Every planar graph without 4-cycles and 6-cycles is (2,9)-colorable

Jianqing Ma*
College of Science
Wuhan University of Science and Technology
Wuhan 430065
China
majianqing@wust.edu.cn

Mingfang Huang
Department of Mathematics
School of Science
Wuhan University of Technology
Wuhan 430070
China
ds_hmf@126.com

Xiaoxia Zhang
Department of Mathematics
Xinyang Normal University
Xinyang 464000
China
xiaoxiazhang_1983@163.com

Abstract. Let $d_1, d_2, \ldots, d_k$ be k nonnegative integers. A graph G is $(d_1, d_2, \ldots, d_k)$-colorable if the vertex set of G can be partitioned into k sets $V_1, V_2, \ldots, V_k$ such that the subgraph $G[V_i]$ has maximum degree at most d_i for $i = 1, \ldots, k$. Sittitrai and Nakprasit proved that every planar graph without 4-cycles and 5-cycles is (2,9)-colorable [Discrete Math., 341 (2018) 2142-2150]. In this paper, we prove that every planar graph without 4-cycles and 6-cycles is (2,9)-colorable.

Keywords: defective coloring, planar graph, cycle.

1. Introduction

Let $d_1, d_2, \ldots, d_k$ be k nonnegative integers. A graph G is $(d_1, d_2, \ldots, d_k)$-colorable if the vertex set can be partitioned into k sets $V_1, V_2, \ldots, V_k$ such that the subgraph $G[V_i]$ has maximum degree at most d_i for $i = 1, \ldots, k$. Any such coloring is called a $(d_1, d_2, \ldots, d_k)$-coloring of G. Thus, Grötzsch's Theorem [12] can be stated as every planar graph without 3-cycles is (0,0,0)-colorable. Improper colorability of graphs has been extensively studied. Cowen, Cowen, and Woodall [6] proved that every planar graph is (2,2,2)-colorable. Eaton and

*. Corresponding author

Hull [10] showed that this is sharp by exhibiting non-$(1, k, k)$-colorable planar graphs for each k.

Recently, many authors are devoted to investigating (d_1, d_2)-colorability of planar graphs. For planar graphs without one short cycle, Montassier and Ochem [15] constructed planar graphs without 3-cycles are not (d_1, d_2)-colorable for given any d_1 and d_2. Choi, Liu and Oum [9] proved that every planar graph without 4-cycles is (5, 5)-colorable and posed the following questions.

Problem 1.1. *What is the minimum d such that every planar graph without 4-cycles is (d, d)-colorable?*

Problem 1.2. *What is the minimum d such that every planar graph without 3-, 4-, 6-cycles is $(0, d)$-colorable?*

For planar graphs without two short cycles, every planar graph without 3-cycles and 4-cycles is $(2, 6)$-, $(4, 4)$-colorable proved by Havet and Sereni [11], $(3, 5)$-colorable proved by Choi and Raspaud [8], and $(1, 10)$-colorable proved by Choi, Choi, Jeong and Suh [7]. Sittitrai and Nakprasit [16] proved that every planar graph without 4-cycles and 5-cycles is $(4, 4)-, (3, 5)-, (2, 9)$-colorable. As Problems 1.1 and 1.2, one naturally asks the following question.

Problem 1.3. *For a pair of integers (i, j) where $3 \leq i < j$, what is the minimum d_1 for a given d_2 such that every planar graph without i-cycles and j-cycles is (d_1, d_2)-colorable?*

For planar graphs without more than two short cycles, more results can be found in [2, 3, 4, 5, 13, 14, 15, 17, 18, 20]. On the other hand, Xu and Wang [19] posed the following conjecture.

Conjecture 1.4. *Every planar graph without 4-cycles and 6-cycles is $(0, 0, 0)$-colorable.*

Motivated by Problem 1.3 and Conjecture 1.4, we present the following result.

Theorem 1.5. *Every planar graph without 4-cycles and 6-cycles is (2,9)-colorable.*

At the end of this section, we give some notation and terminology. For a plane graph G, we use $V(G)$, $E(G)$ and $F(G)$ to denote its vertex set, edge set and face set, respectively. A vertex v is called a *k-vertex* (*k^+-vertex* or *k^--vertex*) if $d(v) = k$ ($d(v) \geq k$ or $d(v) \leq k$). A similar notation will be used for faces. For a face $f \in F(G)$, we use $b(f)$ to denote the boundary walk of f and write $f = [u_1 u_2 \ldots u_m]$ if $u_1, u_2, \ldots, u_m$ are the boundary vertices of f in a cyclic order. A k-face $f = [u_1 u_2 \ldots u_k]$ is called an $(a_1, a_2, \ldots, a_k)$-face if $d(u_i) = a_i$ for $i = 1, 2, \ldots, k$. A 2-vertex is *bad* if it is incident with a 3-face and good otherwise. A good 2-vertex v is called a *type-I* good 2-vertex if v is incident to a 5-face and *type-II* good 2-vertex otherwise. A 3-face is *bad* if it is incident to a bad 2-vertex and good otherwise. A *pendent* 3-face of a vertex v is a 3-face which does not contain v but contains a 3-vertex u adjacent to v, and v is called a *pendent neighbor* of u. A 5_2*-face* is a 5-face incident with two 2-vertices. A vertex h is the *head* of a 5_2-face $f = [hv_1v_2v_3v_4]$ if $d(v_1) = d(v_4) = 2$. If a vertex v is incident to two 5-faces g_1 and g_2 such that g_1 and g_2 have a common edge vu, then we say v is incident with a *double 5-hole*. If v is incident with k 5-faces $f_1, f_2, \dots, f_k$ such that f_j and f_{j+1} is a double 5-hole of v for $j = 1, \dots k-1$, then we say v is incident with consecutive k 5-holes. In particular, let f be a 5-face incident with v, we say v is incident with a lonely 5-hole f if v is not incident with any 5-face g such that f and g have a common edge incident with v, where $f \neq g$. A 4-vertex v is *semi-poor* if v is incident with a consecutive three 5-holes and two type-I good 2-vertices. A 4-vertex v is *poor* if v is incident with a double 5-hole with $f_1 = [vv_1v_2v_3v_4], f_2 = [vv_4v_5v_6v_7]$ such that $d(v_1) = d(v_7) = 2$, and v is incident with a type-II good 2-vertex or a pendent 3-neighbor. A 4-vertex is *rich* if it is neither poor nor semi-poor.

We organize this paper as follows. In Section 2, we show the reducible configurations useful in our proof. In Section 3, we complete the proof of Theorem 1.5 by discharging process.

2. Reducible configurations

Suppose that Theorem 1.5 is not true. Let G be a counterexample with $|V(G)| + |E(G)|$ minimized, then G is connected. A face is called *simple* if its boundary forms a cycle. Otherwise, f is called *non-simple*. Since there are neither 4-cycles nor 6-cycles in G, we have the following lemma.

Lemma 2.1 ([1])**.** (1) *There are no adjacent 3-faces.*
(2) *There are no 4-faces and no simple 6-faces.*
(3) *There are no non-simple 5-faces.*
(4) *Any 5-face is not adjacent to 3-faces.*
(5) *If f is a non-simple 6-face, then its boundary consists of two edge-disjoint triangles and the two triangles have a common vertex.*
(6) *There are no non-simple 7-faces.*

Lemma 2.2. (1) *(Lemmas 1 and 2, [8]) If v is a 3^--vertex, then v is adjacent to at least two 4^+-vertices, one of which is a 11^+-vertex.*

(2) *(Lemma 3, [8]) If v is a 12^--vertex, then v is adjacent to at least one 4^+-vertex.*

Lemma 2.3. (1) *If a 2-vertex v is incident with a bad 3-face f_1, then the other face f_2 incident with v is a 8^+-face.*

(2) *If a 2-vertex v is incident with a 5-face, then the other face incident with v is a 7^+-face.*

Proof. (1) Let $f_1 = [uvw]$ be the bad 3-face. Suppose to the contrary that f_2 is a 7^--face. By Lemma 2.2(1), no two 2-vertices is adjacent, then there are no non-simple 6-faces by Lemma 2.1(5). By Lemma 2.1, we only need to consider the cases that f_2 is a simple 7-face. Let f_2 be a simple 7-face and $f_2 = [v_1v_2v_3v_4uvw]$, then $v_1v_2v_3v_4uw$ is a 6-cycle in G, a contradiction.

(2) It follows from that there are neither 4-cycles nor 6-cycles in G. □

Lemma 2.4. *If f is a k-face with $k \geq 8$, then f is incident to at most $k-7$ bad 2-vertices.*

Proof. Similar to the proof of Lemma 2.3, a cycle $C_{m-1} (m \geq 4)$ can be obtained from C_m incident to a bad 2-vertex. It follows that we can obtain C_6 from a k-face incident to $k-6$ bad 2-vertices. This completes the proof. □

Lemma 2.5. *Let $f = [v_1vv_2]$ be a bad 3-face with $d(v) = 2$. Then one of the following statements is true.*
(1) One of v_1, v_2 is a 5^+-vertex, the other is a 11^+vertex; or
(2) One of v_1, v_2 is a 4-vertex, the other is a 12^+-vertex.

Proof. By the minimality of G, $G - v$ admits a (2,9)-coloring φ. If $\varphi(v_1) = \varphi(v_2)$, then we can color v with the color $\{1, 2\} \setminus \varphi(v_1)$, a contradiction. By symmetry, we assume that $\varphi(v_1) = 1$ and $\varphi(v_2) = 2$. If v_1 has at most one neighbor colored 1, then we can color v with 1 and obtain a $(2, 9)$-coloring of G, a contradiction. Thus, v_1 has two neighbors colored 1. If v_2 has at most eight neighbors colored 2, then we can color v with 2 and obtain a $(2, 9)$-coloring of G, a contradiction. Thus, v_2 has nine neighbors colored 2. Hence, $d(v_1) \geq 4$ and $d(v_2) \geq 11$. If $d(v_1) = 4$ and $d(v_2) = 11$, we can recolor v_1 with 2, recolor v_2 with 1 and color v with 1 or 2, then we obtain a $(2, 9)$-coloring of G, a contradiction. This implies that $d(v_1) \geq 5$ or $d(v_2) \geq 12$. The lemma is proved. □

3. Proof of Theorem 1.5

To prove Theorem 1.5, we are going to derive a contradiction by a discharging procedure.

We assume that G is a counterexample to Theorem 1.5 with $|V(G)|+|E(G)|$ minimized. The initial charge function ch is defined as: $ch(v) = 2d(v) - 6$ for $v \in V(G)$ and $ch(f) = d(f) - 6$ for $f \in F(G)$. By the Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$ and the Handshaking lemma,

$$\sum_{v \in V(G)} (2d(v) - 6) + \sum_{f \in F(G)} (d(f) - 6) = -12.$$

We then define suitable discharging rules and redistribute charge accordingly. Note that any discharging procedure preserves the total charge sum of G. If the final charge function on G such that $ch'(x) \geq 0$ for all $x \in V(G) \cup F(G)$, then we get an obvious contradiction.

$$0 \leq \sum_{x \in V(G) \cup F(G)} ch'(x) = \sum_{x \in V(G) \cup F(G)} ch(x) = -12.$$

Our discharging rules are defined as follows:

(R1) Every 4-vertex sends 1 to each incident 3-face; $\frac{1}{2}$ to each adjacent good 2-vertex; $\frac{1}{2}$ to each pendent 3-face.

(R2) Every k-vertex for $5 \leq k \leq 10$ sends $\frac{3}{2}$ to each incident 3-face; $\frac{1}{2}$ to each adjacent good 2-vertex; $\frac{1}{2}$ to each pendent 3-face.

(R3) Every 11-vertex sends $\frac{5}{2}$ to each incident 3-face; $\frac{3}{2}$ to each adjacent good 2-vertex; $\frac{3}{2}$ to each pendent 3-face.

(R4) Every 12^+-vertex sends 3 to each incident 3-face; $\frac{3}{2}$ to each adjacent good 2-vertex; $\frac{3}{2}$ to each pendent 3-face.

(R5) Every rich 4-vertex v sends $\frac{1}{2}$ to each incident 5-face f such that v has at most one 2-neighbor on $b(f)$; Every poor 4-vertex sends $\frac{1}{4}$ to each incident 5-face; Every semi-poor 4-vertex sends $\frac{1}{3}$ to each incident 5-face.

(R6) Every 5^+-vertex v sends $\frac{1}{2}$ to each incident 5-face f such that v has at most one 2-neighbor on $b(f)$

(R7) Every 8^+-face sends 1 to each incident bad 2-vertex. Every 7^+-face adjacent to a 5-face such that these two faces are both incident with a good 2-vertex sends $\frac{1}{3}$ to the 5-face.

(R8) Every bad 3-face sends 1 to each incident 2-vertex.

We shall show that each $x \in F(G) \cup V(G)$ has final charge $ch'(x) \geq 0$. First consider the vertices. Let v be a k-vertex and let t be the number of 3-faces incident with v. Let v be incident with k_1 lonely 5-holes such that v has at most one 2-neighbor on every 5-face, k_2 incident 5_2-faces such that v is the head of every 5_2-face. Note that each 5_2-face has exactly one head by Lemma 2.2(1).

(1) $k = 2$. Initially $ch(v) = -2$. By Lemma 2.2(1), v is adjacent to two 4^+-vertices, one of which is a 11^+-vertex. If v is a good 2-vertex, then $ch'(v) \geq ch(v) + \frac{1}{2} + \frac{3}{2} = 0$ by (R1)-(R4). If v is a bad 2-vertex, then v is incident to an 8^+-face by Lemma 2.3(1). Hence $ch'(v) \geq ch(v) + 1 + 1 = 0$ by (R7) and (R8).

(2) $k = 3$. Since each 3-vertex is not involved in redistributing discharge procedure, $ch'(v) = ch(v) = 0$.

(3) $k = 4$. Assume that v is incident with consecutive four 5-holes, then $ch'(v) \geq ch(v) - 4 \times \frac{1}{2} = 0$ by (R5). Assume that v is incident with consecutive three 5-holes. If v is a rich 4-vertex, then $ch'(v) \geq ch(v) - \frac{1}{2} - 3 \times \frac{1}{2} = 0$ by (R1) and (R5). If v is a semi-poor 4-vertex, then $ch'(v) \geq ch(v) - 2 \times \frac{1}{2} - 3 \times \frac{1}{3} = 0$ by (R1) and (R5). Assume that v is incident with a double 5-hole. If v is a rich 4-vertex, then $ch'(v) \geq ch(v) - 2 \times \frac{1}{2} - 2 \times \frac{1}{2} = 0$. If v is a poor 4-vertex, then $ch'(v) \geq ch(v) - \frac{1}{2} - 2 \times \frac{1}{2} - 2 \times \frac{1}{4} = 0$. Thus v is not incident with the consecutive 5-holes. By (R1) and (R5), $ch'(v) \geq ch(v) - [t + (\frac{1}{2} + \frac{1}{2})k_1 + 2 \times \frac{1}{2}k_2 + \frac{1}{2}(4 - 2t - 2k_1 - 2k_2)] = 0$.

(4) $5 \leq k \leq 10$. If v is incident with consecutive k 5-holes, then $ch'(v) \geq ch(v) - k \times \frac{1}{2} = \frac{3}{2}k - 6 > 0$ by (R6). If v is incident with the consecutive i 5-holes with $2 \leq i \leq k-1$, then v sends at most $\frac{1}{2}i + 2 \times \frac{1}{2}$ to the incident consecutive i 5-holes together with the 2-neighbors on the consecutive i 5-holes. We can think that v sends charge equality $\frac{\frac{i}{2}+2\times\frac{1}{2}}{i+1}$ to every neighbor incident to the consecutive i 5-holes. Define a real value function $f(i) = \frac{\frac{i}{2}+2\times\frac{1}{2}}{i+1}$ where $2 \leq i \leq k-1$. It is easy to check that $f_{max}(i) = f(2)$. Therefore, in order to obtain the maximal charge v sends, we only consider the case that v is incident with the double 5-hole about the consecutive 5-holes. Let v be incident with q_2 double 5-holes. Since $2t + 2q_2 \leq d(v) = k$, hence, by (R2) and (R6),

$$\begin{aligned}
ch'(v) &\geq ch(v) - [\frac{3}{2}t + (\frac{1}{2} + \frac{1}{2})k_1 + 2 \times \frac{1}{2}k_2 + (2 \times \frac{1}{2} + 2 \times \frac{1}{2})q_2 \\
&\quad + \frac{1}{2}(k - 2t - 2k_1 - 2k_2 - 3q_2)] \\
&= 2k - 6 - (\frac{1}{2}t + \frac{1}{2}q_2 + \frac{1}{2}k) \\
&= \frac{3}{2}k - 6 - \frac{1}{2}(t + q_2) \\
&\geq \frac{3}{2}k - \frac{1}{2} \times \frac{k}{2} - 6 \\
&= \frac{5}{4}k - 6 > 0.
\end{aligned}$$

(5) $k = 11$. If v is incident with consecutive k 5-holes, then $ch'(v) \geq ch(v) - k \times \frac{1}{2} = \frac{3}{2}k - 6 > 0$ by (R6). If v is incident with the consecutive i 5-holes with $2 \leq i \leq k-1$, then v sends at most $\frac{1}{2}i + 2 \times \frac{3}{2}$ to the incident consecutive i 5-holes together with the 2-neighbors on the consecutive i 5-holes. We can think that v sends charge equality $\frac{\frac{i}{2}+2\times\frac{3}{2}}{i+1}$ to every neighbor incident to the consecutive i 5-holes. Define a real value function $f(i) = \frac{\frac{i}{2}+2\times\frac{3}{2}}{i+1}$ where $2 \leq i \leq k-1$. It is easy to check that $f_{max}(i) = f(2)$. Therefore, in order to obtain the maximal charge v sends, we only consider the case that v is incident with the double 5-hole about the consecutive 5-holes. Let v be

incident with q_2 double 5-holes. If $t \geq 1$ or $k_1 \geq 1$ or $q_2 \geq 1$, then by (R3) and (R6),

$$\begin{aligned} ch'(v) &\geq ch(v) - [\frac{5}{2}t + (\frac{1}{2} + \frac{3}{2})k_1 + 2 \times \frac{3}{2}k_2 + (2 \times \frac{3}{2} + 2 \times \frac{1}{2})q_2 \\ &\quad + \frac{3}{2}(11 - 2t - 2k_1 - 2k_2 - 3q_2)] \\ &= \frac{t}{2} + k_1 + \frac{1}{2}q_2 - \frac{1}{2} \geq 0. \end{aligned}$$

If $t = k_1 = q_2 = 0$, then v is adjacent to at least one 4^+-vertex by Lemma 2.2(2). Thus, v is adjacent to at most ten 3^--vertices. Hence, $ch'(v) \geq ch(v) - (2 \times \frac{3}{2})k_2 - \frac{3}{2} \times (10 - 2k_2) = 1 > 0$.

(6) $k \geq 12$. Let v be adjacent to s type-II good 2-vertices, p pendent 3-faces, and let v be incident with q_i precisely consecutive i 5-holes, for $2 \leq i \leq k-1$. If v is incident with k 5-faces, then $ch^{'}(v) \geq ch(v) - k \times \frac{1}{2} = \frac{3}{2}k - 6 > 0$ by (R6). Thus, assume that v is incident with at most $(k-1)$ 5-faces. In this case, $2(k_1 + k_2 + t) + p + s + \sum_{i=2}^{k-1}(i+1)q_i \leq d(v)$. Since $i \geq 2$, $\frac{3}{2}(i+1) \geq \frac{i}{2} + 3$. Hence,

$$\begin{aligned} ch'(v) &\geq ch(v) - [(\frac{3}{2} + \frac{1}{2})k_1 + (2 \times \frac{3}{2})k_2 \\ &\quad + \sum_{i=2}^{k-1}(i \times \frac{1}{2} + 2 \times \frac{3}{2})q_i + \frac{3}{2}s + 3t + \frac{3}{2}p] \\ &\geq 2d(v) - 6 - \frac{3}{2}[2k_1 + 2k_2 + 2t + p + s + \sum_{i=2}^{k-1}(i+1)q_i] \\ &\geq 2d(v) - 6 - \frac{3}{2}d(v) = \frac{1}{2}d(v) - 6 \geq 0. \end{aligned}$$

We next analyze the final charge for $f \in F(G)$.

(1) $d(f) = 6$. Since each 6-face is not involved in redistributing discharge procedure, $ch'(f) = ch(f) = 0$.

(2) $d(f) = 7$. By Lemma 2.2(1), f is incident with at most three 2-vertices. Thus, $ch'(f) \geq ch(f) - 3 \times \frac{1}{3} = 0$ by (R7).

(3) $d(f) = k = 8^+$. By Lemma 2.4, every k-face with $k \geq 8$ is incident with at most $k - 7$ bad 2-vertices. By Lemma 2.2(1), f is incident with at most $\lfloor \frac{k}{2} \rfloor$ 2-vertices. Let f be incident with n_1 bad 2-vertices. Then $n_1 \leq k - 7$ and f is incident with at most $\lfloor \frac{k}{2} \rfloor - n_1$ good 2-vertices. Thus, $ch'(f) \geq ch(f) - n_1 - \frac{1}{3} \times (\lfloor \frac{k}{2} \rfloor - n_1) \geq \frac{5}{6}k - 6 - \frac{2}{3}n_1 \geq \frac{5}{6}k - 6 - \frac{2}{3} \times (k-7) = \frac{k-8}{6} \geq 0$ by (R7).

(4) $d(f) = 3$. If f is a bad 3-face, then f is a $(2, 5^+, 11^+)$-face or f is a $(2, 4, 12^+)$-face by Lemma 2.5. In the former case, $ch'(f) \geq ch(f) + \frac{3}{2} + \frac{5}{2} - 1 = 0$ by (R2)-(R4) and (R8). In the latter case, $ch'(f) \geq ch(f) + 1 + 3 - 1 = 0$ by (R1),(R4) and (R8).

We now consider that f is a good 3-face. If f is incident to a 3-vertex, then f is incident to a 4^+-vertex by Lemma 2.2(1). Thus f is a $(3, 3, k)-$, $(3, 3, 11^+)-$, $(3, 4^+, k)-$, $(3, 4^+, 11^+)-$ or a $(4^+, 4^+, 4^+)-$face where $4 \leq k \leq 10$. Let $f = [v_1v_2v_3]$ with $d(v_1) \leq d(v_2) \leq d(v_3)$. If f is a $(3, 3, k)$-face with $4 \leq k \leq 10$, then the pendent neighbor of a 3-vertex is a 11^+-vertex by Lemma 2.2(1). By (R1)-(R4), f receives $2 \times \frac{3}{2}$ from the pendent neighbors of v_1, v_2 and gets 1 from v_3. Hence $ch'(f) \geq ch(f) + 2 \times \frac{3}{2} + 1 > 0$. If f is a $(3, 3, 11^+)$-face, then the pendent neighbor of a 3-vertex is a 4^+-vertex by Lemma 2.2(1). By (R1)-(R4), f receives $2 \times \frac{1}{2}$ from the pendent neighbors of v_1, v_2 and gets $\frac{5}{2}$ from v_3. Hence $ch'(f) \geq ch(f) + 2 \times \frac{1}{2} + \frac{5}{2} > 0$. If f is a $(3, 4^+, k)$-face with $4 \leq k \leq 10$, then the pendent neighbor of v_1 is a 11^+-vertex by Lemma 2.2(1). By (R1)-(R4), f receives $\frac{3}{2}$ from the pendent neighbors of v_1 and gets 1×2 from v_2, v_3. Hence $ch'(f) \geq ch(f) + \frac{3}{2} + 1 \times 2 > 0$. If f is a $(3, 4^+, 11^+)$-face, then f gets 1 from v_2 and gets $\frac{5}{2}$ from v_3. Hence $ch'(f) \geq ch(f) + \frac{5}{2} + 1 > 0$. Finally, if f is a $(4^+, 4^+, 4^+)$-face, then f gets at least 1 from each of v_1, v_2, v_3 by (R1)-(R4). Hence $ch'(f) \geq ch(f) + 3 \times 1 = 0$.

(5) $d(f) = 5$. First assume that there exists neither poor 4-vertices nor semi-poor 4-vertices on $b(f)$. If there exists exactly two 2-vertices on $b(f)$, then f is a 5_2-face. By Lemma 2.2(1), a 2-vertex is adjacent to at least two 4^+-vertices, one of which is a 11^+-vertex. If the head of f is a k-vertex with $4 \leq k \leq 10$, then f is a $(2, k, 2, 11^+, 11^+)$-face. By (R6), each of the two 11^+-vertices sends at least $\frac{1}{2}$ to f. Thus, $ch'(f) \geq ch(f) + 2 \times \frac{1}{2} = 0$. If the head of f is a 11^+-vertex, then f is a $(2, 11^+, 2, 4^+, 4^+)$-face. By (R5) and (R6), each of the two 4^+-vertices sends $\frac{1}{2}$ to f. Thus, $ch'(f) \geq ch(f) + 2 \times \frac{1}{2} = 0$. If there exists exactly one 2-vertex v on $b(f)$, then v is adjacent to at least two 4^+-vertices, one of which is a 11^+-vertex by Lemma 2.2(1). By (R5) and (R6), the 4^+-vertex and 11^+-vertex send $\frac{1}{2} + \frac{1}{2} = 1$ to f. Hence $ch'(f) \geq ch(f) + \frac{1}{2} + \frac{1}{2} = 0$. Thus, we assume that there is no 2-vertex on $b(f)$. If there exists one 3-vertex v on $b(f)$, then v is adjacent to at least two 4^+-vertices, one of which is a 11^+-vertex by Lemma 2.2(1). Let $f = [vv_1v_2v_3v_4]$. Assume first that $d(v_1) \geq 4$ and $d(v_4) \geq 4$. By (R5) and (R6), each of v_1 and v_4 sends $\frac{1}{2}$ to f and thus $ch'(f) \geq ch(f) + \frac{1}{2} + \frac{1}{2} = 0$. Next, assume that $d(v_1) \geq 4$ and $d(v_4) = 3$ by symmetry. Then $d(v_3) \geq 4$ by Lemma 2.2(1). By (R5) and (R6), each of v_1 and v_3 sends $\frac{1}{2}$ to f and hence $ch'(f) \geq ch(f) + \frac{1}{2} + \frac{1}{2} = 0$. If there exists no 3^--vertex v on $b(f)$, then $ch'(f) \geq ch(f) + 2 \times \frac{1}{2} = 0$ by (R5) and (R6).

Next, assume that there exists a poor 4-vertex v on $b(f)$. We first assume that f is a 5_2-face. By the definition of the poor 4-vertex, v is not the head of f. Let $f = [vv_1v_2v_3v_4]$ and $d(v_2) = d(v_4) = 2$. Then $d(v_1) \geq 4$ by

Lemma 2.2(1). Let f_1 be the other face rather than f incident with v_2 and let f_2 be the other face rather than f incident with v_4. By Lemma 2.3(2), both f_1 and f_2 are 7^+-faces. By (R7), each of f_1 and f_2 sends $\frac{1}{3}$ to f. By (R5), poor 4-vertex v sends $\frac{1}{4}$ to f and 4^+-vertex v_1 sends at least $\frac{1}{4}$ to f . Hence, $ch'(f) \geq ch(f)+2\times\frac{1}{3}+\frac{1}{4}+\frac{1}{4}=\frac{1}{6}>0$. We then assume that f is not a 5_2-face. Let $f=[vv_1v_2v_3v_4]$ and $d(v_4)=2$. By Lemma 2.2(1), $d(v_3)\geq 11$. Let f_2 be the other face rather than f incident with v_4. By Lemma 2.3(2), f_2 is a 7^+-face. By (R7), f_2 sends $\frac{1}{3}$ to f. By (R5), poor 4-vertex v sends $\frac{1}{4}$ to f. By (R6), v_3 sends $\frac{1}{2}$ to f. Hence $ch'(f)\geq ch(f)+\frac{1}{3}+\frac{1}{4}+\frac{1}{2}=\frac{1}{12}>0$.

Finally, assume that there exists a semi-poor 4-vertex v on $b(f)$. Let three consecutive 5-faces incident to v be $f_1=[vv_1v_2v_3v_4]$, $f_2=[vv_4v_5v_6v_7]$ and $f_3=[vv_7v_8v_9v_{10}]$. By symmetry, we only need to consider 5-face f_1 and f_2. By the definition of semi-poor 4-vertex, $d(v_1)=d(v_{10})=2$. We first assume that f_1 is not a 5_2-face. By Lemma 2.2(1), $d(v_2)\geq 11$. Let $f_1^{'}$ be the other face rather than f_1 incident with v_1. By Lemma 2.3(2), $f_1^{'}$ is a 7^+-face. By (R7), $f_1^{'}$ sends $\frac{1}{3}$ to f. By (R5), semi-poor 4-vertex v sends $\frac{1}{3}$ to f_1. By (R6), 11^+-vertex v_2 sends $\frac{1}{2}$ to f_1. Hence, $ch'(f_1)\geq ch(f_1)+2\times\frac{1}{3}+\frac{1}{2}=\frac{1}{6}>0$.

Next we calculate the final charge of f_2. Suppose firstly $d(v_4)=3$. Assume that $d(v_5)=3$, then $d(v_6)\geq 4$ by Lemma 2.2(1). By the definition of poor 4-vertex, neither v_6 nor v_7 is a poor 4-vertex. By (R5) and (R6), each of v_6 and v_7 sends at least $\frac{1}{3}$ to f_2, v sends $\frac{1}{3}$ to f_2. Hence $ch'(f_2)\geq ch(f_2)+3\times\frac{1}{3}=0$. Assume that $d(v_5)\geq 4$. If v_5 is a poor 4-vertex, then $d(v_6)=2$ by the definition of poor 4-vertex. Hence $d(v_7)\geq 11$ by Lemma 2.2(1). Let $f_6^{'}$ be the other face incident with v_6. By Lemma 2.3(2), $f_6^{'}$ is a 7^+-face. By (R7), $f_6^{'}$ sends $\frac{1}{3}$ to f_2. By (R6), v_7 sends $\frac{1}{2}$ to f_2. By (R5), semi-poor 4-vertex v sends $\frac{1}{3}$ to f_1. Hence $ch'(f_2)\geq ch(f_2)+\frac{1}{3}+\frac{1}{3}+\frac{1}{2}>0$. Next suppose that v_5 is not a poor 4-vertex. By (R5) and (R6), v_5 sends at least $\frac{1}{3}$ to f_2. If $d(v_6)=2$, let $f_6^{'}$ be the other face incident with v_6. By Lemma 2.3(2), $f_6^{'}$ is a 7^+-face. By (R7), $f_6^{'}$ sends $\frac{1}{3}$ to f_2. Thus $ch'(f_2)\geq ch(f_2)+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=0$. If $d(v_6)\geq 3$, by Lemma 2.2(2), v is adjacent to at least one 4^+-vertex, then v_7 is a 4^+-vertex and v_7 is not a poor 4-vertex. By (R5) and (R6), v_7 sends at least $\frac{1}{3}$ to f_2. Thus $ch'(f_2)\geq ch(f_2)+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=0$.

Suppose $d(v_4)\geq 4$. It is clear that v_4 is not a poor 4-vertex. If v_4 is a semi-poor 4-vertex, then $d(v_5)=2$. let $f_5^{'}$ be the other face incident with v_5. By Lemma 2.3(2), $f_5^{'}$ is a 7^+-face. By (R7), $f_5^{'}$ sends $\frac{1}{3}$ to f_2. By (R5), each of v and v_4 sends $\frac{1}{3}$ to f_2. Thus $ch'(f_2)\geq ch(f_2)+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=0$. If v_4 is not a semi-poor 4-vertex, then v_4 sends $\frac{1}{2}$ to f_2 by (R6). Assume that $d(v_5)=2$, then the other face incident with v_5 other than f_2 sends $\frac{1}{3}$ to f_2 by Lemma 2.3(2) and (R7). Hence $ch'(f_2)\geq ch(f_2)+\frac{1}{2}+\frac{1}{3}+\frac{1}{3}=\frac{1}{6}>0$. Assume that $d(v_5)=3$. If $d(v_6)=2$, then the other face incident with v_6 other than f_2 sends $\frac{1}{3}$ to f_2 by Lemma 2.3(2) and (R7). Hence $ch'(f_2)\geq ch(f_2)+\frac{1}{2}+\frac{1}{3}+\frac{1}{3}=\frac{1}{6}>0$. If $d(v_6)=3$, then $d(v_7)\geq 4$ by Lemma 2.2(1). And v_7 is not a poor 4-vertex by the definition of the poor 4-vertex. By (R5) and (R6), v_7 sends at least $\frac{1}{3}$ to f_2. Thus $ch'(f_2) \geq ch(f_2)+\frac{1}{3}+\frac{1}{3}+\frac{1}{2} = \frac{1}{6} > 0$. If $d(v_6) \geq 4$, then v_6 is not a poor 4-vertex. By (R5) and (R6), v_6 sends at least $\frac{1}{3}$ to f_2. Thus $ch'(f_2) \geq ch(f_2) + \frac{1}{3} + \frac{1}{3} + \frac{1}{2} = \frac{1}{6} > 0$. Assume that $d(v_5) \geq 4$, then v_5 sends at least $\frac{1}{4}$ to f_2 by (R5) and (R6). Hence $ch'(f_2) \geq ch(f_2) + \frac{1}{3} + \frac{1}{2} + \frac{1}{4} = \frac{1}{12} > 0$.

Next, assume that f_1 is a 5_2-face, then $d(v_3) = 2$. Let $f_1^{'}$ be the other face incident with v_1 and $f_3^{'}$ be the other face incident with v_3. By Lemma 2.3(2), $f_1^{'}$ and $f_3^{'}$ are 7^+-faces. By (R7), each of $f_1^{'}$ and $f_3^{'}$ sends $\frac{1}{3}$ to f_1. Hence $ch'(f_1) \geq ch(f_1) + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 0$.

Next we calculate the final charge of f_2. By Lemma 2.2(1), $d(v_4) \geq 4$. If v_4 is a poor 4-vertex, then $d(v_5) = 2$. By Lemma 2.2(1), $d(v_6) \geq 11$. By (R5), poor 4-vertex v_4 sends $\frac{1}{4}$ to f_2. By (R6), 11^+-vertex v_6 sends $\frac{1}{2}$ to f_2. Hence $ch'(f_2) \geq ch(f_2) + \frac{1}{3} + \frac{1}{2} + \frac{1}{4} = \frac{1}{12} > 0$. If v_4 is a semi-poor 4-vertex, then v_4 sends $\frac{1}{3}$ to f_2 by (R5). Assume that $d(v_5) = 3$. If $d(v_6) = 3$, then $d(v_7) \geq 4$ by Lemma 2.2(1) and v_7 is not a poor 4-vertex. Thus v_7 sends at least $\frac{1}{3}$ to f_2 by (R5) and (R6). Hence $ch'(f_2) \geq ch(f_2) + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 0$. If $d(v_6) \geq 4$, then v_6 is not a poor 4-vertex. Thus v_6 sends at least $\frac{1}{3}$ to f_2. Hence $ch'(f_2) \geq ch(f_2) + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 0$. Assume that $d(v_5) \geq 4$. If v_5 is not a poor 4-vertex, then v_5 sends at least $\frac{1}{3}$ to f_2 by (R5) and (R6). Hence $ch'(f_2) \geq ch(f_2) + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 0$. If v_5 is a poor 4-vertex, then $d(v_6) = 2$. Thus the other face incident with v_6 is a 7^+-face and sends $\frac{1}{3}$ to f_2 by Lemma 2.3(2) and (R7). Hence $ch'(f_2) \geq ch(f_2) + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 0$. If v_4 is neither a poor 4-vertex nor a semi-poor 4-vertex, then v_4 sends $\frac{1}{2}$ to f_2 by (R5) and (R6). Assume that $d(v_5) = 2$, then the other face incident with v_5 is a 7^+-face and sends $\frac{1}{3}$ to f_2 by Lemma 2.3(2) and (R7). Hence $ch'(f_2) \geq ch(f_2) + \frac{1}{3} + \frac{1}{2} + \frac{1}{3} = \frac{1}{6} > 0$. Assume that $d(v_5) = 3$. If $d(v_6) = 3$, then $d(v_7) \geq 4$. By (R5) and (R6), v_7 sends at least $\frac{1}{4}$ to f_2. Hence $ch'(f_2) \geq ch(f_2) + \frac{1}{3} + \frac{1}{2} + \frac{1}{4} = \frac{1}{12} > 0$. If $d(v_6) \geq 4$, then v_6 sends at least $\frac{1}{4}$ to f_2. Hence $ch'(f_2) \geq ch(f_2) + \frac{1}{3} + \frac{1}{2} + \frac{1}{4} = \frac{1}{12} > 0$. Assume that $d(v_5) \geq 4$. Then v_5 sends at least $\frac{1}{4}$ to f_2. Hence $ch'(f_2) \geq ch(f_2) + \frac{1}{3} + \frac{1}{2} + \frac{1}{4} = \frac{1}{12} > 0$.

4. Acknowledgements

The research of the first author is supported by Guiding Project of Hubei Provincial Department of Education (B2019012) and supported in part by the NSFC (11861069), the second author is supported by the Fundamental Research Funds for the Central Universities(WUT: 2020IB010), the third author is supported by the Natural Science Foundation of China (Grant No.11701496) and Nanhu Scholars Program for Young Scholars of XYNU.

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Accepted: March 12, 2021