Numerical integration by using polynomial $P_5(x)$ of Newton-Gregory forward interpolation formula

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Abstract. In this paper, we will appear that the classical quadrature method for the numerical solution by using polynomial $P_5(x)$ of Newton-Gregory forward interpolation formula in numerical integration.
Keywords: Numerical integration, classical quadrature formula, trapezoidal rule, Simpson's $\frac{1}{3}$ rule, Boole's rule, Weddle's rule.

1. Introduction

Throughout this paper, let $f(x)$ be continuous on the interval $[a, b]$. Then evaluation of the definite integral maybe practically impossible or at least difficult, in this two cases we tried to solve it by numerical integration (NI) is that the study of how the numerical value of an integral are often found. It is also called as quadrature which refers to find a square whose area is the same as the area

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under the curve. The inverse process to differentiation in calculus is represented by:

$$I = \int_a^b f(x)dx, \tag{1.1}$$

which means that the integral of a function $f(x)$ with respect to the independent variable x evaluated between the initial values $x = a$ and $x = b$ (Equation (1.1)). In other word, it is mean evaluate the area under the curve of the function $f(x)$ between the two vertical lines $x = a$ to $x = b$ and the x-axis. We will show that graphically in fig. 1.

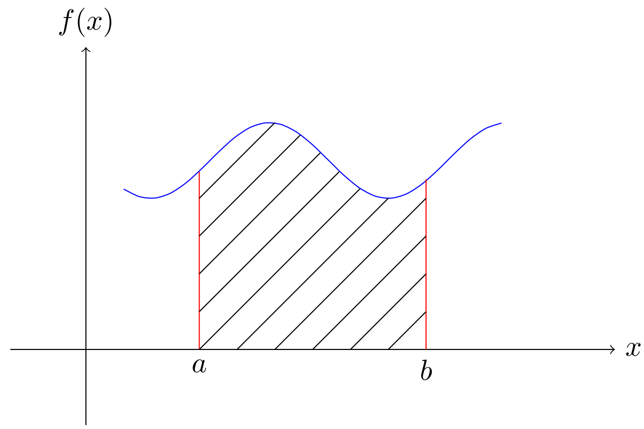

Figure 1: Graphical representation of integral of a function $f(x)$

In the integration, the problem is how can we find the shaded area. Might be the better alternative approach by using simple arithmetic operations to evaluate the area.

The problem of integration is just reduced to the matter of finding shaded area. A better alternative approach might be to use a way that uses simple arithmetic operations to compute the area. This approach is named as NI or numerical quadrature. NI method uses an interpolating polynomial (IP) $P_n(x)$ in the place of $f(x)$ which can be integrated analytically. We have seen several ways of numerical integration which replace the function $f(x)$ by straight line or polynomial $P_n(x)$. In this paper, we interest by the polynomial $P_5(x)$ of order 5. Thus:

$$I = \int_a^b f(x)dx = \int_a^b P_5(x)dx. \tag{1.2}$$

After this brief introduction, we include a section of preliminaries and notation regarding Newton-Gregory forward interpolation formula. We also, review some rules which have been deriving from this formula like Trapezoidal, Simpson's $\frac{1}{3}$, Simpson's $\frac{3}{8}$, Boole and Weddle. In Section 3, we derive the **5A** rule from this formula and we generalize it in Section 4. Finally, in section 5 we discuss and comparing some problems.

2. Preliminaries

In this section, we use the general formula for solving NI, it is also called general quadrature formula.

Choosing interval size $h = \frac{b-a}{n}$, divide the interval $[a, b]$ in to n intervals by means $(n+1)$ equally spaced point $x_0 = a$, $x_n = b$, $x_i = x_0 + ih$, $i = 1, 2, ..., n-1$ and let $y_i = f(x_i)$ for $i = 0, 1, 2, ..., n$, the basic idea as we have seen in numerical integration is to replace the unknown tabulated function $y = f(x)$ by an nth degree polynomial $P_n(x)$ say **Newton-Gregory** forward interpolation formula and carry on the integration. Thus

$$
\begin{aligned}
I &= \int_{a=x_0}^{b=x_n} f(x)dx \\
&\simeq \int_{x_0}^{x_n} P_n(x)dx \\
&= h\int_{q=0}^{n} [y_0 + q\Delta y_0 + \frac{q(q-1)}{2!}\Delta^2 y_0 + \frac{q(q-1)(q-2)}{3!}\Delta^3 y_0 + ... \\
&+ \frac{q(q-1)(q-2)...(q-n+1)}{n!}\Delta^n y_0]dq.
\end{aligned} \tag{2.1}
$$

Here, the new variable is $q = \frac{x-x_0}{h}$, so $dq = \frac{dx}{h}$ where $h = \frac{b-a}{n}$.
By changing n to different values various formulae is used to solve NI, they are:

(i) If $n = 1$, then derivation of Trapezoidal rule formula.

(ii) If $n = 2$, then derivation of Simpson's $\frac{1}{3}$rd rule formula.

(iii) If $n = 3$, then derivation of Simpson's $\frac{3}{8}$th rule formula.

(iv) If $n = 4$, then derivation of Boole's rule formula.

(v) If $n = 6$, then derivation of Weddle's rule formula.

In this paper, we mainly focus on to demonstrating of the rule when $n = 5$ in NI.

3. Main results

We start by using a function $y = f(x)$ on the interval $[x_0, x_5]$, and put the equation (1.2) in the equation (2.1) by taking $n = 5$, this means we have six points and five intervals, so the equation (2.1) becomes:

$$
\begin{aligned}
&\int_{x_0}^{x_5} ydx = h\int_0^5 [y_0 + q\Delta y_0 + \frac{q(q-1)}{2!}\Delta^2 y_0 + \frac{q(q-1)(q-2)}{3!}\Delta^3 y_0 \\
&+ \frac{q(q-1)(q-2)(q-3)}{4!}\Delta^4 y_0 + \frac{q(q-1)(q-2)(q-3)(q-4)}{5!}\Delta^5 y_0]dq \\
&= h\int_0^5 [y_0 + q\Delta y_0 + (\frac{q^2}{2} - \frac{q}{2})\Delta^2 y_0 + (\frac{q^3}{6} - \frac{q^2}{2} + \frac{q}{3})\Delta^3 y_0 \\
&+ (\frac{q^4}{24} - \frac{q^3}{4} + \frac{11q^2}{24} - \frac{q}{4})\Delta^4 y_0 + (\frac{q^5}{120} - \frac{q^4}{12} + \frac{7q^3}{24} - \frac{5q^2}{12} + \frac{q}{5})\Delta^5 y_0]dq \\
&= h[y_0 q + \frac{q^2}{2}\Delta y_0 + (\frac{q^3}{6} - \frac{q^2}{4})\Delta^2 y_0 + (\frac{q^4}{24} - \frac{q^3}{6} + \frac{q^2}{6})\Delta^3 y_0 \\
&+ (\frac{q^5}{120} - \frac{q^4}{16} + \frac{11q^3}{72} - \frac{q^2}{8})\Delta^4 y_0 + (\frac{q^6}{720} - \frac{q^5}{60} + \frac{7q^4}{96} - \frac{5q^3}{36} + \frac{q^2}{10})\Delta^5 y_0]_0^5 \\
&= h[5y_0 + \frac{25}{2}\Delta y_0 + \frac{175}{12}\Delta^2 y_0 + \frac{75}{8}\Delta^3 y_0 + \frac{425}{144}\Delta^4 y_0 + \frac{95}{288}\Delta^5 y_0] \\
&= \frac{h}{288}[1440y_0 + 3600(y_1 - y_0) \\
&+ 4200(y_2 - 2y_1 + y_0) + 2700(y_3 - 3y_2 + 3y_1 - y_0) \\
&+ 850(y_4 - 4y_3 + 6y_2 - 4y_1 + y_0) \\
&+ 95(y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0)] \\
&= \frac{h}{288}[95y_0 + 375y_1 + 250y_2 + 250y_3 + 375y_4 + 95y_5] \\
&= \frac{5h}{288}[19y_0 + 75y_1 + 50y_2 + 50y_3 + 75y_4 + 19y_5].
\end{aligned} \tag{3.1}
$$

Put $A = \frac{1}{288}$ in above equation. That is

$$\int_{x_0}^{x_5} ydx = 5Ah[19y_0 + 75y_1 + 50y_2 + 50y_3 + 75y_4 + 19y_5]. \tag{3.2}$$

So, we name equation (3.2) as $5A$ rule formula.

4. Composite formula

Now, we generalize this result by using the composite integration methods. We divide the interval $[a, b]$ into a number of sub-intervals and evaluate the integral in each sub-interval by the $5A$ rule formula.

We can derive composite formulae from average rule. If the range of integration is from a to $a + nh = b$, then this rule can be improved by dividing the interval $[a, b]$ into sub-intervals of width $5h$ and apply 3.2 for each of the sub-interval. The sum of areas of all sub-intervals is the integral of the interval

$[a, b]$. Thus, equation 3.2 becomes:

$$\begin{aligned}
\int_{x_0}^{x_n} y dx &= \int_{x_0}^{x_5} y dx + \int_{x_5}^{x_{10}} y dx + ... + \int_{x_{n-5}}^{x_n} y dx \\
&= 5Ah[19y_0 + 75y_1 + 50y_2 + 50y_3 + 75y_4 + 19y_5] \\
&+ 5Ah[19y_5 + 75y_6 + 50y_7 + 50y_8 + 75y_9 + 19y_{10}] + ... \\
&+ 5Ah[19y_{n-5} + 75y_{n-4} + 50y_{n-3} + 50y_{n-2} + 75y_{n-1} + 19y_n] \\
&= 5Ah[19(y_0 + y_n) + 38(y_5 + y_{10} + ... + y_{n-5}) \\
&+ 75(y_1 + y_4 + y_6 + y_9 + ... + y_{n-4} + y_{n-1}) \\
&+ 50(y_2 + y_3 + y_7 + y_8 + ... + y_{n-3} + y_{n-2})].
\end{aligned} \tag{4.1}$$

5. Problems

Problem 5.1. In this problem, we will find the approximate value of $\int_1^{11} \frac{dx}{x}$, using Table 1. Also, we find the exact solution and find the solution by using

Table 1: value of x and $y = f(x) = \frac{1}{x}$

x	1	2	3	4	5	6	7	8	9	10	11
y	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	$\frac{1}{9}$	$\frac{1}{10}$	$\frac{1}{11}$

other method.

Finally, we comparing $5A$ rule with other method by finding the absolute errors.

Solution. We have $a = 1$, $b = 11$ and 10 intervals, then $h = \frac{b-a}{n} = \frac{11-1}{10} = 1$. Thus, we use (4.1) to find the Solution:

$$\begin{aligned}
\int_{x_0=1}^{x_{10}=11} \frac{dx}{x} &= 5Ah[19(y_0 + y_n) + 38(y_5 + y_{10} + ... + y_{n-5}) \\
&+ 75(y_1 + y_4 + y_6 + y_9 + ... + y_{n-4} + y_{n-1}) \\
&+ 50(y_2 + y_3 + y_7 + y_8 + ... + y_{n-3} + y_{n-2})] \\
&= \frac{5}{288}[19(y_0 + y_{10}) + 38(y_5) + 75(y_1 + y_4 + y_6 + y_9) \\
&+ 50(y_2 + y_3 + y_7 + y_8)] \\
&= \frac{5}{288}[19(1 + \frac{1}{11}) + 38(\frac{1}{6}) + 75(\frac{1}{2} + \frac{1}{5} + \frac{1}{7} + \frac{1}{10}) \\
&+ 50(\frac{1}{3} + \frac{1}{4} + \frac{1}{8} + \frac{1}{9})] = \frac{5}{288}(138.747114) \\
&= 2.408804.
\end{aligned} \tag{5.1}$$

Now, we find the exact solution

$$\int_1^{11} \frac{dx}{x} = ln(x)|_1^{11} = ln(11) - ln(1) = 2.397895. \tag{5.2}$$

So, the absolute error of the above solution is: $e = |2.408804 - 2.397895| = 0.010909$. The solution by using Trapezoidal rule is:

$$\begin{aligned}\int_{x_0=1}^{x_{10}=11} \frac{dx}{x} &= \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + ... + y_{n-1})] \\ &= \frac{1}{2}[1 + \frac{1}{11} + 2(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10})] \\ &= 2.474423.\end{aligned} \tag{5.3}$$

Here, the absolute error of Trapezoidal rule is: $e = |2.474423 - 2.397895| = 0.076528$. We observe that the solution which is in (5.1) gives more accuracy than the solution which is in (5.3). Hence the $5A$ rule is the better.

Problem 5.2. Find the approximate value of $\int_1^3 f(x)dx$, using Table2 below.

Table 2: value of x and $y = f(x)$

x	1	1.2	1.4	1.6	1.8	2
y	2.287355	3.094479	3.996196	4.950921	5.891435	6.718850
x	2.2	2.4	2.6	2.8	3	
y	7.296691	7.445750	6.940575	5.508762	2.834471	

Solution. Since the number of intervals is 10, we can not use Simpson$\frac{1}{3}$ rule, Simpson$\frac{3}{8}$ rule, Boole's rule and Weddle's rule. We have $a = 1$, $b = 3$ and 10 intervals, then $h = \frac{b-a}{n} = \frac{3-1}{10} = \frac{1}{5}$. Thus we use $5A$ rule to find the Solution:

$$\begin{aligned}&\int_{x_0=1}^{x_{10}=3} f(x)dx = \frac{5h}{288}[19(y_0 + y_n) + 38(y_5 + y_{10} + ... + y_{n-5}) \\ &+ 75(y_1 + y_4 + y_6 + y_9 + ... + y_{n-4} + y_{n-1}) \\ &+ 50(y_2 + y_3 + y_7 + y_8 + ... + y_{n-3} + y_{n-2})] \\ &= \frac{5 \times \frac{1}{5}}{288}[19(y_0 + y_{10}) + 38(y_5) \\ &+ 75(y_1 + y_4 + y_6 + y_9) + 50(y_2 + y_3 + y_7 + y_8)] \\ &= \frac{1}{288}[19(2.287355 + 2.834471) + 38(6.718850) \\ &+ 75(3.094479 + 5.891435 + 7.296691 + 5.508762) \\ &+ 50(3.996196 + 4.950921 + 7.445750 + 6.940575)] \\ &= 10.950193.\end{aligned} \tag{5.4}$$

Problem 5.3. In this problem we will find the approximate value of $\int_0^2 e^{x^2} dx$, using $5A$ rule two times. First one with $h = 0.2$ and the second with $h = 0.1$. Furthermore, we comparing the result with other methods which are already calculated from state-of-the-art studies like [7].

Solution. (i) Using $5A$ rule with $h = 0.2$, the estimate is:

$$\begin{aligned}
\int_0^2 e^{x^2} dx &= \frac{5h}{288}[19(y_0 + y_n) + 38(y_5 + y_{10} + ... + y_{n-5}) \\
&+ 75(y_1 + y_4 + y_6 + y_9 + ... + y_{n-4} + y_{n-1}) \\
&+ 50(y_2 + y_3 + y_7 + y_8 + ... + y_{n-3} + y_{n-2})] \\
&= \frac{5 \times \frac{2}{10}}{288}[19(y_0 + y_{10}) + 38(y_5) \\
&+ 75(y_1 + y_4 + y_6 + y_9) + 50(y_2 + y_3 + y_7 + y_8)] \\
&= \frac{1}{288}[19(1 + 54.598150) + 38(2.718282) \\
&+ 75(1.040811 + 1.896481 + 4.220696 + 25.533722) \\
&+ 50(1.173511 + 1.433329 + 7.099327 + 12.935817)] = 16.470962.
\end{aligned} \tag{5.5}$$

(ii) Using $5A$ rule with $h = 0.1$, the estimate is:

$$\begin{aligned}
\int_0^2 e^{x^2} dx &= \frac{5h}{288}[19(y_0 + y_n) + 38(y_5 + y_{10} + ... + y_{n-5}) \\
&+ 75(y_1 + y_4 + y_6 + y_9 + ... + y_{n-4} + y_{n-1}) \\
&+ 50(y_2 + y_3 + y_7 + y_8 + ... + y_{n-3} + y_{n-2})] \\
&= \frac{5 \times \frac{1}{10}}{288}[19(y_0 + y_{20}) + 38(y_5 + y_{10} + y_{15}) + 75(y_1 + y_4 + y_6 + y_9 + y_{11} \\
&+ y_{14} + y_{16} + y_{19}) + 50(y_2 + y_3 + y_7 + y_8 + y_{12} + y_{13} + y_{17} + y_{18})] \\
&= \frac{1}{576}[19(1 + 54.598150) + 38(1.284025 + 2.718282 + 9.487736) \\
&+ 75(1.01005 + 1.173511 + 1.433329 + 2.247908 + 3.353485 + 7.099327 \\
&+ 12.935817 + 36.966053) + 50(1.040811 + 1.094174 + 1.632316 \\
&+ 1.896481 + 4.220696 + 5.419481 + 17.993310 + 25.533722)] \\
&= 16.453120.
\end{aligned} \tag{5.6}$$

(iii) We have already the Exact Integration of $\int_0^2 e^{x^2} dx$ which is equal to 16.45263. In addition, we have been seen some methods like Trapezoidal rule (TR), Simpson's $\frac{1}{3}$ rule ($SR_{1/3}$), Simpson's $\frac{3}{8}$ rule ($SR_{3/8}$), Boole's rule (BR), Weddle's rule (WR) and proposed NI method of $\int_0^2 e^{x^2} dx$ (for reference see, [7]), which are given below:

(1) The estimate of the given integral (I) by using TR is equal to 16.95311.

(2) The estimate of the given I by using $SR_{1/3}$ is equal to 16.47144.

(3) The estimate of the given I by using $SR_{3/8}$ is equal to 16.49168.

(4) The estimate of the given I by using BR is equal to 16.45674.

(5) The estimate of the given I by using WR is equal to 16.45524.

(6) The estimate of the given I by using the proposed method when $k = 20$ is equal to 16.453890.

Table 3: Evaluating $\int_0^2 e^{x^2} dx$ using different numerical integration methods, n is the arrangement of methods that has small error.

Integration method	Area	Absolute error	n
Exact (Numerical integration)	16.45263	0	0
Trapezoidal rule	16.95311	0.50048	8
Simpson's $\frac{1}{3}$ rule	16.47144	0.01881	6
Simpson's $\frac{3}{8}$ rule	16.49168	0.03905	7
Boole's rule	16.45674	0.00411	4
Weddle's rule	16.45524	0.00261	3
Proposed method when $k = 20$	16.45389	0.00126	2
proposed method when $h = 0.2$	16.470962	0.018332	5
proposed method when $h = 0.1$	16.45312	0.00049	1

6. Conclusion

Comparing our proposed method ($5A$ rule) to the various numerical integration formulas from Table3 using the absolute errors, when $h = 0.2$, the estimation of the area under the curve are better than the $SR_{1/3}$, $SR_{3/8}$ and TR. Farther, when $h = 0.1$, the estimates of the area under the curve are better than the $SR_{1/3}$, $SR_{3/8}$, TR, BR, WR and the method which proposed when $k = 20$.

Hence, $5A$ rule with $h = 0.1$ gives a better estimate with lesser errors as compared to the $SR_{1/3}$, $SR_{3/8}$, TR, BR, WR, and the method which proposed when $k = 20$.

References

[1] M. A. Ullah, *Numerical integration and a proposed rule,* American Journal of Engineering Research, 4 (2015), 120-123.

[2] J. Casaletto, M. Pickett, J. Rice, *A comparison of some numerical integration programs,* ACM SIGNUM Newsletter, ACM New York, NY, USA, 4 (1969), 30-40.

[3] M. Chalpuri, J. Sucharitha, *Numerical integration by using straight line interpolation formula,* Global Journal of Pure and Applied Mathematics, 13 (2017), 2123-2132.

[4] M. Evans, T. Swartz, *Methods for approximating integrals in statistics with special emphasis on Bayesian integration problems,* Statist. Sci., Inst. Math. Statist., Bethesda, MD., 10 (1995), 254-272.

[5] E. Isaacson, H. B. Keller, *Analysis of numerical methods,* Courier Corporation, 2012.

[6] L. Kaishun, L. Peter *Reliability analysis by numerical integration and curve fitting,* Structural safety, Elsevier, 3 (1985), 29-36.

[7] F. O. Mettle, E. N. Quaye, L. Asiedu, K. A. Darkwah, *A proposed method for numerical integration,* Journal of Advances in Mathematics and Computer Science, 17 (2016), 1-15.

[8] R. K. Sinha, R. Kumar, *Numerical method for evaluating the integrable function on a finite interval,* International Journal of Engineering Science and Technology, 2 (2010).

[9] P. V. Ubale, *Numerical solution of Boole's rule in numerical integration by using general quadrature formula,* Bulletin of Society for Mathematical Services & Standards (B SO MA SS), 1 (2012), 1-5.

Accepted: April 3, 2021