L^p-boundedness of fractional maximal operator

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Abstract. In this paper, we are introducing a new generalised one-sided fractional maximal operator $M^+_{g,\alpha}$ and our aim is to study its boundedness between weighted Lebesgue spaces L^p_w.
Keywords: maximal operator, fractional maximal operator, weighted Lebesgue spaces, $S^+_{p,g,\alpha}$ class.

1. Introduction

The Hardy-Littlewood maximal function for a locally integrable function f on R^n is defined as

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|}\left(\int_Q |f(t)|dt\right),$$

where the sup is taken over all the cubes Q in R^n and $|Q|$ denotes the Lebesgue measure of Q.

A weight w is measurable, locally integrable function which is finite and positive almost everywhere on $(0,\infty)$. The space L^p_w consists of all measurable functions f for which the quantity

$$\|f\|_{L^p_w} = \left(\int |f|^p w\right)^{\frac{1}{p}}$$

is finite.

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B. Muckenhoupt [7] in 1972, proved that for $1 < p < \infty$,

$$\int_{-\infty}^{+\infty} (Mf(x))^p \, w(x)dx \leq C \int_{-\infty}^{+\infty} |f(x)|^p w(x)dx,$$

for all f holds if and only if $w \in A_p$, i.e.,

$$\left(\frac{1}{h}\int_a^{a+h} w(x)dx\right)\left(\frac{1}{h}\int_a^{a+h} w(x)^{\frac{-1}{p-1}}dx\right)^{p-1} \leq C.$$

E.T. Sawyer [9] in 1982 proved that if w and v are non-negative functions on R^n and $1 < p \leq q < \infty$, then

$$\left(\int (Mf(x))^q w(x)dx\right)^{\frac{1}{q}} \leq C\left(\int f^p(x)v(x)dx\right)^{\frac{1}{p}},$$

for all f in $L^p(v)$ if and only if

$$\int \left((M(\chi_Q v^{1-p'})(x))^q w(x)dx\right)^{\frac{1}{q}} \leq C\left(\int_Q v^{1-p'}(x)dx\right)^{\frac{1}{p}} < \infty,$$

for all cubes Q.

In [2], the authors studied the operator M_α^+ defined by

$$M_\alpha^+ f(x) = \sup_{h>0} h^{\alpha-1}\left(\int_x^{x+h} |f(t)|dt\right)$$

and gave a new characterisation for the pairs of weights in which M_α^+ is bounded.

Consider now the maximal operator defined as

$$M_g^+ f(x) = \sup_{h>0}\left(\int_x^{x+h} |f(t)|g(t)dt\right)\left(\int_x^{x+h} g(t)dt\right)^{-1},$$

where g is a positive and locally integrable function on R.

In [5], F.J.Martin-Reyes et al characterised the pairs of weights such that the operator M_g^+ where g is a locally integrable and positive function in R is of strong and weak type with different sets of weights.

In this paper, we are introducing a new one-sided fractional maximal operator

$$M_{g,\alpha}^+ f(x) = \sup_{h>0}\left(\int_x^{x+h} |f(t)|g(t)dt\right)\left(\int_x^{x+h} g(t)dt\right)^{\alpha-1}, 0 \leq \alpha < 1.$$

Remark 1.1. For $\alpha = 0$, $M_{g,\alpha}^+$ becomes the operator M_g^+ studied in the paper [5].

For $g = 1$, $M_{g,\alpha}^+$ reduces to the operator M_α^+ studied in the paper [2].

For $\alpha = 0$ and $g = 1$, $M_{g,\alpha}^+$ gives the operator M^+ studied in [8].

For the further development of these topics, one may refer to [1], [3], [4], [10] and [6].

Our primal aim here is to characterise the pairs of non-negative functions (u, v) for which the operator $M^+_{g,\alpha}$ is bounded from L^p_v to L^p_u.

Throughout the paper, $|Q|$ and χ_Q denote the Lebesgue measure and the characteristic function of the set Q, respectively, and $0.\infty$ is assumed to be 0. The letter C denotes a positive constant which is not necessarily the same at each occurence. The conjugate exponent of p is p' where $\frac{1}{p} + \frac{1}{p'} = 1$ and $p \geq 1$.

Firstly, we introduce the following definitions for $0 \leq \alpha < 1$.

Definition 1.1. *Let $p > 1$ and u, v be nonnegative functions. We say that a pair (u, v) satisfies condition $S^+_{p,g,\alpha}$ if there exists a constant $C > 0$ such that for every $I = (a, b)$ with $\int_{(-\infty,a)} u > 0$,*

$$\int_a^b \left(M^+_{g,\alpha}\left(\chi_I g^{\frac{1}{p-1}}\sigma\right)\right)^p u \leq C \int_a^b g^{p'}\sigma < \infty,$$

where $\sigma = v^{\frac{-1}{p-1}}$.

Definition 1.2. *Let $p > 1$ and u, v be nonnegative functions. We say that a pair (u, v) satisfies condition $S^-_{p,g,\alpha}$ if there exists a constant $C > 0$ such that for every $I = (a, b)$ with $\int_{(-\infty,a)} u > 0$,*

$$\int_a^b \left(M^-_{g,\alpha}\left(\chi_I g^{\frac{1}{p-1}}\sigma\right)\right)^p u \leq C \int_a^b g^{p'}\sigma < \infty,$$

where $\sigma = v^{\frac{-1}{p-1}}$.

Definition 1.3. *Let $p > 1$ and u, v be nonnegative functions. We say that a pair (u, v) satisfies condition $S_{p,g,\alpha}$ if there exists a constant $C > 0$ such that for every $I = (a, b)$ with $\int_{(-\infty,a)} u > 0$,*

$$\int_a^b \left(M_{g,\alpha}\left(\chi_I g^{\frac{1}{p-1}}\sigma\right)\right)^p u \leq C \int_a^b g^{p'}\sigma < \infty,$$

where $\sigma = v^{\frac{-1}{p-1}}$.

2. Strong boundedness

We prove the following:

Theorem 2.1. *Let u and v be weight functions and $p > 1$. The operator $M^+_{g,\alpha}$ is bounded from L^p_v to L^p_u if and only if $(u, v) \in S^+_{p,g,\alpha}$.*

Proof. Let $I=(a,b)$. The necessity follows by taking $f=\chi_I g^{\frac{1}{p-1}} v^{\frac{-1}{p-1}}$.

For the sufficiency, without loss of generality, we assume the function f in L_v^p to be non-negative bounded function having compact support. Let $O_k = \{x \in R : M_{g,\alpha}^+ f(x) > 2^k\} \cap (-N,+\infty)$, where N is any positive integer and k is any positive real number. We note that O_k is open for each k, therefore, there exists a sequence $\{I_{jk}\}$ of open pairwise disjoint intervals with the following properties: $O_k = \cup_j I_{jk}$ and

$$\int_x^{b_{jk}} gf \geq 2^k \left(\int_x^{b_{jk}} g\right)^{(1-\alpha)} \tag{1}$$

holds for all $x \in I_{jk} = (a_{jk}, b_{jk})$.

Let $A_{jk} = \{x \in I_{jk} : \int_x^{b_{jk}} g^{p'}\sigma = \infty\}$, for every j and k.

If $A_{jk} \neq \varnothing$, let $x_{jk} = \sup A$; if $A_{jk} = \varnothing$, let $x_{jk} = a_{jk}$. It is clear that $\int_x^{b_{jk}} g^{p'}\sigma < \infty$ if $x > x_{jk}$ and $u=0$ a.e. x in (a_{jk}, x_{jk}) by $S_{p,g,\alpha}^+$. Let $E_{jk} = I_{jk} \cap \left\{x : M_{g,\alpha}^+ f(x) \leq 2^{k+1}\right\}$ and $F_{jk} = (x_{jk}, b_{jk}) \cap E_{jk}$.

Clearly, the sets E_{jk} are pairwise disjoint and I_{jk}, $\cup_j E_{jk} = \{x : 2^k < M_{g,\alpha}^+ f(x) \leq 2^{(k+1)}\} \cap (-N,\infty)$ for every k. Then

$$\int_{-N}^{+\infty} \left(M_{g,\alpha}^+ f(x)\right)^p u(x)dx = \sum_k \int_{\cup_j E_{jk}} \left(M_{g,\alpha}^+ f(x)\right)^p u(x)dx$$
$$= \sum_{k,j} \int_{E_{jk}} \left(M_{g,\alpha}^+ f(x)\right)^p u(x)dx = \sum_{k,j} \int_{F_{jk}} \left(M_{g,\alpha}^+ f(x)\right)^p u(x)dx.$$

Using (2.1) and the definition of F_{jk}, we get

$$\left(M_{g,\alpha}^+ f\right)^p u \leq 2^p \left(\int_x^{b_{jk}} gf\right)^p \left(\int_x^{b_{jk}} g\right)^{(\alpha-1)p}.$$

Now, by the last argument, we have

$$\int_{-N}^{\infty} \left(M_{g,\alpha}^+ f\right)^p u(x)dx \leq 2^p \sum_{k,j} \int_{F_{jk}} \left(\int_x^{b_{jk}} gf\right)^p \left(\int_x^{b_{jk}} g\right)^{(\alpha-1)p} u(x)dx$$

(2)

$$= 2^p \sum_{k,j} \int_{F_{jk}} \left(\int_x^{b_{jk}} gf\right)^p \left(\int_x^{b_{jk}} g^{p'}\sigma\right)^{-p} \left(\int_x^{b_{jk}} g^{p'}\sigma\right)^p \left(\int_x^{b_{jk}} g\right)^{(\alpha-1)p} u(x)dx.$$

Let m and ν denote the Lebesgue measure on R and the counting measure on Z respectively.

Define a function ϕ on X as

$$\phi(j,k,x) = \chi_{F_{jk}(x)} \left(\int_x^{b_{jk}} g^{p'}\sigma\right)^p \left(\int_x^{b_{jk}} g\right)^{(\alpha-1)p} u(x)$$

and

$$Th(j,k,x) = \int_x^{b_{jk}} hg^{p'}\sigma \left(\int_x^{b_{jk}} g^{p'}\sigma\right)^{-1},$$

where $X = Z \times Z \times R$ and $w = \nu \times \nu \times m$ is the product measure. The inequality (2.2) transfers to

$$\int_{-N}^{\infty} \left(M_{g,\alpha}^{+} f\right)^p u \le 2^p \int_X \left(T(f(g^{-1}v)^{\frac{1}{p-1}})\right)^p \phi dw.$$

Next, we establish that the operator T is bounded. According to Marcinkiewicz interpolation theorem, it is sufficient to show that

$$\int_{\{(j,k,x)\in X: Th(j,k,x)>\lambda\}} \phi dw \le \frac{C}{\lambda}\int_{-\infty}^{+\infty} hg^{p'}\sigma dx.$$

Let

$$A_{jk}(\lambda) = F_{jk} \cap \{x : Th(j,k,x) > \lambda\},$$

$$s_{jk}(\lambda) = \inf A_{jk}(\lambda)$$

and

$$J_{jk} = J_{jk}(\lambda) = [s_{jk}(\lambda), b_{jk}).$$

We observe that $A_{jk}(\lambda)$ are pairwise disjoint and any two intervals J_{lm} and J_{jk} are either disjoint or one of them is contained in the other. In addition

$$\int_{J_{jk}} hg^{p'}\sigma \ge \lambda \int_{J_{jk}} g^{p'}\sigma. \tag{3}$$

The intervals J_{jk} have bounded sizes which allows us to extract a maximal subcollection $\{J_i\}$ in such a way that each J_i verify (2.3). Then by using the facts that $(u,v) \in S_{p,g,\alpha}^{+}$ and J_i are maximal elements and (2.3), we have

$$\begin{aligned}
&\int_{\{(j,k,x)\in X: Th(j,k,x)>\lambda\}} \phi dw \\
&= \sum_{(k,j)} \int_{A_{jk(\lambda)}} \left(\int_x^{b_{jk}} g^{p'}\sigma\right)^p \left(\int_x^{b_{jk}} g\right)^{(\alpha-1)p} u(x)dx \\
&\le \sum_i \sum_{\{(k,j):J_{jk}\subset J_j\}} \int_{A_{jk(\lambda)}} \left(\int_x^{b_{jk}} g^{p'}\sigma\right)^p \left(\int_x^{b_{jk}} g\right)^{(\alpha-1)p} u(x)dx \\
&\le \sum_i \left(M_{g,\alpha}^{+}(\chi_{J_i} g^{\frac{1}{p-1}}\sigma)(x)\right)^p u(x)dx \\
&\le C\sum_i \int_{J_i} g^{p'}(x)\sigma(x)dx \le \frac{C}{\lambda}\sum_i \int_{J_i} h(x)g^{p'}(x)\sigma(x)dx \\
&\le \frac{C}{\lambda}\int_{-\infty}^{+\infty} h(x)g^{p'}(x)\sigma(x)dx
\end{aligned}$$

which proves T is bounded from $L^p(g^{p'}\sigma dx)$ to $L^p(X, \phi dw)$. As N approaches to infinity,

$$\int_{-N}^{+\infty} \left(M_{g,\alpha}^{+} f\right)^p u \leq C.2^p \int_{-\infty}^{+\infty} \left(f(g^{-1}v)^{\frac{1}{p-1}}\right)^p g^{p'}\sigma dx = C.2^p \int_{-\infty}^{+\infty} f^p v$$

we have the desired inequality. □

Remark 2.1. Analogous result hold changing $M_{g,\alpha}^{+}$ by $M_{g,\alpha}^{-}$ and $S_{p,g,\alpha}^{+}$ by $S_{p,g,\alpha}^{-}$ where

$$M_{g,\alpha}^{-} f(x) = \sup_{h>0} \left(\int_{x-h}^{x} |f(t)|g(t)dt\right) \left(\int_{x-h}^{x} g(t)dt\right)^{\alpha-1}, 0 \leq \alpha < 1.$$

Proposition 2.1. *For* $1 < p < \infty$,

$$S_{p,g,\alpha} = S_{p,g,\alpha}^{+} \cap S_{p,g,\alpha}^{-}.$$

Proposition 2.2.

$$\frac{1}{2}\left(M_{g,\alpha}^{+} + M_{g,\alpha}^{-}\right) \leq M_{g,\alpha} \leq M_{g,\alpha}^{+} + M_{g,\alpha}^{-}.$$

Remark 2.2. Let u and v be two weights on R. If (u, v) satisfies strong condition then (u, v) also satisfies the weak condition from the fact that strong convergence implies weak convergence.

3. Conclusion

Theorem 3.1. *If* $p > 1$ *and* u *and* v *are weight functions, then the operator* $M_{g,\alpha}$ *is bounded from* L_v^p *to* L_u^p *if and only if* $(u, v) \in S_{p,g,\alpha}$.

Proof. The proof follows from the Propositions 2.1 and 2.2. □

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