 0$ as $n \to \infty$; and hence by (ii) of Lemma 2.1, the sequence $\{x_n\}$ is a Cauchy sequence in (X,d). Since (X,d) is complete there is a $z \in X$ such that $\lim_{n\to\infty} x_n = z$, proving part (i) of the theorem.

To prove part (ii), first suppose $z \in X$ is such that $p(z, fz) = p(fz, z)$ and $p(z,z) = 0$ and we now prove that z is the unique fixed point of f.

For any $n \geq 1$, we have by (4), that

$$\begin{aligned} p(fz, x_{n+1}) &= p(fz, fx_n) \leq \alpha(p(z, x_n), p(fz, z), p(fx_n, x_n)) \\ &= \alpha(p(z, x_n), p(fz, z), P_n) \end{aligned}$$

in which letting $n \to \infty$, using the continuity of α on $\mathbb{R}_+^3$ as well as the lower semi continuity of $p(z, .)$ and (6) we get

$$p(fz, z) \leq \alpha(p(z,z), p(fz,z), 0) = \alpha(0, p(fz,z), 0)$$

so that,by (ii) of Definition 1.4, we find $p(fz,z) \leq k.0 = 0$ showing $p(fz,z) = 0$. Then, $p(fz,z) = p(z,fz) = 0$. Thus $p(z,fz) = p(z,z) = 0$ so, that by (i) of Lemma 2.1, we get $fz = z$ showing z is a fixed point of f.

If $u \in X$ is another fixed point of f, then, by (4),

$$p(u,u) = p(fu, fu) \leq \alpha(p(u,u), p(fu,u), p(fu,u)) = \alpha(p(u,u), p(u,u), p(u,u))$$

which gives $p(u,u) \leq k.p(u,u)$, so that $p(u,u) = 0$. Also,

$$\begin{aligned} p(u,z) = p(fu, fz) &\leq \alpha(p(u,z), p(fu,u), p(fz,z)) \\ &= \alpha(p(u,z), p(u,u), p(z,z)) \\ &= \alpha(p(u,z), 0, 0) \text{ gives } p(u,z) \leq k.0 = 0 \end{aligned}$$

and hence $p(u,z) = 0$. Thus, $p(u,u) = p(u,z) = 0$ showing $u = z$ by (i) of Lemma 2.1. Thus z is the unique fixed point of f.

Conversely, if $z \in X$ with $p(z, fz) = p(fz, z)$ is the unique fixed point of f then $p(z,z) = p(fz, fz) \leq \alpha(p(z,z), p(fz,z), p(fz,z))$ gives $p(z,z) \leq k.p(z,z)$ and hence $p(z,z) = 0$. □

Remark 2.3. In the case $p = d$, the metric on the set X then the condition (ii) of Theorem 2.2 holds and hence every A-contraction on a complete metric space has a unique fixed point in X which is Theorem 5 of [1].

3. Common fixed point theorems

In this section we prove common fixed point theorems for A-contractions relative to a weak distance on complete metric spaces. We introduce a notation.

Definition 3.1. Suppose (X,d) is a metric space with weak distance p on it.

If an ordered pair (F,G) of self maps of X is such that there is an $\alpha \in A$ for which $p(Fx, Gy) \leq \alpha(p(x,y), p(Fx,x), p(Gy,y))$, for all $x, y \in X$, holds then, we write $(F,G) \in \mathscr{C}_\alpha^p$.

Observe that $f : X \to X$ is a A-contraction relative to p if $(f,f) \in \mathscr{C}_\alpha^p$.

Remark 3.2. It is easy to show that $(F,G) \in \mathscr{C}_\alpha^p$ if and only if $(G,F) \in \mathscr{C}_\alpha^p$ in case p is such that $p(x,y) = p(y,x)$ for all $x, y \in X$. In particular if δ is a metric on X then $(F,G) \in \mathscr{C}_\alpha^\delta \Leftrightarrow (G,F) \in \mathscr{C}_\alpha^\delta$.

Theorem 3.3. *Suppose (X,d) is a complete metric space with weak distance p on it for which*

(i) $d(x,y) \leq p(x,y)$ for all $x, y \in X$.

Suppose f and g are self maps of X; of which g is continuous on (X,d), and

(ii) $(f,g) \in \mathscr{C}_\alpha^p$ and $(g,f) \in \mathscr{C}_\alpha^p$, for some $\alpha \in A$

then, for any $x_0 \in X$ the sequence $\{x_n\}$ defined for $n \geq 1$ by $x_n = fx_{n-1}$ or gx_{n-1} according as n is even or odd,converges to a point $z \in X$. If $p(z,z) = 0$ then z is the unique common fixed point of f and g.

Proof. Take any $x_0 \in X$ and for any $n \geq 1$, let

$$x_n = \begin{cases} fx_{n-1}, & \text{if } n \text{ is even,} \\ gx_{n-1}, & \text{if } n \text{ is odd.} \end{cases}$$

Write

$$p(x_{n+1}, x_n) = \begin{cases} P_n, & \text{if } n \text{ is odd,} \\ Q_n, & \text{if } n \text{ is even.} \end{cases}$$

Now, if n is odd then since $(f, g) \in \mathscr{C}_\alpha^p$

$$\begin{aligned} P_n &= p(x_{n+1}, x_n) = p(fx_n, gx_{n-1}) \\ &\leq \alpha(p(x_n, x_{n-1}), p(fx_n, x_n), p(gx_{n-1}, x_{n-1})) \\ &= \alpha(Q_{n-1}, P_n, Q_{n-1}) \leq k.Q_{n-1}, \end{aligned} \tag{8}$$

for some $k \in [0, 1)$, and if n is even then since $(g, f) \in \mathscr{C}_\alpha^p$ we get

$$\begin{aligned} Q_n &= p(x_{n+1}, x_n) = p(gx_n, fx_{n-1}) \\ &\leq \alpha(p(x_n, x_{n-1}), p(gx_n, x_n), p(fx_{n-1}, x_{n-1})) \\ &= \alpha(P_{n-1}, Q_n, P_{n-1}) \leq k.P_{n-1} \end{aligned} \tag{9}$$

so that (8) and (9) give

$$P_n \leq k^2.P_{n-2} \text{ and } Q_n \leq k^2.Q_{n-2}, \quad \text{for each } n \geq 1. \tag{10}$$

That is for each $n \geq 1$

$$p(x_{n+1}, x_n) \leq k^2.p(x_{n-1}, x_{n-2}) \tag{11}$$

which on repeated use gives

$$p(x_{n+1}, x_n) \leq k^n.p(x_1, x_0) \text{ or } p(x_{n+1}, x_n) \leq k^n.p(x_0, x_1)$$

according as n is even or odd so that $p(x_{n+1}, x_n) \to 0$ as $n \to \infty$, since $0 \leq k < 1$. Therefore, if $b_n = p(x_{n+r}, x_{n+r-1}) + p(x_{n+r-1}, x_{n+r-2}) + ... + p(x_{n+1}, x_n)$ for any $n \geq 1$ and $r \geq 1$ then

$$\lim_{n\to\infty} b_n = 0. \tag{12}$$

Now, for any $m = n + r > n \geq 1$, by (a) of Definition 1.1, we have that $p(x_m, x_n) = p(x_{n+r}, x_n) \leq p(x_{n+r}, x_{n+r-1}) + p(x_{n+r-1}, x_{n+r-2}) + ... + p(x_{n+1}, x_n) = b_n$ and therefore, in view of (12), we get by (ii) of Lemma 2.1 that $\{x_n\}$ is a Cauchy sequence in the complete metric space(X, d) so that there is a $z \in X$ with $\lim_{n\to\infty} x_n = z$. That is,

$$\lim_{n\to\infty} d(x_n, z) = 0. \tag{13}$$

Therefore, the continuity of $d(x,y)$ in x; the continuity of g on X (by the hypothesis) and (13) give

$$\begin{aligned} 0 &= \lim_{n\to\infty} d(x_{2n+1}, z) = \lim_{n\to\infty} d(gx_{2n}, z) \\ &= d(\lim_{n\to\infty} gx_{2n}, z) = d(g(\lim_{n\to\infty} x_n), z) = d(gz, z) \end{aligned}$$

which shows $gz = z$. That is, z is a fixed point of g.

Also, $p(fz,z) = p(fz, gz) \le \alpha(p(z,z), p(fz,z), p(gz,z)) = \alpha(p(z,z), p(fz,z), p(z,z))$ gives $p(fz,z) \le k.p(z,z)$ so that $p(fz,z) = 0$ if $p(z,z) = 0$. Hence, by (i) of the Theorem 3.3, $d(fz,z) = 0$ giving that $fz = z$. Thus, z is a common fixed point of f and g if $p(z,z) = 0$.

To prove the uniqueness,we first observe that $p(u,u) = 0$, for any common fixed point of f and g. In fact, $p(u,u) = p(fu, gu) \le \alpha(p(u,u), p(fu,u), p(gu,u)) = \alpha(p(u,u), p(u,u), p(u,u))$ gives $p(u,u) \le k.p(u,u)$, so that $p(u,u) = 0$.

Now, if u and z are two common fixed points of f and g then $p(u,z) = p(fu, gz) \le \alpha(p(u,z), p(fu,u), p(gz,z)) = \alpha(p(u,z), 0, 0)$, so that $p(u,z) \le k.0 = 0$ giving $p(u,z) = 0$. Thus, $p(u,z) = p(u,u) = 0$ shows $u = z$ completing the proof of the theorem. □

Remark 3.4. In case $p = \delta$, another metric on (X,d) then in view of Remark 3.2, we get Theorem 7 of [1].

Theorem 3.5. *Suppose (X,d) is a complete metric space with weak distance p on it such that $p(x,y) = p(y,x)$ for all $x, y \in X$ and $p(x,.)$ is continuous for each $x \in X$. Suppose $\{f_n\}$ is a sequence of self maps of X such that $(f_i, f_j) \in \mathscr{C}_\alpha^p$, for some $\alpha \in A$ and for any $i \ne j$. Then*

(i) for any $x_0 \in X$ the sequence $\{x_n\}$ defined by $x_n = f_n x_{n-1}$ for $n \ge 1$ converges to some $z \in X$;

(ii) any such $z \in X$ with $p(z,z) = 0$ is the unique common fixed point of $\{f_n\}$.

Proof. By hypothesis there is an $\alpha \in A$ such that

$$p(f_i x, f_i y) \le \alpha(p(x,y), p(f_i x, x), p(f_j y, y)), \tag{14}$$

for all $x, y \in X$ and for $i \ne j$. Take any $x_0 \in X$ and let $x_n = f_n x_{n-1}$ for $n \ge 1$. Now, for any $n \ge 1$ we have by (14),

$$\begin{aligned} p(x_{n+1}, x_n) &= p(f_{n+1} x_n, f_n x_{n-1}) \\ &\le \alpha(p(x_n, x_{n-1}), p(f_{n+1} x_n, x_n), p(f_n x_{n-1}, x_{n-1})) \\ &= \alpha(p(x_n, x_{n-1}), p(x_{n+1}, x_n), p(x_n, x_{n-1})) \end{aligned}$$

which shows that $p(x_{n+1}, x_n) \le k.p(x_n, x_{n-1})$, for some $k \in [0,1)$, which on repeated use gives

$$p(x_{n+1}, x_n) \le k.p(x_n, x_{n-1}) \le ..... \le k^n.p(x_1, x_0). \tag{15}$$

Now, (15) implies

$$\lim_{n\to\infty} p(x_{n+1}, x_n) = 0 \tag{16}$$

and that $\{x_n\}$ is a Cauchy sequence in the complete metric space (X, d). Hence there is a $z \in X$ such that $\lim_{n\to\infty} x_n = z$, proving part (i) of the theorem.

(ii) Suppose such $z \in X$ is with $p(z, z) = 0$. To prove z is a common fixed point of$\{f_n\}$, fix n and consider

$$\begin{aligned} p(z, f_n z) &\le p(z, x_{m+1}) + p(x_{m+1}, f_n z) = p(z, x_{m+1}) + p(f_{m+1}x_m, f_n z) \\ &\le p(z, x_{m+1}) + \alpha(p(x_m, z), p(x_{m+1}, x_m), p(f_n z, z)) \\ &= p(z, x_{m+1}) + \alpha(p(z, x_m), p(x_{m+1}, x_m), p(z, f_n z)), \end{aligned} \tag{17}$$

where we used the hypothesis that $p(x, y) = p(y, x)$, for all $x, y \in X$.

In (17) letting $m \to \infty$,using the continuity of α as well as the continuity of $p(z, .)$ and (16) we get $p(z, f_n z) \le p(z, z) + \alpha(p(z, z), 0, p(z, f_n z)) = 0 + \alpha(0, 0, p(z, f_n z))$ showing $p(z, f_n z) \le k.0$ which implies $p(z, f_n z) = 0$. Thus, $p(z, f_n z) = p(z, z) = 0$, so that $f_n z = z$. That is z is a common fixed point of $\{f_n\}$.

The uniqueness can be proved on the lines similar to earlier theorem. □

Remark 3.6. In case $p = d$, the metric on the space X, our Theorem 3.5 gives Theorem 6 of [1].

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