# Cusa-Huygens type inequality and a correction for generalized trigonometric functions

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**Abstract.** In this paper, we mainly give several Cusa-Huygens and Kober type inequalities for generalized trigonometric functions by using classical analysis. Finally, we also show that the proof of Theorem 3.9 given by Huang et.al. in the paper "Some Wilker and Cusa-Huygens type inequality for generalized trigonometric functions" is incorrect. In addition, we present a weak result by using l'Hôspital monotone rule.
**Keywords:** generalized trigonometric functions, Cusa-Huygens type inequality.

## 1. Introduction

It is well known from calculus that

$$\arcsin x = \int_0^x \frac{1}{(1-t^2)^{1/2}}dt$$

for $0 \le x \le 1$ and

$$\frac{\pi}{2} = \arcsin 1 = \int_0^1 \frac{1}{(1-t^2)^{1/2}}dt.$$

For $1 < p < \infty$ and $0 \le x \le 1$, the arcsine may be generalized as

$$\arcsin_p x = \int_0^x \frac{1}{(1-t^p)^{1/p}}dt \tag{1}$$

and

$$\frac{\pi_p}{2} = \arcsin_p 1 = \int_0^1 \frac{1}{(1-t^p)^{1/p}}dt. \tag{2}$$

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The inverse of $\arcsin_p$ on $[0, \frac{\pi_p}{2}]$ is called the generalized sine function, denoted by $\sin_p$ and may be extended to $(-\infty, \infty)$. See [5] and closely related references therein.

For $x \in [0, \frac{\pi_p}{2}]$, the generalized cosine function $\cos_p x$ is defined by

$$\cos_p x = \frac{\mathrm{d} \sin_p x}{\mathrm{d}x}. \tag{3}$$

It is easy to see that

$$\cos_p x = (1 - \sin_p^p x)^{1/p} \tag{4}$$

and

$$\frac{\mathrm{d} \cos_p x}{\mathrm{d}x} = -\cos_p^{2-p} x \sin_p^{p-1} x. \tag{5}$$

The generalized tangent function $\tan_p x$ is defined as

$$\tan_p x = \frac{\sin_p x}{\cos_p x}, \quad x \in \mathbb{R} \backslash \left\{ k\pi_p + \frac{\pi_p}{2} : k \in \mathbb{Z} \right\}. \tag{6}$$

From (1.6), it follows that

$$\frac{\mathrm{d} \tan_p x}{\mathrm{d}x} = 1 + |\tan_p x|^p, \quad x \in \left( -\frac{\pi_p}{2}, \frac{\pi_p}{2} \right). \tag{7}$$

In present, there has been a vivid interest on the generalized trigonometric and hyperbolic functions, numerous papers have been published on the studies of generalized trigonometric functions and their inequalities. Interested readers can refer to the recent review paper [12] or references [1, 2, 3, 6, 11].

The classical Cusa-Huygens inequality for trigonometric functions asserts that

$$\frac{\sin(x)}{x} < \frac{\cos(x) + 2}{3}, x \in (0, \frac{\pi}{2}), \tag{8}$$

while the Kober inequality states that

$$1 - \frac{2x}{\pi} < \cos x < 1 - \frac{x^2}{\pi}, x \in (0, \frac{\pi}{2}). \tag{9}$$

for many references related to both inequalities (8) and (9), see e.g.[9]. Due to Cusa-Huygens inequality, Klén, Vuorinen and Zhang[6] proved the Cusa-Huygens type inequalities for generalized trigonometric and hyperbolic functions in 2012. Later, Nantomah et.al. have done some meaningful work. The reader may refer to references [7, 8]. For the generalization of Kober inequality to generalized trigonometric and hyperbolic functions, the results in this aspect are rare limited to the author's knowledge.

In this note, we mainly give several new Cusa-Huygens type inequalities and Kober type inequalities for generalized trigonometric functions. Finally, we point out a mistake in proof of [4]. In addition, we present a weak result by using l'Hôspital monotone rule.

## 2. Cusa-Huygens type inequalities

**Lemma 2.1** ([6, Theorem 3.22])**.** *For $p \in (1,2]$, the following inequality*

$$\frac{\sin_p x}{x} < \frac{\cos_p x + p}{1+p} \leq \frac{\cos_p x + 2}{3} \tag{10}$$

*holds for all $x \in \left(0, \frac{\pi_p}{2}\right)$.*

Next, two theorems show the Cusa-Huygens type inequalities for the generalized trigonometric functions.

**Theorem 2.1.** *For $p \in (1,2]$, the function $\delta(x) = \frac{x - \sin_p x}{x^{p+1}}$ is strictly decreasing from $(0, \frac{\pi_p}{2})$ onto $\left(\frac{2^p(\pi_p - 2)}{\pi_p^{p+1}}, \frac{1}{p(p+1)}\right)$. In particular, for $x \in \left(0, \frac{\pi_p}{2}\right)$, we have*

$$1 - \alpha x^p < \frac{\sin_p x}{x} < 1 - \beta x^p, \tag{11}$$

*where the constants $\alpha = \frac{1}{p(p+1)}$ and $\beta = \frac{2^p(\pi_p - 2)}{\pi_p^{p+1}}$ are the best possible.*

**Proof of Theorem 2.1.** Simple computation yields

$$\delta'(x) = \frac{(p+1)\sin_p x - x\cos_p x - px}{x^{p+1}},$$

which is negative by Lemma 2.1. Hence, $\delta(x)$ is strictly decreasing in $x \in \left(0, \frac{\pi_p}{2}\right)$. The limits follows by applying l'Hospital rule.

**Lemma 2.2** ([6, Theorem 3.6])**.** *For all $p \in (1, \infty)$ and $x \in \left(0, \frac{\pi_p}{2}\right)$,*

$$\cos_p^{\alpha} x < \frac{\sin_p x}{x} < 1 \tag{12}$$

*with the best constant $\alpha = \frac{1}{p+1}$.*

**Lemma 2.3.** *For $p \geqslant 2$, the function $\beta(x) = \frac{x^p}{p+1} + \cos_p^{\frac{p}{p+1}} x$ is strictly decreasing from $(0, \frac{\pi_p}{2})$ onto $\left(\frac{\pi_p^2}{4(p+1)}, 1\right)$. In particular, for $x \in \left(0, \frac{\pi_p}{2}\right)$, we have*

$$\left[\frac{\pi_p^2 - 4x^p}{4(p+1)}\right]^{\frac{p+1}{p}} < \cos_p x < \left(1 - \frac{x^p}{p+1}\right)^{\frac{p+1}{p}}. \tag{13}$$

**Proof of Lemma 2.3.** By direct calculation, we get

$$\begin{aligned} \beta'(x) &= \frac{p}{p+1}\left(x^{p-1} - \cos_p^{2-p-\frac{1}{p+1}} x \sin_p^{p-1} x\right) \\ &= \frac{px^{p-1}}{p+1}\left[1 - \frac{(\frac{\sin_p x}{x})^{p-1}}{\left(\cos_p^{\frac{p^2-p-1}{p^2-1}} x\right)^{p-1}}\right]. \end{aligned}$$

Considering to $p \geqslant 2$, we easily obtain $\frac{p^2-p-1}{p^2-1} \geqslant \frac{1}{p+1}$. (In fact, this is equivalent to $p(p-2) \geqslant 0$.) Therefore, we have

$$\frac{\sin_p x}{x} > (\cos_p x)^{\frac{1}{p+1}} \geqslant (\cos_p x)^{\frac{p^2-p+1}{p^2-1}},$$

where we apply the function $\cos_p x$ is decreasing on $\left(0, \frac{\pi_p}{2}\right)$ and Lemma 2.2. This implies that $\beta'(x) \leqslant 0$.

So, $\beta(x)$ is decreasing on $\left(0, \frac{\pi_p}{2}\right)$ and the limiting values are trivial. □

By Lemma 2.3, we easily obtain the following corollary.

**Corollary 2.1.** *For $p \geqslant 2$ and $x \in \left(0, \frac{\pi_p}{2}\right)$, we have*

$$\frac{x^p \cos_p^{\frac{1}{p+1}} x + (p+1)\cos_p x}{p+1} < \cos_p^{\frac{1}{p+1}} x < \frac{4}{\pi_p^2}\Big(\frac{x^p \cos_p^{\frac{1}{p+1}} x + (p+1)\cos_p x}{p+1}\Big).$$

Completely similar to the proof of Lemma 2.3, we can get the following proposition.

**Proposition 2.1.** *For $1 < p \leqslant 2$, the function $\beta(x)$ is strictly increasing from $\left(0, \frac{\pi_p}{2}\right)$ onto $\left(1, \frac{\pi_p^2}{4(p+1)}\right)$. As a result, we have*

$$\Big(1 - \frac{x^p}{p+1}\Big)^{\frac{p+1}{p}} < \cos_p x < \Big(\frac{\pi_p^2 - 4x^p}{4(p+1)}\Big)^{\frac{p+1}{p}}$$

*for $x \in \left(0, \frac{\pi_p}{2}\right)$.*

**Remark.** Lemma 2.3, Corollary 2.1 and Proposition 2.1 show the Kober type inequality for the generalized trigonometric functions.

**Theorem 2.2.** *For $p > 1$, the function $r(x) = p - 1 + \cos_p^{\frac{p^2}{p+1}} x - \frac{p \sin_p^p x}{x^p}$ is strictly increasing from $(0, \frac{\pi_p}{2})$ onto $\left(0, p-1-\frac{p\cdot 2^p}{\pi_p^p}\right)$. As a result, for $x \in \left(0, \frac{\pi_p}{2}\right)$, we have*

$$\frac{\frac{p\cdot 2^p}{\pi_p^p} + \cos_p^{\frac{p^2}{p+1}} x}{p} < \frac{\sin_p^p x}{x^p} < \frac{p - 1 + \cos_p^{\frac{p^2}{p+1}} x}{p}. \tag{14}$$

**Proof of Theorem 2.2.** Differiating the function $r(x)$, we get

$$h'(x) = -\frac{p^2 \sin_p^{p-1} x}{x^p}\Big[\frac{x^p \cos_p^{\frac{1}{p+1}} x}{p+1} + \cos_p x - \frac{\sin_p x}{x}\Big].$$

Using Corollary 2.1, we have

$$\frac{x^p \cos_p^{\frac{1}{p+1}} x}{p+1} + \cos_p x < \cos_p^{\frac{1}{p+1}} x.$$

This implies that

$$h'(x) > -\frac{p^2 \sin_p^{p-1} x}{x^p}\Big(\cos_p^{\frac{1}{p+1}} x - \frac{\sin_p x}{x}\Big) > 0,$$

which is strictly increasing for any $p > 1$ by Lemma 2.2. The limiting values follow by easy computation.

Similar to the proof of Theorem 2.2, the following theorem holds true.

**Theorem 2.3.** *For $2 \leqslant p \leqslant \sqrt{2}+1$, the function $\Delta(x) = p-1+\cos_p^{\frac{2p}{p+1}} x - \frac{2\sin_p^p x}{x^p}$ is strictly increasing from $\big(0, \frac{\pi_p}{2}\big)$ onto $\Big(p-2, p-1-\frac{2^{p+1}}{\pi_p^p}\Big)$. In particular, for $x \in \big(0, \frac{\pi_p}{2}\big)$, we have*

$$\frac{\frac{2^{p+1}}{\pi_p^p} + \cos_p^{\frac{2p}{p+1}} x}{2} < \frac{\sin_p^p x}{x^p} < \frac{1+\cos_p^{\frac{2p}{p+1}} x}{2} \tag{15}$$

*or*

$$\frac{2\sin_p^p x}{x^p} - 1 < \cos_p^{\frac{2p}{p+1}} x < \frac{2\sin_p^p x}{x^p} - \frac{2^{p+1}}{\pi_p^p}. \tag{16}$$

**Proof of Theorem 2.3.** By easy calculation, we get

$$\Delta'(x) = \frac{-2p\sin_p^{p-1} x}{x^{p+1}}\Big(\frac{x^{p+1}\cos_p^{\frac{p-1}{p+1}+2-p} x}{(p+1)x} + x\cos_p x - \sin_p x\Big). \tag{17}$$

For $p \geqslant 2$ and $x \in \big(0, \frac{\pi_p}{2}\big)$, Corollary 2.1 states

$$\frac{x^p \cos_p^{\frac{p-1}{p+1}+2-p} x + (p+1)\cos_p x}{p+1} < \cos_p^{\frac{1}{p+1}} x.$$

Therefore, we get

$$\Delta'(x) > -\frac{2p\sin_p^{p-1} x}{x^p}\Big(\cos_p^{\frac{1}{p+1}} x - \frac{\sin_p x}{x}\Big) > 0, \tag{18}$$

where we applied Lemma 2.2. Thus, the function $\Delta(x)$ is strictly increasing on $\big(0, \frac{\pi_p}{2}\big)$ and the limiting values follow.

Finally, we give a conjecture about Kober type inequality.

**Conjecture.** For $x \in (0, \frac{\pi}{2})$ and $p \geqslant 2$, the function $f(x) = \frac{x^p(2p+1+\cos_p x)}{1-\cos_p x}$ is strictly increasing from $\big(0, \frac{\pi_p}{2}\big)$ onto $\Big(2p(p+1), \frac{(2p+1)\pi_p^p}{2^p}\Big)$? In particular, for $p \geqslant 2$ and $x \in \big(0, \frac{\pi_p}{2}\big)$, we have the following inequalities

$$\frac{\alpha - (2p+1)x^p}{\alpha + x^p} < \cos_p x < \frac{\beta - (2p+1)x^p}{\beta + x^p} \tag{19}$$

or

$$\frac{2p+2}{1+\frac{x^p}{\alpha}} - (2p+1) < \cos_p x < \frac{2p+2}{1+\frac{x^p}{\beta}} - (2p+1), \tag{20}$$

where the constants $\alpha = 2p(p+1)$ and $\beta = \frac{(2p+1)\pi_p^p}{2^p}$ are the best possible?

## 3. A correction

Lemma 2.1 shows the Cusa-Huygens type inequality for generalized trigonometric functions. Later, Huang et al.[4] extended and sharpened inequality (10). Their result is as stated below:

**Theorem 3.1.** *For $1 < p \leq 2$ and $x \in (0, \frac{\pi_p}{2})$, we have*

$$\left(\frac{p+\cos_p x}{p+1}\right)^{\alpha} < \frac{\sin_p x}{x} < \left(\frac{p+\cos_p x}{p+1}\right)^{\beta} \tag{21}$$

*with the best constants $\alpha = \frac{\ln\left(\frac{2}{\pi_p}\right)}{\ln\left(\frac{p}{p+1}\right)}$ and $\beta = 1$.*

In their proof, it obtained that

$$\begin{aligned} &\tfrac{x\cos_p x - \sin_p x}{x\sin_p x}\ln\left(\tfrac{p+\cos_p x}{p+1}\right) + \tfrac{\cos_p x \tan_p^{p-1} x}{p+\cos_p x}\ln\left(\tfrac{\sin_p x}{x}\right) \\ &> \ln\left(\tfrac{\sin_p x}{x}\right)\left(\tfrac{x\cos_p x - \sin_p x}{x\sin_p x} + \tfrac{\cos_p x \tan_p^{p-1} x}{p+\cos_p x}\right). \end{aligned} \tag{22}$$

Next, we explained how intermediate the above inequality (21) is logically incorrect. In fact, using inequality (10), we easily get

$$\ln\left(\frac{\sin_p x}{x}\right) < \ln\left(\frac{p+\cos_p x}{p+1}\right)$$

which gives

$$\frac{x\cos_p x - \sin_p x}{x\sin_p x}\ln\left(\frac{\sin_p x}{x}\right) > \frac{x\cos_p x - \sin_p x}{x\sin_p x}\ln\left(\frac{p+\cos_p x}{p+1}\right)$$

since $x\cos_p x - \sin_p x < 0$. This is a mathematical mistake. So their proof as they claimed can not be considered as a correct proof of inequality (21).

**Lemma 3.1** ([6])**.** *For $p > 1$, the function $g_1(x) = \frac{\sin_p x}{x}$ is strictly decreasing from $(0, \frac{\pi_p}{2})$ onto $(\frac{2}{\pi_p}, 1)$.*

**Lemma 3.2.** *For $p > 1$, the function $g_2(x) = (\cos_p x)^{1-p}$ is strictly increasing on $(0, \frac{\pi_p}{2})$.*

**Proof of Lemma 3.2.** Direct calculation yields

$$(g_2(x))' = (p-1)\,(\cos_p x)^{2-2p}\,(\sin_p x)^{p-1} > 0.$$

This implies the above result. □

**Lemma 3.3.** *For $1 < p \leq 2$, the function $g_3(x) = \cos_p x + (\cos_p x)^{1-p}$ is strictly increasing on $(x_p^*, \frac{\pi_p}{2})$ where $x_p^*$ is the unique root of the equation $(\cos_p x)^p + (1-p) = 0$.*

**Proof of Lemma 3.3.** Simple calculation yields

$$(g_3(x))' = -(\cos_p x)^{2-2p}(\sin_p x)^{p-1}\left[(\cos_p x)^p + (1-p)\right] > 0.$$

We complete the proof. □

**Lemma 3.4.** *For $1 < p \leq 2$, the function $g_4(x) = p\cos_p x + (p-2)(\cos_p x)^{1-p}$ is strictly decreasing on $(0, \frac{\pi_p}{2})$.*

**Proof of Lemma 3.4.** Using Lemma 3.2, we easily obtain the function $g_4(x)$ is strictly decreasing on $(0, \frac{\pi_p}{2})$. □

**Lemma 3.5** (l'Hôspital monotone rule[6])**.** *For $-\infty < a < b < \infty$, let $f, g : [a,b] \to \mathbb{R}$ be continuous and differentiable on $[a,b]$. Let $g'(x) \neq 0$ on $[a,b]$. If $f'(x)/g'(x)$ is increasing(decreasing) on $[a,b]$, then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \text{ and } \frac{f(x)-f(b)}{g(x)-g(b)}.$$

*If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.*

**Theorem 3.2.** *For $1 < p \leq 2$ and $x \in (x_p^*, \frac{\pi_p}{2})$, the inequality (21) holds true with the best constants $\alpha = \frac{\ln\left(\frac{2}{\pi_p}\right)}{\ln\left(\frac{p}{p+1}\right)}$ and $\beta = 1$.*

**Proof of Theorem 3.2.** Let

$$f(x) = \frac{\ln\left(\frac{\sin_p x}{x}\right)}{\ln\left(\frac{p+\cos_p x}{p+1}\right)} = \frac{f_1(x)}{f_2(x)}$$

where $f_1(x) = \ln\left(\frac{\sin_p x}{x}\right)$ and $f_2(x) = \ln\left(\frac{p+\cos_p x}{p+1}\right)$ with $f_1(0^+) = 0$ and $f_2(0^+) = 0$.

Direct computation results in

$$\frac{f_1'(x)}{f_2'(x)} = \frac{(\sin_p x - x\cos_p x)(p+\cos_p x)}{x(\cos_p x)^{2-p}(\sin_p x)^p} = \frac{f_3(x)}{f_4(x)}$$

with $f_3(0^+) = 0$ and $f_4(0^+) = 0$.

Calculating again, we obtain

$$f_3'(x) = (\cos_p x)^{2-p}(\sin_p x)^{p-1}\left[px - \sin_p x + 2x\cos_p x\right]$$

and

$$f_4'(x) = (\cos_p x)^{2-p}(\sin_p x)^{p-1}\left[\sin_p x - (2-p)x(\cos_p x)^{1-p} + px\cos_p x\right].$$

Therefore, we get

$$\begin{aligned}
\frac{f_3'(x)}{f_4'(x)} &= \frac{px - \sin_p x + 2x\cos_p x}{\sin_p x - (2-p)x\,(\cos_p x)^{1-p} + px\cos_p x} \\
&= 1 + \frac{p - \frac{2\sin_p x}{x} + (2-p)\left[\cos_p x + (\cos_p x)^{1-p}\right]}{\frac{2\sin_p x}{x} + p\cos_p x + (p-2)\,(\cos_p x)^{1-p}} \\
&= 1 + \frac{\alpha_p(x)}{\beta_p(x)}
\end{aligned}$$

where

$$\alpha_p(x) = p - \frac{2\sin_p x}{x} + (2-p)\left[\cos_p x + (\cos_p x)^{1-p}\right]$$

and

$$\beta_p(x) = \frac{2\sin_p x}{x} + p\cos_p x + (p-2)\,(\cos_p x)^{1-p}.$$

Applying to Lemma 3.1-3.4, we easily obtain the function $\alpha_p(x)$ is positive and strictly increasing and $\beta_p(x)$ is positive and strictly decreasing on $(x_p^*, \frac{\pi_p}{2})$. So we get the function $\frac{f_3'(x)}{f_4'(x)}$ is strictly increasing on $(x_p^*, \frac{\pi_p}{2})$. Using Lemma 2.5, we know the function $f(x)$ is strictly increasing on $(x_p^*, \frac{\pi_p}{2})$.

Using Hôspital rule, we easily obtain

$$f(\frac{\pi_p}{2}) = \frac{\ln\left(\frac{2}{\pi_p}\right)}{\ln\left(\frac{p}{p+1}\right)}.$$

So, the left of inequality (21) is proved. On the other hand, the right of inequality (21) is trivial by applying (10). This completes the proof. □

If $p = 2$ in Theorem 3.2, we easily obtain the following corollary.

**Corollary 3.1** (Theorem 1.1, [10]). *For $x \in (0, \frac{\pi}{2})$, we have*

$$\left(\frac{2+\cos x}{3}\right)^{\alpha} < \frac{\sin x}{x} < \left(\frac{2+\cos x}{3}\right)^{\beta} \tag{23}$$

*with the best constants $\alpha = \frac{\ln(\frac{2}{\pi})}{\ln(\frac{2}{3})}$ and $\beta = 1$.*


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