# Study of generalised solutions for piezoelectric material with long memory and damage

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**Abstract.** We consider a dynamic problem that describes a frictional contact with damage between piezoelectric body and a conductive foundation. The constitutive law used is the electro viscoelastic with long term memory. The contact is supposed bilateral and frictional, it is modeled with Tresca's law. The damage is described by a function, its evolution is given by an inclusion of parabolic type. We derive a variational formulation for the model which is in a form of a coupled system for the displacement, the electric potential and damage function. The existence of a unique weak solution for a contact problem is established. The proof is based on the evolutionary variational inequalities, evolution inclusion and Banach's fixed point theorem.

**Keywords:** evolutionary variational inequality, fixed point, frictional contact, piezoelectric material.

## 1. Introduction

The piezoelectric materials are characterized by the combination of mechanical and electrical proprieties. The mechanical stress is generated when the electrical potential is applied and conversely the electric potential is created when the mechanical stress is present. Certain crystals, such as quartz, tourmaline, Rochelle salt, when subjected to a stress, become electrically polarized [7]. A general models for electro elastic problem can found in [13, 18, 19]. We consider material electro viscoelastic with long term memory. For problems involving such constutive law we can see for instance [21, 6]. We are interested to Models taking into account the influence of interne state variable which describe a damage of system. The subject of damage is extremely important in design engineering since it affects directly the useful life of the designed structure or component. There exists a very large engineering literature on it. General novel models for damage were derived from the virtual power principle. A mathematical analysis has been investigated for example in [9, 10]. A considerable interest in contact problems involving a damage see for instance [11, 12]. The process of damage introduce a function $\beta \in [0,1]$ , There is no damage when $\beta = 1$, and it is completely damaged when $\beta = 0$.

In this paper we study a contact between electro-viscoelastic body with long term memory and a deformable conductive foundation. Our interest is to describe the evolution of the deformation of the body, the function of damage and of the electric potential on the time interval $[0, T]$. The frictional contact is bilateral and it is modeled with Tresca's law, we suppose that the acceleration of system is not negligible so that the process is dynamic. Dynamic contact problems for elastic and viscoelastic body were considered in [2, 14, 15] and in the references therein. A few mathematical results arising in the study of a dynamic frictional contact of piezoelectric bodies may be found in [1, 8, 17].

Our study serves two purposes, the first one is to obtain variational formulation of the problem and second one to prove existence and uniqueness of weak solutions. The novelly in this work is to study the case of dynamic frictional contact for piezoelectric body taking into account the influence of the internal damage of the material on the process.

So in section 2 the piezoelectric problem is stated together with three variational formulations, in section 3 we state the existence and uniqueness result of the problem $\mathcal{P}_V$ (Theorem 3.1). The proof is based on the theory of evolution equations with monotone operators, on the evolutionary inclusions related to the variational inequalities and Banach's fixed point theorem.

## 2. Problem statement and notations

We assume that the body occupies the bounded domain $\Omega$ , and assume that the boundary $\Gamma$ of $\Omega$ is Lipschitz continuous and partitioned into three disjoint measurable open parts $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$, and a partition $\Gamma_1 \sqcup \Gamma_2$ into open parts $\Gamma_a$ and $\Gamma_b$. We assume that $meas\Gamma_2 > 0$ and $meas\Gamma_a > 0$. The body is clamped on $\Gamma_1$, therefore the displacement field vanishes there. A volume force of density $f_0$ acts in $\Omega \times (0, T)$ and surface traction of density $f_2$ act in $\Gamma_2 \times (0, T)$. The body may arrive in contact on $\Gamma_3 \times (0, T)$ with an obstacle, we assume that the contact is bilateral and frictional. The electric effects leads to the appearance of charges of density $q_0$. The process is be assumed electrically static.

We denote by $\mathbb{S}^d$ the space of second order of symétric tensors on $\mathbb{R}^d$ ($d = 1, 2, 3$) and by $(\cdot)$ et $|\cdot|$ respectively the scalar product and the Euclidean norm in $\mathbb{S}^d$ (resp in $\mathbb{R}^d$).

$$\mathbf{u}.\boldsymbol{v} = \mathbf{u}_i v_i \qquad |\mathbf{u}| = (\mathbf{u}.\mathbf{u})^{\frac{1}{2}} \quad \forall \mathbf{u}, \boldsymbol{v} \in \mathbb{R}^d, i = 1, \cdots d,$$
$$\boldsymbol{\sigma}.\boldsymbol{\tau} = \sigma_{ij}\tau_{ij} \qquad |\tau| = (\tau.\tau)^{\frac{1}{2}} \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d, i = 1, \cdots d, j = 1, \cdots d.$$

Here and bellow the indices $i$, $j$ run between 1 and $d$ and the summation convention over repeated indices is adopted Let $\Omega \subset \mathbb{R}^d$. we shall use the notation

$$\begin{aligned}
&H = \left\{\mathbf{u} = (u_i) \mid u_i \in L^2(\Omega)\right\} = \left(L^2(\Omega)\right)^d,\\
&\mathcal{W} = \left\{\boldsymbol{D} \in H \mid div\boldsymbol{D} \in L^2(\Omega)\right\},\\
&\mathcal{H} = \left\{\boldsymbol{\sigma} = (\sigma)_{ij} \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\right\},\\
&H_1 = \left\{\mathbf{u} = (u_i) \mid \boldsymbol{\varepsilon}(\mathbf{u}) \in \mathcal{H}\right\},\\
&\mathcal{H}_1 = \left\{\boldsymbol{\sigma} \in \mathcal{H} \mid Div\boldsymbol{\sigma} \in H\right\},
\end{aligned}$$

with $\boldsymbol{\varepsilon} : H \to \mathcal{H}$ and $Div : \mathcal{H} \to H$ are respectively operators of deformation and divergence defined by :

$$\boldsymbol{\varepsilon}(\mathbf{u}) = (\boldsymbol{\varepsilon}_{ij}(\mathbf{u})), \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}) \text{ and } Div\boldsymbol{\sigma} = (\sigma_{ij,j}).$$

The tensors $\mathcal{E} = (e_{ijk})$ and its transpose $\mathcal{E}^* = (e_{kij})$ satisfy the equality

$$\mathcal{E}\boldsymbol{\sigma} \cdot \boldsymbol{v} = \boldsymbol{\sigma} \cdot \mathcal{E}^* \boldsymbol{v},$$

where the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable. We assume that the mass density $\rho$ satisfy

(1) $\rho \in L_\infty(\Omega)$ and there exist $\rho_* > 0$ such that $\rho(\boldsymbol{x}) \geq \rho_*$ a.e. in $\Omega$,

then the space $H$ is a Hilbert space endowed with a new inner product

$$(\mathbf{u}, \boldsymbol{v})_H = \int_\Omega \rho u_i v_i \, dx.$$

The space $H$, $\mathcal{H}$, $H^1$ and $\mathcal{H}^1$ are Hilbert spaces endowed with the inner products given by

$$\begin{aligned}
(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} &= \int_\Omega \sigma_{ij}\tau_{ij} \, dx,\\
(\mathbf{u}, \boldsymbol{v})_{H_1} &= (\mathbf{u}, \boldsymbol{v})_H + (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}},\\
(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} &= (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (Div\boldsymbol{\sigma}, Div\boldsymbol{\tau})_H.
\end{aligned}$$

The associated norms on those spaces respectively are $|\cdot|_H$, $|\cdot|_{\mathcal{H}}$, $|\cdot|_{H^1}$ and $|\cdot|_{\mathcal{H}^1}$. Since $\Gamma$ is assumed be Lipschitz continuous then the unit outward normal vector $\nu$ is defined a.e., for every vector $\boldsymbol{v} \in H_1$, we use the notation $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ on $\Gamma$ and we denote by $\boldsymbol{v}_\nu$ and $\boldsymbol{v}_\tau$ the normal and tangential components of $\boldsymbol{v}$ on $\Gamma$, given by $v_\nu = \boldsymbol{v} \cdot \nu$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \nu$. For regular stress field $\boldsymbol{\sigma}$ (say $C^1$), the application of its trace to $\nu$ is the Cauchy stress vector $\boldsymbol{\sigma}\nu$. We define the normal and tangential components of $\boldsymbol{\sigma}$ by $\sigma_\nu = (\boldsymbol{\sigma}\nu) \cdot \nu$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\nu - \boldsymbol{\sigma}_\nu \nu$, and recall that the Green's formula holds

(2) $$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (Div\boldsymbol{\sigma}, \boldsymbol{v})_H = \int_\Gamma \boldsymbol{\sigma}\nu \cdot \boldsymbol{v}, \quad \forall \boldsymbol{v} \in H_1.$$

Let the Banach spaces $L^p(0,T;H)$ and $W^{1,p}(0,T;H)$ $1 \leq p \leq +\infty$, defined by

$$L^p(0,T;H) = \left\{ \mathbf{u} \mid \mathbf{u} : (0,T) \to H \ , \|\mathbf{u}\|_{L^p(0,T;H)} = \left( \int_0^T \|\mathbf{u}\,(t)\|_H^p \, dt \right)^{1/p} \right\},$$

$$W^{1,p}(0,T;H) = \Big\{ \mathbf{u} \in L^p(0,T;H), \ \dot{\mathbf{u}} = \tfrac{d\boldsymbol{u}(t)}{dt} \in L^p(0,T;H) \text{ such that } \|\mathbf{u}\|_{W^{1,p}} = \left( \int_0^T \|\dot{\mathbf{u}}\,(t)\|_H^p \, dt \right)^{1/p} + \left( \int_0^T \|\mathbf{u}\,(t)\|_H^p \, dt \right)^{1/p} \Big\}.$$

Here and every where in this paper the dot above $\mathbf{u}$ is the derivative with respect to the time variable.

The physical model for the process is as follows :

**Problem** $\mathcal{P}$. Find a displacement field $\boldsymbol{u} : \Omega \times [0,T] \to \mathbb{R}^d$, an electric potential $\varphi : \Omega \times [0,T] \to \mathbb{R}$, an electric displacement field and a damage function $\beta : \Omega \times [0,T] \to \mathbb{R}$

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{B}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \beta) + \int_0^t \mathcal{C}(t-s, \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \beta(s))ds \tag{3}$$

$$+\mathcal{E}^* \nabla\varphi \text{ in } \Omega \times (0,T)\,, \tag{4}$$

$$\boldsymbol{D} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \gamma\nabla\varphi \text{ in } \Omega \times (0,T)\,, \tag{5}$$

$$\dot{\beta} - \hat{k} \bigtriangleup \beta + \partial\varphi_K(\beta) \ni \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \beta), \tag{6}$$

$$\rho\ddot{\mathbf{u}} = Div\ \boldsymbol{\sigma} + \mathbf{f}_0 \qquad \text{in } \Omega \times (0,T)\,, \tag{7}$$

$$div\ \boldsymbol{D} = \boldsymbol{q}_0 \text{ in } \Omega \times (0,T)\,, \tag{8}$$

$$\boldsymbol{u} = 0 \qquad \text{on } \Gamma_1 \times (0,T)\,, \tag{9}$$

$$\boldsymbol{\sigma}\nu = \boldsymbol{f}_2 \qquad \text{on } \Gamma_2 \times (0,T)\,, \tag{10}$$

$$\sigma_\nu = 0 \quad \text{on } \Gamma_3 \times (0,T), \tag{11}$$

$$\|\boldsymbol{\sigma}_\tau\| \leq g,$$

$$\dot{\mathbf{u}}_\tau \neq \mathbf{0} \Rightarrow \boldsymbol{\sigma}_\tau = -g\frac{\dot{\mathbf{u}}_\tau}{\|\dot{\mathbf{u}}_{\boldsymbol{\tau}}\|} \quad \text{on } \Gamma_3 \times (0,T), \tag{12}$$

$$\varphi = 0 \quad \text{on } \Gamma_a \times (0,T)\,, \tag{13}$$

(14) $$\boldsymbol{D}\cdot\nu = q_b \quad \text{on}\, \Gamma_b \times (0,T)\,,$$

(15) $$\boldsymbol{D}\cdot\nu = (\varphi - \varphi_0) \;\text{ on } \Gamma_3 \times (0,T)\,,$$

(16) $$\frac{\partial\beta}{\partial\nu} = 0, \text{ on } \Gamma \times (0,T),$$

(17) $$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0 \quad \beta(0) = \beta_0 \quad \text{in } \Omega.$$

During the process of contact the normal component of the electric displacement field or the free charge is assumed to be proportional to the difference between the potential of the foundation and the body's surface potential, with $k$ as the proportionality factor.

The equations (3)-(8) and below, in order to simplify we do not indicate explicitly the dependence of various functions on the variables $x \in \Omega \cup \Gamma$ and $t \in [0,T]$. The general viscoelastic constitute law with damage and with electric effects is given by (3), where $\mathcal{A}$ is nonlinear viscosity function, $\mathcal{B}$ is nonlinear elasticity, depends on strain tensor $\boldsymbol{\varepsilon}(\boldsymbol{u})$ and the damage function $\beta$ . The stress is depends on electric field $-\nabla\varphi$. The relation ( 5) is the electric displacement it is a linear function of strain and electric field. The expression (6) describes the evolution of the damage function, $K$ denotes the set of admissible damage functions defined by $K = \left\{\xi \in H^1(\Omega) \mid 0 \leq \xi \leq 1 \text{ a.e. on } \Omega\right\}$, $H^1(\Omega)$ is the classical Sobolev space. The set $\partial\varphi_K$ is the subdifferential of the indicator function $\chi_K$ of the convex $K$. The equations (7) and (8) are the equilibrium equations, in equation (7) we suppose the process is dynamic with a masse density $\rho$. Here the conditions (9) and (10) are the displacement and traction boundary conditions, respectively conditions (11) and (12) represents the bilateral frictional contact condition, it is modeled with Tresca's law of dry friction with prescribed friction bound $g$ (hence $g \geq 0$ and the nonseparation condition $u_\nu = 0$. The boundary is electricly conductive (13) and (14) are boundary conditions on electric potential $\varphi$ and displacement field $D$ on $\Gamma_a$ and $\Gamma_b$. On a part of boundary $\Gamma_3$, and during the process of contact the normal of electric displacement field is assumed be proportional to the difference between the potential of foundation $\varphi_0$ and the body's surface potential, thus the condition (15). The equations given in (17) are the initial condition on displacement, the velocity field and the damage function. To present variational formulation of the above problem we need additional notations. Let us consider the subspaces of $H_1$ and $H^1$defined by

$$V = \{\boldsymbol{v} \in H_1 \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1\}\,,$$
$$W = \left\{\xi \in H^1 \mid \xi = 0 \text{ on } \Gamma_a\right\}$$

We recall since $meas\Gamma_1 > 0$ and $meas\Gamma_a > 0$, Korn's and Friederichs-Poincaré inequalities hold, thus there exist respectively a constant $C_K > 0$ and $c_F > 0$ which depends respectively only on $\Gamma_1, \Gamma_a$ and $\Omega$ such that

$$|\boldsymbol{\varepsilon}(\boldsymbol{v})|_{\mathcal{H}_1} \geq c_K |\boldsymbol{v}|_{H_1}, \quad |\nabla\xi|_H \geq c_F |\xi|_{H^1}, \quad \forall \boldsymbol{v} \in V, \forall \xi \in W.$$

For $\boldsymbol{u}$, $\boldsymbol{v} \in V$ and $\forall \varphi, \xi \in W$ , we have

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}_1}, \qquad (\varphi, \xi)_W = (\nabla\varphi, \nabla\xi)_H,$$

and we have $|\boldsymbol{u}|_V = (\boldsymbol{u},\boldsymbol{u})_V^{1/2}$ , $|\varphi|_V = (\varphi, \varphi)_W^{1/2}$, therefore $(V, |\cdot|_V)$ and $(W, |\cdot|_W)$ are real Hilbert spaces. Moreover, by Sobolev trace theorem, there exist a constants $c_0$, $\tilde{c}_0$ depending only on $\Omega$ and $\Gamma_1, \Gamma_a$ such that

$$|\xi|_{L^2(\Gamma_a)} \leq c_0 |\xi|_W, \qquad \forall \xi \in W, \tag{18}$$

$$|\boldsymbol{v}|_{L^2(\Gamma_1)^d} \leq \tilde{c}_0 |\boldsymbol{v}|_V, \qquad \forall \boldsymbol{v} \in V. \tag{19}$$

Let note $V'$ and $W'$ the dual spaces of $V$ and $W$, so we have continuous and dense imbedding $V \subset H \subset V'$ rep ( $W \subset L^2(\Omega) \subset W'$). To study problem $\mathcal{P}$ we must make some assumptions.The viscosity operator $\mathcal{A}$ and the elasticity one $\mathcal{B}$ satisfy conditions

$$\left\{\begin{array}{l}
\text{(a) } \mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d \\
\text{(b) there exist } L_{\mathcal{A}} > 0 \text{ such that} \\
|\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)| \leq L_{\mathcal{A}} |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| \ \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega, \\
\text{(c) there exist } m_{\mathcal{A}} > 0 \text{ such that} \\
(\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathcal{A}} |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^2, \\
\forall \boldsymbol{\varepsilon}_1, \ \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega, \\
\text{(d) the mapping } \boldsymbol{x} \longmapsto \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega, \\
\text{(e) the mapping } \boldsymbol{x} \longmapsto \mathcal{A}(\boldsymbol{x}, \boldsymbol{0}) \text{ belongs to } \mathcal{H}.
\end{array}\right. \tag{20}$$

$$\left\{\begin{array}{l}
\text{(a) } \mathcal{B} : \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d, \\
\text{(b) } \mathcal{B}(\mathbf{x}, \boldsymbol{\tau}, \zeta) = b(\mathbf{x}, \zeta)\boldsymbol{\tau}, \ \forall \boldsymbol{\tau} \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\
\text{(c) } b(\mathbf{x}, \zeta) = (b_{ijkl}), \ b_{ijkl} = b_{klij} \text{ belongs to } L^\infty(\Omega \times \mathbb{R}). \forall i, j, k, l. \\
\text{(d) There exists } m_{\mathcal{B}} > 0 \text{ such that} \\
\qquad \mathcal{B}(\mathbf{x}, \boldsymbol{\xi}, \zeta)\ \boldsymbol{\xi} \geq m_{\mathcal{B}} \|\boldsymbol{\xi}\|^2, \text{ for all } \mathbf{x} \in \Omega, \ \boldsymbol{\xi} \in \mathbb{S}^d, \zeta \in \mathbb{R}.
\end{array}\right. \tag{21}$$

The relaxation function $\mathcal{C} : \Omega \times (0, T) \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$, satisfies

$$\left\{\begin{array}{l}
|\mathcal{C}(t, \boldsymbol{x}, \boldsymbol{\varepsilon}_1, \beta_1) - \mathcal{C}(t, \boldsymbol{x}, \boldsymbol{\varepsilon}_2, \beta_2)| \leq L_{\mathcal{C}} |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| + |\beta_1 - \beta_2|, \\
\forall \boldsymbol{\varepsilon}_1, \ \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \ \forall \beta_1, \beta_2 \in \mathbb{R} \text{ a.e. } \boldsymbol{x} \in \Omega, \forall t \in (0, T). \\
\text{(c) the mapping } \boldsymbol{x} \longmapsto \mathcal{C}(\boldsymbol{x}, \boldsymbol{\varepsilon}, \beta) \text{ is Lebesgue measurable on } \Omega, \\
\qquad \forall \boldsymbol{\varepsilon} \in \mathbb{S}^d, \ \forall \beta \in \mathbb{R}. \\
\text{(c) the mapping } t \longmapsto \mathcal{C}(t, \boldsymbol{x}, \boldsymbol{\varepsilon}, \beta) \text{ is continious on } (0, T), \\
\qquad \forall \boldsymbol{\varepsilon} \in \mathbb{S}^d, \ \forall \beta \in \mathbb{R}. \\
\text{(d) the mapping } \boldsymbol{x} \longmapsto \mathcal{C}(t, \boldsymbol{x}, \boldsymbol{0}, 0) \text{ belongs to } \mathcal{H}.
\end{array}\right. \tag{22}$$

The damage source function satisfies

$$(23)\quad \begin{cases} \text{(a) } \phi : \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R} \\ \left|\phi\left(\boldsymbol{x},\ \boldsymbol{\varepsilon}_1, \beta_1\right) - \phi\left(\boldsymbol{x}, \boldsymbol{\varepsilon}_2, \beta_2\right)\right| \le L_\phi \left|\boldsymbol{\varepsilon}_1 -\ \boldsymbol{\varepsilon}_2\right| + \left|\beta_1 - \beta_2\right|, \\ \ \forall \boldsymbol{\varepsilon}_1,\ \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d,\ \forall \beta_1, \beta_2 \in \mathbb{R} \text{ a.e. } \boldsymbol{x} \in \Omega, \\ \text{(c) the mapping } \boldsymbol{x} \longmapsto \phi\left(\boldsymbol{x}, \boldsymbol{\varepsilon}, \beta\right) \text{ is Lebesgue measurable on } \Omega, \\ \forall \boldsymbol{\varepsilon} \in \mathbb{S}^d,\ \forall \beta \in \mathbb{R} \\ \text{(d) the mapping } \boldsymbol{x} \longmapsto \phi\left(\boldsymbol{x}, \boldsymbol{0}, 0\right) \text{ belongs to } \mathcal{H}. \end{cases}$$

The piezoelectic tensor $\mathcal{E}$ and the permeability tensor $\gamma$ satisfy

$$(24)\quad \begin{cases} \text{(a) } \mathcal{E} : \Omega \times \mathbb{S}^d \to \mathbb{R}^d \\ \text{(b) } \mathcal{E}\left(\boldsymbol{x}, \zeta\right) = \left(e_{ijk}(\boldsymbol{x})\zeta_{jk}\right), \quad \forall \zeta = (\zeta_{ij}) \in \mathbb{S}^d, \text{ a.e. } \ \boldsymbol{x} \in \Omega \\ \text{(c) } e_{ijk} = e_{ikj} \in L^\infty\left(\Omega\right) \end{cases}$$

$$(25)\quad \begin{cases} \text{(a) } \gamma : \Omega \times \mathbb{R}^d \to \mathbb{R}^d, \\ \text{(b) } \gamma\left(\boldsymbol{x}, \mathbf{E}\right) = \left(\gamma_{ij}(\boldsymbol{x})E_j\right), \quad \forall \mathbf{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. } \ \boldsymbol{x} \in \Omega, \\ \text{(c) } \gamma_{ik} = \gamma_{ji} \in L^\infty\left(\Omega\right), \\ \text{(d) there exist a constant } m_\gamma > 0 \text{ such that} \\ \gamma_{ij}(\boldsymbol{x})E_iE_j \ge m_\gamma \left\|\mathbf{E}\right\|^2, \forall \mathbf{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \end{cases}$$

The body forces and surfaces traction and free charges densities satisfy

$$(26)\quad \mathbf{f}_0 \in W^{1;1}\left(0, T; H\right),$$

$$(27)\quad \boldsymbol{f}_2 \in L^2\left(0, T; L^2\left(\Gamma_2\right)^d\right),$$

$$(28)\quad q_0 \in W^{1;2}\left(0, T; L^2\left(\Omega\right)\right),$$

$$(29)\quad q_b \in W^{1;2}\left(0, T; L^2\left(\Gamma_b\right)\right).$$

We assume $\mathbf{v}_0$ belongs to $\partial j$ the subdifferential of $j$ in convex sens and satisfies

$$(30)\quad \begin{aligned} &(\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{v}_0),\ \boldsymbol{\varepsilon}(\mathbf{v}) - \varepsilon(\mathbf{v}_0))_{\mathcal{H}} + (\mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}_0), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{v}_0))_{\mathcal{H}} + \\ &\quad j(\mathbf{v}) - j(\mathbf{v}_0) \ge \langle \mathbf{f}_0(0), \mathbf{v} - \mathbf{v}_0 \rangle_H \ \forall v \in V. \end{aligned}$$

We also suppose that

$$(31)\quad \varphi_0 \in L^2(\Gamma_3),$$

$$(32)\quad \hat{k} > 0, \quad \beta_0 \in K,$$

$$(33)\quad \boldsymbol{u}_0 \in V, \quad \boldsymbol{v}_0 \in V.$$

By using Riesz's representation theorem we define un element $q(t) \in W$ by

$$(q(t),\xi)_W = -\left(q_0(t),\xi\right)_{L^2(\Omega)} - \left(q_b(t),\xi\right)_{L^2(\Gamma_b)}, \ \ \forall\xi\in W, \text{ a.e. } t\in(0,T), \tag{34}$$

Let $j : V \to \mathbb{R}$ be the functional such that

$$j(\mathbf{v}) = \int_{\Gamma_3} g\|\mathbf{v}_\tau\|\, da + \int_{\Gamma_2} f_2\mathbf{v} da, \tag{35}$$

$h : W\times W \to \mathbb{R}$ by

$$h\left(\varphi,\xi\right) \ \ = \ \ \int_{\Gamma_3} \left(\varphi-\varphi_0\right)\xi\ da, \ , \forall\xi,\, \varphi\in W.$$

and a bilinear form $a : W \times W \to \mathbb{R}$ such that

$$a\left(\zeta,\eta\right) = \hat{k}\int_\Omega \nabla\zeta\nabla\eta\ dx, \quad \forall\zeta,\eta\in W. \tag{36}$$

Note that the integrals in (35) and in (36) are well defined. If $\{\boldsymbol{u},\varphi,\beta\}$ are regular functions satisfying (3)-(17), this imply that $\boldsymbol{u}(t)\in V$, $\varphi(t)\in W$, and $\beta(t)\in H^1$, and keeping in mind the relations (2), (35), (36), we deduce the variational formulation of problem $\mathcal{P}$, noted $\mathcal{P}_V$.

**Problem** $\mathcal{P}_V$. Find a displacement field $\boldsymbol{u} : \Omega\times[0,T]\to\mathbb{R}^d$, an electric potential $\varphi : \Omega\times[0,T]\to\mathbb{R}$ and a damage function $\beta : \Omega\times[0,T]\to\mathbb{R}$ such that

$$\begin{aligned} &(\ddot{\mathbf{u}}(t),\mathbf{v}-\dot{\mathbf{u}})_{V'} + (\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)),\boldsymbol{\varepsilon}(\mathbf{v})-\varepsilon(\dot{\mathbf{u}}(t)))_{\mathcal{H}} \\ &+ \left(\mathcal{B}\left(\varepsilon(\mathbf{u}(t)),\beta\left(t\right)\right) + \left(\int_0^t \mathcal{C}(t-s,\boldsymbol{\varepsilon}(\boldsymbol{u}(s)),\beta(s))ds, \boldsymbol{\varepsilon}(\mathbf{v})-\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t))\right)_{\mathcal{H}} \\ &+ (\mathcal{E}^*\nabla\varphi(t),\boldsymbol{\varepsilon}(\mathbf{v})-\boldsymbol{\varepsilon}(\ \dot{\mathbf{u}}(t)))_{\mathcal{H}} \ + j(\mathbf{v})-j(\dot{\mathbf{u}}(t)) \geq (\mathbf{f}_0(t),\mathbf{v}-\dot{\mathbf{u}}(t))_H, \end{aligned} \tag{37}$$

$$\begin{gathered} (\gamma\nabla\varphi(t),\nabla\xi)_{\mathcal{H}} - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t),\nabla\xi)_{\mathcal{H}} + h\left(\varphi(t),\xi\right) = (q(t),\xi)_W \\ \forall\xi\in W, \text{ a.e. on } (0,T), \end{gathered} \tag{38}$$

$$\begin{aligned} \beta(t)\in K, \quad &\left(\dot{\beta}(t),\zeta-\beta(t)\right)_{L^2(\Omega)} + a\left(\beta(t),\zeta-\beta(t)\right) \\ &\geq \left(\phi\left(\ \boldsymbol{\varepsilon}\left(\boldsymbol{u}(t)\right),\beta(t)\right),\zeta-\beta(t)\right)_{L^2(\Omega)}, \\ &\qquad \forall\zeta\in K, \text{ a.e. on } (0,T), \end{aligned} \tag{39}$$

$$\boldsymbol{u}(0)=\boldsymbol{u}_0,\ \dot{\boldsymbol{u}}(0)=\boldsymbol{v}_0,\ \beta(0)=\beta_0. \tag{40}$$

## 3. An existence and uniqueness result for the problem $\mathcal{P}_V$

**Theorem 3.1.** *Assume that the conditions (20)-(33) hold. Then, there exists a unique solution of the problem $\mathcal{P}_V$. Moreover the solution satisfies*

$$\begin{aligned} \boldsymbol{u} &\in W^{1;\infty}(0,T;V) \cap W^{2;\infty}\left(0,T;V'\right). \\ \varphi &\in W^{1;2}(0,T;W), \\ \beta &\in K, \quad \beta \in W^{1;2}\left(0;T,L^2(\Omega)\right) \cap L^2\left(0,T;H^1(\Omega)\right). \end{aligned} \tag{41}$$

The proof of Theorem 3.1 will be carried out in several steps. It is based on results of evolution equations with monotone operators and Banach's fixed point theorem and classical results on parabolic equations bellow, see ([3]).

**Theorem 3.2.** *Let $V$ and $H$ be real Hilbert spaces satisfying $V \subset H \subset V'$, with continuous and dense injection, and let $B : V \to V'$ be a linear continuous and symmetric operator satisfying the coerciveness condition*

$$\exists \alpha_0 > 0,\ \alpha_1 \in \mathbb{R} \ \ \textit{such that}\ \ (Bu,u)_{V',V} + \alpha_1 |u|_H^2 \geq \alpha_0 |u|_V^2, \forall u \in V.$$

*Let $M : V \to 2^{V'}$ a multivalued maximal monotone operator, $f$ a given function in $W^{1,1}(0,T;V')$ and $u_0$, $v_0$ two datas satisfying*

$$u_0 \in V,\ v_0 \in D(M)\ \textit{and}\ (Bu_0 + Mv_0) \cap H \neq \emptyset. \tag{42}$$

*Then, there exist a unique function $u \in W^{1;\infty}(0,T;V) \cap W^{2;\infty}(0,T;V')$ solution to the problem*

$$\begin{cases} \ddot{u}(t) + Bu(t) + M\dot{u}(t) \ni f(t),\ \text{a.e. } t \in (0,T), \\ u(0) = 0, \quad \dot{u}(0) = v_0. \end{cases} \tag{43}$$

**Theorem 3.3.** *Let $V \subset H \subset V'$ be a Gelfand triple. Let $K$ a be nonempty and closed set of $V$. Let define a bilinear and symmetric and continuous form $a(.,.) : V \times V \to R$ which satisfy*

$$\exists \alpha_0 > 0,\ \alpha_1 \in \mathbb{R} \quad \textit{such that}\ a(u,u)_{V',V} \geq \alpha_0 |u|_V^2 + \alpha_1 |u|_H^2,\ \forall u \in V, \tag{44}$$

*Then, for a given $u_0 \in K$ and $f \in L^2(0,T;H)$, there exist a unique function $u$ which satisfies*

$$u \in L^2(0,T;V) \cap H^1([0,T];H)$$

*for all $t \in [0,T]$, $u(t) \in K$, and for almost $t \in (0,T)$*

$$(\dot{u}(t), v - u(t)) + a(u(t), v - u(t)) \geq (f(t), v - u(t))_H, \qquad \forall u \in K, \tag{45}$$

$$u(0) = u_0. \tag{46}$$

Now, assume that assumptions (1) and (20)-(29) hold and let $\eta = \left(\eta^1, \eta^2\right) \in W^{1;1}(0,T;V') \times L^2(0,T;L^2(\Omega))$. We consider now the following problems.

**Problem** $\mathcal{P}_1^\eta$. Find a displacement field $\boldsymbol{u}_\eta : \Omega \times [0,T] \to \mathbb{R}^d$, such that

$$\begin{gathered}(\ddot{\mathbf{u}}_\eta(t), \mathbf{v} - \dot{\mathbf{u}}_\eta(t))_{V'} + (\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\eta(t)), \boldsymbol{\varepsilon}(\mathbf{v}) - \varepsilon(\dot{\mathbf{u}}_\eta(t)))_{\mathcal{H}} + \\ (\mathcal{B}\left(\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)), \beta\left(t\right)\right), \boldsymbol{\varepsilon}(\mathbf{v}) - \ \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\eta(t)))_{\mathcal{H}} + (\eta^1(t), \mathbf{v} - \dot{\mathbf{u}}_\eta(t))_{V'} + \\ j(\mathbf{v}) - j(\dot{\mathbf{u}}_\eta(t)) \geq (\mathbf{f}_0(t), \mathbf{v} - \dot{\mathbf{u}}_\eta(t))_{V'}, \\ \forall \boldsymbol{v} \in V, \text{ a.e. } t \in (0,T)\,,\end{gathered} \tag{47}$$

$$\boldsymbol{u}_\eta(0) = \boldsymbol{u}_0, \;\; \dot{\boldsymbol{u}}_\eta(0) = \boldsymbol{v}_0, \tag{48}$$

**Problem** $\mathcal{P}^\eta$. Find an electric potential $\varphi : \Omega \times [0,T] \to \mathbb{R}$, such that

$$\begin{gathered}(\gamma \nabla \varphi_\eta(t), \nabla \xi)_{\mathcal{H}} - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(t), \nabla \xi)_{\mathcal{H}} + \left(h\left(\boldsymbol{u}_\eta(t), \varphi_\eta(t)\right), \xi\right)_W \\ = (q(t), \xi)_W \; \forall \xi \in W, \text{ a.e. on } (0,T)\,,\end{gathered} \tag{49}$$

**Problem** $\mathcal{P}_2^\eta$. Find a damage function $\beta_\eta : \Omega \times [0,T] \to \mathbb{R}$, such that

$$\begin{aligned}\beta_\eta(t) \in K, \; \left(\dot{\beta}_\eta(t), \zeta - \beta_\eta(t)\right)_{L^2(\Omega)} &+ a\left(\beta_\eta(t), \zeta - \beta_\eta(t)\right) \\ &\geq \left(\eta^2(t), \zeta - \beta_\eta(t)\right)_{L^2(\Omega)}, \;\; \forall \zeta \in K, \text{ a.e. on } (0,T)\,.\end{aligned} \tag{50}$$

$$\beta_\eta(0) = \beta.$$

Note that last problem is decoupled, so we can established separately existence and uniqueness solution for each one. We denote by $c$ all different constants given in the estimates below.

**Lemma 3.1.** *There exists a unique solution to the problem* $\mathcal{P}_1^\eta$. *Moreover it satisfies*

$$\boldsymbol{u}_\eta \in W^{1;\infty}(0,T;V) \cap W^{2;\infty}\left(0,T;V'\right). \tag{51}$$

*If* $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ *are two solutions of the problem* $\mathcal{P}_1^\eta$ *corresponding to the data* $\eta_1^1$ $\eta_2^1$ *of* $\eta^1$, *then there exist a constant* $c > 0$ *such that*

$$\left|\mathbf{u}_1(t) - \mathbf{u}_2(t)\right|_V \leq c\left|\eta_2^1 - \ \eta_1^1\right|_{L^2(0,T;V')}, \tag{52}$$

**Proof.** First let rewrite the equation (47) in the operator form equation like in (43) of Theorem 3.2 where

$$M\dot{\mathbf{u}}(t) = A\dot{\mathbf{u}}(t) + \partial j \;\, \dot{\mathbf{u}}(t)\,,$$

where $A : V \to V'$

$$(A\dot{\mathbf{u}}(t), \mathbf{v})_V = (\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\eta(t)), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}},$$

Using (20(a),(d)), (20(c)) $A$ is bounded and monotone on $V$,

$$(A\mathbf{u}_1 - A\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)_V \geq m_A \|\mathbf{u}_1 - \mathbf{u}_2\|_V^2 \quad \forall\, \mathbf{u}_1, \mathbf{u}_2 \in V.$$

The operator $A$ is hemicontinuous, $\forall u, v \in V$, indeed let $(s_n)_n \in [0,1]$ such that $s_n \to s$ when $n \to \infty$, we have

$$\begin{aligned} &|\langle A(\mathbf{u} + s_n\mathbf{v}) - A(\mathbf{u} + s\mathbf{v}), \mathbf{v}\rangle_V| \\ &= |(\mathcal{A}\varepsilon(\dot{\mathbf{u}} + s_n\dot{\mathbf{v}}) - \mathcal{A}\varepsilon(\dot{\mathbf{u}} + s\dot{\mathbf{v}}),\ \varepsilon(\mathbf{v}))_{\mathcal{H}}| \\ &\leq\ C_{\mathcal{A}}\,|s_n - s|\,|\dot{\mathbf{v}}|_V\,|\mathbf{v}|_V \underset{n\to\infty}{\to} 0. \end{aligned} \tag{53}$$

In the second see that the functional $j$ defined by (35) is convex lower semi continuous. Its subdifferentiel in the convex sens $\partial j$ is a maximal monotone operator. The multivalued $M : V \to 2^{V'}$

$$M\dot{\mathbf{u}}\,(t) = A\dot{\mathbf{u}}(t) + \partial j\ \dot{\mathbf{u}}\,(t)\,,$$

is maximal monotone with respect to $\dot{\mathbf{u}}$, Indeed it is the sum of two operators, the first $A$ which is assumed to be monotone and continuous (20 (a)-(e)), therefore $A$ is hemicontinuous and bounded, and the second $\partial j$ the subdiffrential of a proper convex lower semi-continuous function, so it is maximal monotone. According to the functional analysis results on the maximal monotone operators we conclude that the sum $M\big(\dot{u}(t)\big)$ is maximal monotone. Let define $B\,(t) : V \to V$, $t \in (0,T)$ where

$$(B\,(t)\,\mathbf{u}(t), \mathbf{v})_V = (\mathcal{B}\,(\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)), \beta\,(t))\,, \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}}.$$

It is clear that by (21(a)-(d)) $B\,(t)$ is a linear operator continuous, symmetric and coercive, for almost $t \in (0,T)$. We recall that $F_\eta = \mathbf{f}_0 -\ \eta^1 \in W^{1;1}\,(0,T;V')$ and $\boldsymbol{v}_0 \in V$, (see the conditions (26), (48)).

Since we have $u_0 \in V$, $v_0 \in D(\partial j)$, the condition (33) and $F_\eta(0) \in H$, this implies that there exists an element both in $(Bu_0 + Mv_0)$ and in $H$. Thus, the condition $(Bu_0 + Mv_0) \cap H \neq \emptyset$ is satisfied. All conditions of the Theorem 3.2 are satisfied, then there exists a unique solution $\mathbf{u}_\eta$ of problem $\mathcal{P}_1^\eta$ with $\mathbf{u}_\eta \in W^{1;\infty}\,(0,T;V) \cap W^{2;\infty}\,(0,T;H)$. Let now $\eta_1^1\ \eta_2^1 \in W^{1;2}\,(0,T;V')$ and $\mathbf{u}_{\eta_1^1}\,(t)$, $\mathbf{u}_{\eta_2^1}\,(t)$ noted respectively $\mathbf{u}_1$ and $\mathbf{u}_2$ two solutions of problems $\mathcal{P}_I^{\eta_1^1}$ and respectively $\mathcal{P}_I^{\eta_2^1}$. Recall that $\mathcal{P}_1^{\eta_i^1}$, $i = 1,2$ can also be written as

$$\begin{aligned} &\langle \ddot{\mathbf{u}}_i, v - \dot{\mathbf{u}}_i\rangle_{V'} + \langle A\mathbf{u}_i, v - \dot{\mathbf{u}}_i\rangle_V + \langle B\mathbf{u}_i, v - \dot{\mathbf{u}}_i\rangle_V + \\ &+ j(v) - j(\dot{\mathbf{u}}_i) \geq (F_{\eta_i}, v - \dot{\mathbf{u}}_i)_{V'},\ \forall v \in V. \end{aligned} \tag{54}$$

Replacing $v$ respectively in (54) by $\dot{\mathbf{u}}_2$ then by $\dot{\mathbf{u}}_1$, and then by adding the obtained inequalities, we find

$$\begin{aligned} &\langle \ddot{\mathbf{u}}_2 - \ddot{\mathbf{u}}_1, \dot{\mathbf{u}}_2 - \dot{\mathbf{u}}_1\rangle_{V'} + \langle A\ \dot{\mathbf{u}}_2 - A\dot{\mathbf{u}}_1, \dot{\mathbf{u}}_2 - \dot{\mathbf{u}}_1\rangle_V + \\ &\langle B\,(\mathbf{u}_2 - \mathbf{u}_1)\,, \dot{\mathbf{u}}_2 - \dot{\mathbf{u}}_1\rangle_V \leq (\eta_1^1 - \eta_2^1, \dot{\mathbf{u}}_2 - \dot{\mathbf{u}}_1)_{V'}, \end{aligned}$$

by using the property of strong monotonicity of $A$, the Lipshitz condition on $B$, we find that

$$\frac{d}{dt}\left|\dot{\mathbf{u}}_2-\dot{\mathbf{u}}_1\right|_H^2+m_A\left|\dot{\mathbf{u}}_1-\dot{\mathbf{u}}_2\right|_V^2\leq\left|\eta_2^1-\eta_1^1\right|_{V'}\left|\dot{\mathbf{u}}_2-\dot{\mathbf{u}}_1\right|_V+$$
$$C_B\left|\mathbf{u}_2-\mathbf{u}_1\right|_V\left|\dot{\mathbf{u}}_2-\dot{\mathbf{u}}_1\right|_V,$$

using the inequality $|ab|\leq\frac{\alpha}{2}|a|^2+\frac{2}{\alpha}|b|^2$ where $\alpha\in\mathbb{R}_+^*$ is fixed, we have

$$\begin{aligned}&\frac{d}{dt}\left|\dot{\mathbf{u}}_2-\dot{\mathbf{u}}_1\right|_H^2+m_A\left|\dot{\mathbf{u}}_1-\dot{\mathbf{u}}_2\right|_V^2\\ \leq\ & c\left\{\alpha\left|\dot{\mathbf{u}}_1-\dot{\mathbf{u}}_2\right|_V^2+\frac{2}{\alpha}\left|\eta_2^1-\eta_1^1\right|_{V'}^2+\frac{2}{\alpha}\left|\mathbf{u}_1-\mathbf{u}_2\right|_V^2\right\},\end{aligned}$$

$$\frac{d}{dt}\left|\dot{\mathbf{u}}_2-\dot{\mathbf{u}}_1\right|_H^2+(m_A-\alpha c)\left|\dot{\mathbf{u}}_1-\dot{\mathbf{u}}_2\right|_V^2\leq c\left\{\left|\eta_2^1-\eta_1^1\right|_H^2+\left|\mathbf{u}_1-\mathbf{u}_2\right|_V^2\right\}.$$

by the choice of $\alpha$ such that $m_A-\alpha c>0$, and by integration from 0 to $T$, we get

$$\begin{aligned}&\left|\dot{\mathbf{u}}_2(t)-\dot{\mathbf{u}}_1(t)\right|_H^2+\left(m_A-\alpha\tfrac{c}{2}\right)\left|\dot{\mathbf{u}}_1-\dot{\mathbf{u}}_2\right|_{L^2(0,t;V)}^2\\ &\quad\leq c\left|\eta_2^1-\eta_1^1\right|_{L^2(0,t;V')}^2+\left|\mathbf{u}_1-\mathbf{u}_2\right|_{L^2(0,t;V)}^2.\end{aligned}$$

$$\left|\dot{\mathbf{u}}_2(t)-\dot{\mathbf{u}}_1(t)\right|_H^2\leq c\left|\eta_2^1-\eta_1^1\right|_{L^2(0,t;V')}^2+\left|\mathbf{u}_1-\mathbf{u}_2\right|_{L^2(0,t;V)}^2$$

hence

$$\left|\dot{\mathbf{u}}_1-\dot{\mathbf{u}}_2\right|_{L^2(0,t;V)}^2\leq c\left\{\left|\ \eta_2^1-\eta_1^1\right|_{L^2(0,t;V')}^2+\left|\mathbf{u}_1-\mathbf{u}_2\right|_{L^2(0,t;V)}^2\right\}.$$

Recall that $\mathbf{u}_1(t)=\int_0^t\dot{\mathbf{u}}_1(s)\,ds+\mathbf{u}_0$, $s<t$, then

$$\begin{aligned}\left|\mathbf{u}_1(t)-\mathbf{u}_2(t)\right|_V^2&\leq\int_0^t\left|\dot{\mathbf{u}}_1(s')-\dot{\mathbf{u}}_2(s')\right|_V^2ds'\\ &\leq c\left|\eta_2^1-\eta_1^1\right|_{L^2(0,t;V')}^2+c\int_0^t\left|\mathbf{u}_1(s)-\mathbf{u}_2(s)\right|_V^2ds,\end{aligned}$$

by Gronwall's Lemma we deduce that

$$\left|\mathbf{u}_1(t)-\mathbf{u}_2(t)\right|_V^2\leq c\left|\eta_2^1-\ \eta_1^1\right|_{L^2(0,T;V')}^2,$$

The Lemma is now proved. □

**Lemma 3.2.** *There exists a unique solution*

(55) $$\varphi_\eta\in W^{1,2}(0,T;W),$$

*of problem $\mathcal{P}^\eta$. If $\varphi_1$, $\varphi_2$ are two solutions of the problem $\mathcal{P}^\eta$ corresponding to the data $\eta_1$ and $\eta_2$ of $\eta$, then there exist a constant $c>0$ such that*

(56) $$|\varphi_1(t)-\varphi_2(t)|_W\leq c\,|\mathbf{u}_1(t)-\mathbf{u}_2(t)|_V.$$

**Proof.** Let define the operator $A_\eta(t) : W \to W$, for $t \in [0,T]$. by

$$(A_\eta(t)\varphi_\eta(t), \xi)_W = (\gamma\nabla\varphi_\eta(t), \nabla\xi)_{\mathcal{H}} - (\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t), \nabla\xi)_{\mathcal{H}} + \int_{\Gamma_3} (\varphi_\eta - \varphi_0)\,\xi, \qquad \forall\xi \in W.$$

Let $\varphi_1$, $\varphi_2 \in W$, will $\gamma$ is coercive (see (25)), thus imply

$$(A_\eta(t)\varphi_1 - A_\eta(t)\varphi_2, \varphi_1 - \varphi_2)_W \geq m_\gamma \left|\varphi_1 - \varphi_2\right|_W^2 .$$

The operator $A_\eta(t)$ is strongly monotone. Now, by Conditions (25) and (18), we have

$$(A_\eta(t)\varphi_1 - A_\eta(t)\varphi_2, \xi)_W \leq c\left|\varphi_1 - \varphi_2\right|_W \left|\xi\right|_W ,$$

thus imply that $A_\eta(t)$ is Lipshitz continuous. The equation $A_\eta(t)\varphi(t) = q(t)$, have a unique solution $\varphi_\eta(t) \in W$, for $q(t) \in W$. The function $\varphi_\eta(t)$ is then the unique solution of the problem $\mathcal{P}^\eta$. It is a classical result of evolutionary elliptic problems see for example [5]

Let now prove that $\varphi_\eta \in W^{1,2}(0,T;W)$, let $t_1$, $t_2 \in [0,T]$ and $\varphi_\eta(t_1) = \varphi_1$, $\varphi_\eta(t_2) = \varphi_2$, $\mathbf{u}_\eta(t_1) = \mathbf{u}_1$, $\mathbf{u}_\eta(t_2) = \mathbf{u}_2$, $q(t_1) = q_1$, $q(t_2) = q_2$. Then, we deduce from conditions (25), (18), (19), (24) that

$$m_\gamma \left|\varphi_1 - \varphi_2\right|_W^2 + k\int_{\Gamma_3} (\varphi_1 - \varphi_2)\,(\varphi_1 - \varphi_2)\,da \leq \left[c_{\mathcal{E}}\left|\mathbf{u}_1 - \mathbf{u}_2\right|_V + \left|q_1 - q_2\right|_W\right]\left|\varphi_1 - \varphi_2\right|_W$$

thus imply that

$$|\varphi_1 - \varphi_2|_W \leq c\left[|\mathbf{u}_1 - \mathbf{u}_2|_V + |q_1 - q_2|_W\right]. \tag{57}$$

Since $q_i \in W^{1,2}(0,T;W)$ and $\mathbf{u}_i \in W^{1,\infty}(0,T;V)$, $i = 1, 2$, then we have $\varphi_\eta \in W^{1,2}(0,T;W)$. We show next that we have the estimation (56).

Let $\eta_1$, $\eta_2 \in W^{1;2}(0,T;H)$ and $\varphi_1$, $\varphi_2$, are respectively solutions of problems $\mathcal{P}^{\eta_i}$, $i = 1, 2$, and $\mathbf{u}_1$, $\mathbf{u}_2$, are solutions of problems $\mathcal{P}_1^{\eta_i}$, $i = 1, 2$. With similar arguments we deduce that $|\varphi_1 - \varphi_2|_W \leq \frac{c_{\mathcal{E}}}{m_\gamma}|\mathbf{u}_1 - \mathbf{u}_2|_V$. □

For the existence and uniqueness of solution $\beta_\eta$ for the problem $\mathcal{P}_2^\eta$, we have

**Lemma 3.3.** *There exist a unique solution $\beta_\eta$ for the problem $\mathcal{P}_2^\eta$ such that*

$$\beta_\eta \in L^2\left(0,T;L^2(\Omega)\right) \cap L^2\left(0,T;H^1(\Omega)\right). \tag{58}$$

*If $\beta_1$, $\beta_2$ are two solutions of the problem $\mathcal{P}_\eta^2$ corresponding to data $\eta^2 = \eta_i^2$, $i = 1, 2$. Then, we have*

$$|\beta_1(t) - \beta_2(t)|_{L^2(\Omega)}^2 \leq \int_0^t \left|\eta_1^2(s) - \eta_2^2(s)\right|_{L^2(\Omega)}^2\,ds. \tag{59}$$

**Proof.** Apply the Theorem 3.3, let $a : W \times W \to \mathbb{R}$ such that

$$a(\zeta, \xi) = \hat{k} \int_{\Omega} \nabla\zeta \nabla\xi \, dx, \quad \forall \zeta, \xi \in H^1(\Omega), \text{ a.e. } t \in (0, T). \tag{60}$$

We have $\hat{k} > 0$, then we can see that $a$ satisfy the conditions of a Theorem 3.3 and then for a given $\beta_0 \in K$ and $, \eta^2 \in L^2(0, T; L^2(\Omega))$, there exist a unique function $\beta_\eta$ which satisfies

$$\beta_\eta \in K, \, \beta_\eta \in W^{1,2}(0, T; L^2(\Omega)) \cap L^2\left(0, T; H^1(\Omega)\right), \tag{61}$$

$$\left(\dot{\beta}_\eta(t), \zeta - \beta_\eta\right) + a\left(\beta_\eta(t), \zeta - \beta_\eta\right) \geq \left(\eta^2(t), \zeta - \beta_\eta\right), \quad \forall \zeta \in K \quad \text{a.e. } t) \tag{62}$$

$$\beta_\eta(0) = \beta_0. \tag{63}$$

Let $t \in [0, T]$ and $\beta_1$, $\beta_2$ the two solutions of problem $\mathcal{P}^{\eta_i}$, corresponding to data $\eta^2 = \eta_i^2$, $i = 1, 2$, then we have

$$|\beta_1(t) - \beta_2(t)|^2_{L^2(\Omega)} + \hat{k} \int_0^t |\nabla(\beta_1(s) - \beta_2(s))|^2_{L^2(\Omega)} \tag{64}$$

$$\leq \int_0^t \left|\eta_1^2 - \eta_1^2\right|^2_{L^2(\Omega)} ds + \int_0^t |\beta_1(t) - \beta_2(t)|^2_{L^2(\Omega)} ds. \tag{65}$$

Will $\tau_i$, $i = 1, 2$, belongs to $L^2(0, T; L^2(\Omega))$, applying Gronwall's lemma to $|\beta_1(t) - \beta_2(t)|^2_{L^2(\Omega)}$, we have

$$|\beta_1(t) - \beta_2(t)|^2_{L^2(\Omega)} \leq \int_0^t \left|\eta_1^2(s) - \eta_1^2(s)\right|^2_{L^2(\Omega)} ds. \tag{66}$$

□

Let now define the operator

$$\Lambda : Q \to Q \text{ by}$$

where $Q = L^2\left(0, T; V' \times L^2(\Omega)\right)$, $Q$ is a Banach space endowed with the norm

$$\left|(\eta^1, \eta^2)\right|_Q = \left|\eta^1\right|_{L^2(0,T;V')} + \left|\eta^2\right|_{L^2(0,T;L^2(\Omega))}$$

$$\Lambda(\eta) = \left(\mathcal{E}^* \nabla\varphi_\eta + \int_0^t \mathcal{C}(t - s, \boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(s)), \beta_\eta(s)) ds \; ; \; \phi\left(\boldsymbol{\varepsilon}\left(\mathbf{u}_\eta\right), \beta_\eta\right)\right), \tag{67}$$

where $\mathbf{u}_\eta$, $\varphi_\eta$, $\beta_\eta$ are respectively the unique solutions of problems $\mathcal{P}_1^\eta$, $\mathcal{P}^\eta$ and $\mathcal{P}_2^\eta$. We have this result.

**Theorem 3.4.** *The operator* $\Lambda$ *has a unique fixed point*

$$\eta^* \in L^2\left(0,T;V'\times L^2(\Omega)\right).$$

**Proof.** Let $t\in[0,T]$, $\eta_i\in Q$, $i=1,2$, $\eta_1=\left(\eta_1^1,\eta_1^2\right)$, $\eta_2=\left(\eta_2^1,\eta_2^2\right)$ ,and use the notation $\mathbf{u}_{\eta_i}=\mathbf{u}_i$, $\varphi_{\eta_i}=\varphi_i$ and $\beta_{\eta_i}=\beta_i$, $i=1,2$, then by applying proprieties (23 (b)), (24),

$$\begin{aligned}|\Lambda\eta_1-\Lambda\eta_2|_{V'\times L^2(\Omega)} &\le C_{\mathcal{E}^*}\left|\varphi_1-\varphi_2\right|_W+\left|\phi\left(\boldsymbol{\varepsilon}\left(\mathbf{u}_1\right),\beta_1\right)-\phi\left(\boldsymbol{\varepsilon}\left(\mathbf{u}_2\right),\beta_2\right)\right|_{L^2(\Omega)}\\ &+\int_0^t\left|\mathcal{C}(t-s,\boldsymbol{\varepsilon}(\boldsymbol{u}_1(s)),\beta_1(s))-\mathcal{C}\left(t,x,\boldsymbol{\varepsilon}\left(\mathbf{u}_2(s)\right),\beta_2(s)\right)\right|_{\mathcal{H}}ds,\end{aligned}$$

and

$$\begin{aligned}|\Lambda\eta_1-\Lambda\eta_2|_{V'\times L^2(\Omega)}\le c\Big\{&|\varphi_1-\varphi_2|_W+|\mathbf{u}_1-\mathbf{u}_2|_V+|\beta_1-\beta_2|_{L^2(\Omega)}\\ &+|\mathbf{u}_1-\mathbf{u}_2|_{L^2(0,t,V)}+|\beta_1-\beta_2|_{L^2(0,t;L^2(\Omega))}\Big\}.\end{aligned}$$

By the estimation (56) we have

$$\begin{aligned}|\Lambda\eta_1-\Lambda\eta_2|_{V'\times L^2(\Omega)}\le c\Big\{&|\mathbf{u}_1-\mathbf{u}_2|_V+|\beta_1-\beta_2|_{L^2(\Omega)}\\ &+|\mathbf{u}_1-\mathbf{u}_2|_{L^2(0,t,V)}+|\beta_1-\beta_2|_{L^2(0,t;L^2(\Omega))}\Big\}.\end{aligned}$$

Apply now the inequalities (52) and (59), this leads to

$$\begin{aligned}|\Lambda\eta_1-\Lambda\eta_2|_{V'\times L^2(\Omega)} &\le c\left\{\left|\eta_1^1-\eta_2^1\right|_{L^2(0,t,V')}+\left|\eta_1^2-\eta_2^2\right|_{L^2(0,t;L^2(\Omega))}\right\}\\ &= c\left|\eta_1-\eta_2\right|_Q.\end{aligned}$$

Reiterating the estimation $n$ times yields

$$|\Lambda^n\eta_1-\Lambda^n\eta_2|_Q\le\frac{(Tc)^n}{n!}\left|\eta_1-\eta_2\right|_Q.$$

Thus imply that the operator $\Lambda$ is a contraction on $Q$. By Banach's fixed point theorem, $\Lambda$ has a unique fixed point $\eta^*\in L^2(0,T;V'\times L^2(\Omega))$. □

**Proof.** *Proof of Theorem* 3.1

**Existence.** Let $\eta^*\in Q$ a unique fixed point of $\Lambda$ and $(\mathbf{u}_{\eta^*};\varphi_{\eta^*};\beta_{\mathcal{T}^*})$ be the solution of the three problems $\mathcal{P}_1^{\eta^*}$, $\mathcal{P}_2^{\eta^*}$ and $\mathcal{P}^{\eta^*}$. Then, using (37), (38), (39) and (67). Keeping in mind that $\Lambda(\eta^*)=\eta^*$, we deduce that $(\mathbf{u}_{\eta^*};\varphi_{\eta^*};\beta_{\mathcal{T}^*})$ is the solution of the problem $\mathcal{P}_V$. The regularity (41) provide from Lemma 3.1, Lemma 3.2 and Lemma 3.3.
**Uniqueness.** The uniqueness of solutions in Theorem 3.1 is the consequence of the uniqueness of the fixed point of the operator $\Lambda$ given by (67). □

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