Bayes estimation of Lomax parameters under different loss functions using Lindley's approximation

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Abstract. This paper discusses Maximum Likelihood and Base estimation of two parameters shape parameter β and scale parameter Θ of the Lomax distribution based on three kinds of Loss Functions; Squared Error Loss Function (SELF), Quadratic Loss Function (QLF), and Linex Loss Function (LLF). Lindley approximation was used to obtain the values for the Lomax distribution parameters and to find the best estimation of the parameters. Many sample sizes (10,20,30,50,100) were used to compare the best estimator using the Monte Carlo simulation and the statistical scale Mean Squared Error (MSE). The results show that the Bayes estimator with both the Linex loss function and the Lindley approximation is the best estimator, and when increasing the values of shape parameter due to decreasing the values of mean square error with various vales of scale parameter.

Keywords: maximum likelihood, base estimation, Lindley approximation, Squared Error Loss Function (SELF), Quadratic Loss Function (QLF), Linex Loss Function (LLF), Monte Carlo, Mean Squared Error (MSE).

1. Introduction

A number of authors have studied Lomax distribution Inference by several authors. Nasiri and Hosseini [9] were used (MLE) with an appropriate prior distribution to obtain an estimate of Bayes based on the use of the result (Quadratic

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Loss and Squared Error Loss) functions. Okasha [10] obtained the Bayes estimator for the Parameter under (the Balanced Squared Error Loss) function. A Comparison was made using the Monte Carlo simulation between the new approach and Maximum Likelihood and Bayes methods. Afaq and others [1] used Jeffery's prior and Jeffery's extension prior to the derivation of the Maximum likelihood and Bayes estimators under use (Squared error loss, Al-Bayyati, and Precautionary loss) functions. Pak and Mahmoudi [11] determined Lomax parameters by using Bayes estimators under the (SEL) function and Maximum likelihood estimator and reached Bayesian with informative priors giving smaller biases and MSE compared to the Maximum likelihood method. Fitrilia, Fithriani and Nurrohmah [3] set the shape parameter based on the assumption of the scale parameters and used Gamma as a conjugate prior distribution with a balanced squared error loss function. The three parameters generalized Lomax distribution were studied by Maurya, Tripathi, Lodhi and Rastogi [7]. Several structural and statistical properties have been investigated, including moments, quantiles, stochastic ordering, and order statistics. They used the Tierney and Kadane method to compute Bayes estimators.

2. Lomax distribution

The distribution of Lomax was introduced by Lomax world in [6], which is appropriate for many issues such as: theory of queuing, business, economics, and internet traffic modeling. The Probability Density Function of the Lomax distribution [1] is defined as:

$$f(x) = \frac{\beta}{\Theta}\left(1+\frac{x}{\Theta}\right)^{-(\beta+1)}, \quad x>0, \Theta, \beta>0 \tag{1}$$

with the (shape parameter) β and the (scale parameter) Θ. The Corresponding Cumulative Distribution Function is given as [12]:

$$F(x) = 1-\left(1+\frac{x}{\Theta}\right)^{-\beta)}, \quad x>0, \Theta, \beta>0. \tag{2}$$

Figure (1) shows probability and cumulative density function curve of the Lomax distribution with several different parameters of (β, Θ). The Reliability and hazard functions are given as follows:

$$R(x) = \left(1+\frac{x}{\Theta}\right)^{-\beta}, \tag{3}$$

$$h(x) = \frac{\beta}{\Theta}\left(1+\frac{x}{\Theta}\right)^{-1}. \tag{4}$$

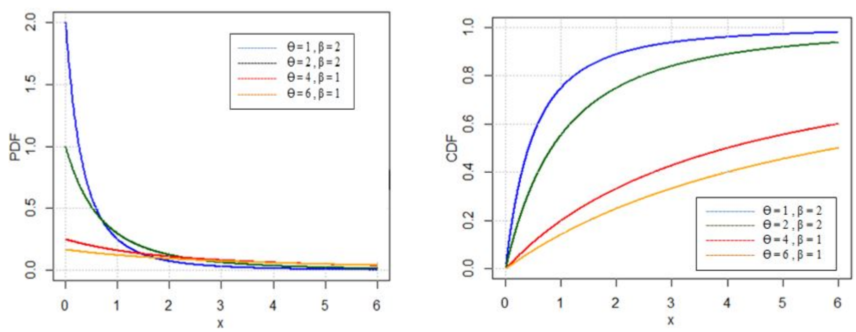

Fig 1: Probability *and* Cumulative Density Function Curve of Lomax Distribution for Various chioce for β, Θ [14]

3. Maximum Likelihood Estimator method (MLE)

Let $x = (x_1, x_2, \ldots, x_n)$ be a random sample (r.s.) of size (n) of the Lomax dist., the likelihood function is: $L(\beta, \Theta; x) = \prod_{i=1}^{n} f(x_i, \beta, \Theta)$

$$L(\beta, \Theta; x) = \frac{\beta^n}{\Theta^n} \prod_{i=1}^{n} (1 + \frac{x_i}{\Theta})^{-(\beta+1)}. \tag{5}$$

Subsequently, the partial derivative for Log Likelihood function is performed with respect to β and equating to zero, and the estimation of MLE of β denoted by $\widehat{\beta}_{MLE}$ is as follows:

$$\widehat{\beta}_{MLE} = \frac{n}{\sum_{i=1}^{n} \ln(1 + \frac{x_i}{\Theta})}. \tag{6}$$

Suppose β is known. We propose to solve Θ by using the Newton-Raphson method. Thus Θ is obtained by taking the initial value for Θ_i and iterating the process till it converges as:

$$\Theta_{i+1} = \Theta_i - \frac{-n + \frac{n}{\sum_{i=1}^{n} \ln(1+\frac{x_i}{\Theta})} \sum_{i=1}^{n} \frac{x_i}{\Theta+x_i} + \sum_{i=1}^{n} \frac{x_i}{\Theta+x_i}}{\frac{n}{\Theta}[\frac{\sum_{i=1}^{n} \frac{x_i}{\Theta+x_i}}{\sum_{i=1}^{n} \ln(1+\frac{x_i}{\Theta})}]^2 + \frac{n}{\sum_{i=1}^{n} \ln(1+\frac{x_i}{\Theta})} \sum_{i=1}^{n} \frac{x_i}{[\Theta+x_i]^2} - \sum_{i=1}^{n} \frac{x_i}{[\Theta+x_i]^2}}. \tag{7}$$

When we determine the value of $\widehat{\Theta}; (\widehat{\widehat{\Theta}}_{MLE})$, we get the value of $\widehat{\beta}; (\widehat{\widehat{\beta}}_{MLE})$.

4. Bayes estimation

In this section, we studied Bayes estimators of parameters β and Θ and reliability function R(t) based on informative prior with different loss functions [4], [13]. Prior distributions for finding Bayes estimators of the Lomax distribution parameters are taken as follows:

$$g_1(\beta) = \frac{b^a \beta^{a-1} e^{-b\beta}}{\Gamma(a)}, \quad \beta, a, b > 0, \tag{8}$$

$$g_2(\Theta) = \frac{1}{\Theta}, \quad \Theta > 0, \tag{9}$$

where $\Gamma(a)$ is the function of gamma. Thus, the joint prior distribution for β and Θ is:

$$g(\beta,\Theta)=\frac{b^a\beta^{a-1}e^{-b\beta}}{\Theta\Gamma(a)},\quad \beta,\Theta,a,b>0, \tag{10}$$

$$\pi(\beta,\Theta\setminus x_1,x_2,...,x_n)=\frac{L(x_1,x_2,...,x_n;\beta,\Theta)g(\beta,\Theta)}{\int_0^\infty\int_0^\infty L(x_1,x_2,...,x_n;\beta,\Theta)g(\beta,\Theta)d\beta d\Theta}. \tag{11}$$

Substituting L(β,Θ;x) and g(β,Θ) of (5) and (10) correspondingly, the joint posterior P(β,Θ;x) as follows:

$$\pi(\beta,\Theta\setminus x_1,x_2,...,x_n)=\frac{\frac{\beta^{n+a-1}}{\Theta^{n+1}}\prod_{i=1}^n(1+\frac{x_i}{\Theta})^{-(\beta+1)}e^{-b\beta}}{\int_0^\infty\int_0^\infty\frac{\beta^{n+a-1}}{\Theta^{n+1}}\prod_{i=1}^n(1+\frac{x_i}{\Theta})^{-(\beta+1)}e^{-b\beta}d\beta d\Theta}. \tag{12}$$

5. Bayes estimation using Lindly approximation

Lindley approximation method proposed by Lindley [5]. The procedure was designed and applied to get an approximate Bayes estimator and compute the ratio of two integrals [2], [3] as follows:

$$I=E[u(\beta,\Theta)]=\frac{\int_{(\beta,\Theta)}u(\beta,\Theta)e^{l(\beta,\Theta)+g(\beta,\Theta)}d(\beta,\Theta)}{\int_{(\beta,\Theta)}e^{l(\beta,\Theta)+g(\beta,\Theta)}d(\beta,\Theta)}, \tag{13}$$

where u(β,Θ)=function of β and Θ,l(β,Θ)=Log Likelihood and g(β,Θ)=joint prior of β and Θ. The Bayesian estimator for β and Θ by using the square error loss function is: R=E[R(β,Θ)], where R(β,Θ) is any function for β and Θ:

$$R=E[R(\beta,\Theta)]=\int_{(\beta,\Theta)}R(\beta,\Theta)\pi(\beta,\Theta\setminus x_1,x_2,...,x_n)d(\beta,\Theta). \tag{14}$$

By using Lindley's approximate method, which accession the ratio of two integrals in order to obtain the Bayes estimators β,Θ , it is possible to considering $\widehat{R}$ as follows:

$$\begin{aligned}\widehat{R}&=u(\widehat{\beta},\widehat{\Theta})+[u_1\alpha_1+u_2\alpha_2+\alpha_3+\alpha_4]\\&\quad+\frac{1}{2}[A(u_1\sigma_{11}+u_2\sigma_{12})+B(u_1\sigma_{21}+u_2\sigma_{22})],\end{aligned} \tag{15}$$

where $\widehat{\beta}$ and $\widehat{\Theta}$ are the MLE of β and Θ respectively

$$\begin{aligned}&\alpha_1=\rho_1\sigma_{11}+\rho_2\sigma_{12},\quad \alpha_2=\rho_1\sigma_{21}+\rho_2\sigma_{22},\\&\alpha_3=\frac{1}{2}[u_{12}\sigma_{12}+u_{21}\sigma_{21}],\quad \alpha_4=\frac{1}{2}[u_{11}\sigma_{11}+u_{22}\sigma_{22}],\\&B=\sigma_{11}L_{112}+2\sigma_{12}L_{122}+\sigma_{22}L_{222},\end{aligned}$$

$$u_1 = \frac{\partial u(\beta,\Theta)}{\partial \beta}, \quad u_2 = \frac{\partial u(\beta,\Theta)}{\partial \Theta}$$
$$u_{11} = \frac{\partial^2 u(\beta,\Theta)}{\partial \beta^2}, \quad u_{12} = \frac{\partial^2 u(\beta,\Theta)}{\partial \beta \partial \Theta},$$
$$u_{21} = \frac{\partial^2 u(\beta,\Theta)}{\partial \beta \partial \Theta}, \quad u_{22} = \frac{\partial^2 u(\beta,\Theta)}{\partial \Theta^2},$$
$$\rho = \ln g(\beta,\Theta) = a \ln b + (a-1)\ln\beta - b\beta - \ln\Theta - \ln\Gamma(a),$$
$$\rho_1 = \frac{\partial \rho}{\partial \beta} = \frac{a-1}{\beta} - b, \quad \rho_2 = \frac{\partial \rho}{\partial \Theta} = -\frac{1}{\Theta},$$
$$L_{11} = \frac{\partial^2 L(\beta,\Theta)}{\partial \beta^2} = \frac{-n^2}{\beta}, \quad L_{12} = \frac{\partial^2 u(\beta,\Theta)}{\partial \beta \partial \Theta} = \sum_{i=1}^{n} \frac{x_i}{(\Theta^2 + \Theta x_i)},$$
$$L_{21} = \frac{\partial^2 u(\beta,\Theta)}{\partial \Theta \partial \beta} = L_{12}, \quad L_{22} = \frac{\partial^2 u(\beta,\Theta)}{\partial \beta^2} = \frac{n}{\Theta^2} - (\beta+1)\sum_{i=1}^{n} \frac{2\Theta x_i + x_i^2}{(\Theta^2 + \Theta x_i)^2},$$
$$L_{211} = \frac{\partial^3 u(\beta,\Theta)}{\partial \Theta \partial \beta^2} = L_{121} = \frac{\partial^3 u(\beta,\Theta)}{\partial \beta^2 \partial \Theta} = L_{112} = \frac{\partial^3 u(\beta,\Theta)}{\partial \beta^2 \partial \Theta} = 0,$$
$$L_{212} = \frac{\partial^3 u(\beta,\Theta)}{\partial \beta \partial \Theta^2} = -\sum_{i=1}^{n} \frac{2\Theta + x_i}{(\Theta^2 + \Theta x_i)^2}$$
$$L_{221} = \frac{\partial^3 u(\beta,\Theta)}{\partial \Theta^2 \partial \beta} = L_{122} = \frac{\partial^3 u(\beta,\Theta)}{\partial \Theta^2 \partial \beta} = -\sum_{i=1}^{n} \frac{2\Theta x_i + x_i^2}{(\Theta^2 + \Theta x_i)^2},$$
$$L_{222} = \frac{\partial^3 u(\beta,\Theta)}{\partial \beta^3} = \frac{-2n}{\Theta^3} - \frac{(\beta+1)\sum_{i=1}^{n} 2x_i(\Theta^2 + \Theta x_i) - 2(2\Theta x_i + x_i^2)(2\Theta + x_i)}{\Theta^2 + \Theta x_i}$$

β and Θ are independent then:

$$\sigma_{12} = \sigma_{\beta\Theta} = \sigma_{\Theta\beta} = \sigma_{21} = 0, \quad \sigma_{11} = -\frac{1}{L_{11}}, \quad \sigma_{22} = -\frac{1}{L_{22}}$$

6. Estimator of Bayes with Squared Error Loss (SEL) function

The squared error loss function is given as: [1]

(16) $$L_{BS}(\widehat{\Theta},\Theta)\alpha(\widehat{\Theta} - \Theta)^2.$$

Then, the Risk function under the Squared Error Loss Function is denoted by $R_{SL}(\widehat{\Theta},\Theta)$ is:

(17) $$R_{SEL}(\widehat{\Theta},\Theta) = \int_0^\infty (\widehat{\Theta} - \Theta)\pi(\Theta \backslash t)d\Theta.$$

Using a partial derivative for $R_{SEL}(\widehat{\Theta},\Theta)$ with regard to $\widehat{\Theta}$ and fixing it equals zero, we get:

(18) $$\widehat{\widehat{\Theta}} = E(\Theta/t)$$

i) If $u(\beta,\Theta)=\widehat{\beta}$ then: $u_1=1, u_2=u_{11}=u_{22}=u_{12}=u_{21}=0$. Then, the estimator β is written as follows:

$$\widehat{\beta}_{BSEL}=\beta-\frac{\rho_1}{L_{11}}+\frac{L_{111}L_{222}+L_{221}}{2L_{11}L_{22}}+\frac{L_{111}}{2L_{11}^2}+\frac{L_{221}L_{222}}{2L_{22}^2}. \tag{19}$$

ii) If $u(\beta,\Theta)=\widehat{\Theta}$ then: $u_2=1, u_2=u_{11}=u_{12}=u_{21}=u_{22}=0$. Then, the estimator Θ is written as follows:

$$\widehat{\widehat{\Theta}}_{BSEL}=\widehat{\Theta}-\frac{\rho_2}{L_{22}}-\frac{L_{111}L_{222}}{2L_{11}L_{22}^2}+\frac{L_{221}L_{222}}{2L_{22}^3}. \tag{20}$$

7. Estimator of Bayes with Quadratic Loss (QL) function

The (QL) function is given as [1]

$$L_{QL}(\widehat{\Theta},\Theta)=\left(\frac{\widehat{\Theta}-\Theta}{\Theta}\right)^2. \tag{21}$$

Then, the function of Risk with (QL) function is denoted as $R_{QL}(\widehat{\Theta}-\Theta)$:

$$R_{QL}(\widehat{\Theta},\Theta)=\int_0^\infty\left(\frac{\widehat{\Theta}-\Theta}{\Theta}\right)^2\pi(\Theta\backslash t)d\Theta. \tag{22}$$

Going to take partial derivative to $R_{QL}(\widehat{\Theta},\Theta)$ with regard to $\widehat{\Theta}$ and to place it equal to zero, we get:

$$\widehat{\Theta}=\left(\frac{E(\Theta^{-1}\backslash t)}{E(\Theta^{-2}\backslash t)}\right) \tag{23}$$

i) If $u(\beta,\Theta)=\frac{1}{\beta}$ then: $u_1=u_{11}=u_{12}=u_{21}=0, u_2=\frac{-1}{\beta^2}, u_{22}=\frac{2}{\beta^3}$

$$E(\beta^{-1}\backslash t)=\frac{1}{\beta}+\frac{\rho_1}{\beta^2L_{22}}-\frac{1}{\beta^3L11}-\frac{L_{221}}{2\beta^2L_{11}^2}-\frac{L_{111}}{2\beta^2L_{11}L_{22}}. \tag{24}$$

ii) If $u(\beta,\Theta)=\frac{1}{\beta^2}$ then: $u_1=\frac{-2}{\beta^3}, u_{11}=\frac{6}{\beta^4}, u_2=u_{22}=u_{12}=u_{21}=0$

$$E(\beta^{-2}\backslash t)=\frac{1}{\beta^2}+\frac{2\rho_1}{\beta^3L_{22}}-\frac{3}{\beta^4L_{11}}-\frac{L_{221}}{\beta^3L_{11}^2}-\frac{L_{111}}{\beta^3L_{11}L_{22}}. \tag{25}$$

Then, the estimator β is written as follows:

$$\widehat{\widehat{\beta}}_{BQL}=\frac{E(\beta^{-1}\backslash t)}{E(\beta^{-2}\backslash t)}=\frac{1+\frac{\rho_1}{\beta L_{22}}-\frac{1}{\beta^2L_{11}}-\frac{L_{221}}{2\beta L_{11}^2}-\frac{L_{111}}{2\beta L_{11}L_{22}}}{\frac{1}{\beta}+\frac{2\rho_1}{\beta^2L_{22}}-\frac{3}{\beta^3L_{11}}-\frac{L_{221}}{\beta^2L_{11}^2}-\frac{L_{111}}{\beta^2L_{11}L_{22}}}. \tag{26}$$

iii) If $u(\beta,\Theta)=\frac{1}{\Theta}$ then: $u_1=\frac{-1}{\Theta^2}, u_{11}=\frac{2}{\Theta^3}, u_2=u_{22}=u_{12}=u_{21}=0$

$$E(\Theta^{-1}\backslash t)=\frac{1}{\Theta}+\frac{\rho_1}{\Theta^2 L_{22}}-\frac{1}{\Theta^3 L11}-\frac{L_{221}}{2\Theta^2 L_{22}^2}-\frac{L_{111}}{2\Theta^2 L_{11}L_{22}}. \quad (27)$$

iv) If $u(\beta,\Theta)=\frac{1}{\Theta^2}$ then: $u_1=\frac{-2}{\Theta^3}, u_{11}=\frac{6}{\Theta^4}, u_2=u_{21}=u_{12}=u_{22}=0$

$$E(\Theta^{-2}\backslash t)=\frac{1}{\Theta^2}+\frac{2\rho_1}{\Theta^3 L_{22}}-\frac{3}{\Theta^4 L_{11}}-\frac{L_{221}}{\Theta^3 L_{11}^2}-\frac{L_{111}}{\Theta^3 L_{11}L_{22}}. \quad (28)$$

Then, the estimator Θ is written as follows:

$$\hat{\hat{\Theta}}_{BQL}=\frac{E(\Theta^{-1}\backslash t)}{E(\Theta^{-2}\backslash t)}=\frac{1+\frac{\rho_1}{\Theta L_{22}}-\frac{1}{\Theta^2 L_{11}}-\frac{L_{221}}{2\Theta L_{11}^2}-\frac{L_{111}}{2\Theta L_{11}L_{22}}}{\frac{1}{\Theta}+\frac{2\rho_1}{\Theta^2 L_{22}}-\frac{3}{\Theta^3 L_{11}}-\frac{L_{221}}{\Theta^2 L_{11}^2}-\frac{L_{111}}{\Theta^2 L_{11}L_{22}}}. \quad (29)$$

8. Bayes Estimator with Linex Loss (LL) function

The (LLF) is indicated as [8]

$$L(\hat{\Theta},\Theta)=e^{c(\hat{\Theta}-\Theta)}-c(\hat{\Theta}-\Theta)-1, \quad c\neq 0. \quad (30)$$

Additionally, $R_{LS}(\hat{\Theta},\Theta)$ defines the Risk function with Linex loss function as follows:

$$R_{LS}(\hat{\Theta},\Theta)=\int_0^\infty [e^{c(\hat{\Theta}-\Theta)}-c(\hat{\Theta}-\Theta)-1]\pi(\Theta\backslash t)d\Theta. \quad (31)$$

Take a partial derivative for $R_{LS}(\hat{\Theta},\Theta)$ with due regard to $\hat{\Theta}$ and making it equal to zero, we obtain:

$$\hat{\hat{\beta}}_{BLL}=-\frac{1}{c}\ln E_\beta[e^{-c\beta}], \quad (32)$$

where E_β is posterior expectation. i) If $u(\hat{\beta},\hat{\Theta})=e^{-c\beta}$ then: $u_1=-ce^{-c\beta}, u_{11}=c^2e^{-c\beta}, u_2=u_{21}=u_{12}=u_{22}=0$

$$E_\beta[e^{-c\beta}]=e^{-c\beta}(1+\frac{c\rho_1}{L_{11}}-\frac{c^2}{2L_{11}}-\frac{c}{2L_{11}^2}+\frac{cL_{221}}{2L_{11}L_{22}}). \quad (33)$$

The estimator β then can be written as follows:

$$\hat{\hat{\beta}}_{BLL}=\hat{\beta}-\frac{1}{c}\ln(1+\frac{c\rho_1}{L_{11}}-\frac{c^2}{2L_{11}}-\frac{c}{2L_{11}^2}+\frac{cL_{221}}{2L_{11}L_{22}}). \quad (34)$$

ii) If $u(\hat{\beta},\hat{\Theta})=e^{-c\Theta}$ then: $u_2=-ce^{-c\Theta}, u_{22}=c^2e^{-c\Theta}, u_1=u_{21}=u_{12}=u_{11}=0$

$$\hat{\hat{\Theta}}_{BLL}=\frac{-1}{c}\ln E_\Theta[e^{-c\Theta}],$$

$$E_\Theta[e^{-c\Theta}]=e^{-c\Theta}(1+\frac{c\rho_2}{L_{22}}-\frac{c^2}{2L_{22}}+\frac{c}{2L_{11}^2}+\frac{cL_{111}L_{222}}{L_{11}L_{22}^2}+\frac{cL_{221}L_{222}}{2L_{22}^3}). \quad (35)$$

Then, the estimator Θ written as follows:

$$\hat{\hat{\Theta}}_{BLL}=\hat{\Theta}-\frac{1}{c}\ln(1+\frac{c\rho_2}{L_{22}}-\frac{c^2}{2L_{22}}+\frac{c}{2L_{11}^2}+\frac{cL_{111}L_{222}}{L_{11}L_{22}^2}+\frac{cL_{221}L_{222}}{2L_{22}^3}). \quad (36)$$

9. Simulation study and results

The simulation of (Monte Carlo) was used in this paper to compare the various methods used to estimate the Lomax distribution parameters. This approach is outlined in the following steps:

1) Select many assumed values for the parameters β and Θ for the Lomax distribution: $\beta = 0.5, 0.75, 1.5$; $\Theta = 0.5, 0.75, 1.5$.

2) Choose a number of different sample sizes: n=10, 20, 30, 50, 100.

3) Assumed that the values of a, b and c in the Lindley approximation method are as follows: a=1.5 ,b=0.75 ,c=2.

4) Random data was generated from the distribution of the uniform to the distribution of Lomax using the inverse transformation process, which was based on the determination of the inverse of the cumulative distribution function as follows: $x_i = \Theta[(1 - u_i)^{\frac{-1}{\beta}} - 1]$.

5) Repeat each experiment (L=1000) times.

6) Compare all estimation methods using (MSE) as follows:

$$MSE(\widehat{\beta}) = \frac{1}{L}\sum_{i=1}^{L}(\beta_i - \widehat{\beta}_i)^2, \quad MSE(\widehat{\Theta}) = \frac{1}{L}\sum_{i=1}^{L}(\Theta_i - \widehat{\Theta}_i)^2. \tag{37}$$

The following tables include sample of the numerical results for the empirical works.

TABLE 1: MLE and Bayes estimation for β *and* Θ by using Lindly when $\beta = 0.5$; $\Theta = 0.5, 0.75, 1.5$

Θ	n	MLE		Bayes under Square Error Loss		Bayes under Quadratic Loss		*Bayes under Linex Loss*		Best
		$\hat{\hat{\beta}}_{MLE}$	$\hat{\hat{\Theta}}_{MLE}$	$\hat{\hat{\beta}}_{BSEL}$	$\hat{\hat{\Theta}}_{BSEL}$	$\hat{\hat{\beta}}_{BQL}$	$\hat{\hat{\Theta}}_{BQL}$	$\hat{\hat{\beta}}_{BLL}$	$\hat{\hat{\Theta}}_{BLL}$	
0.5	10	0.55953	0.45823	0.33287	0.41155	0.42884	0.44991	0.11967	0.23189	BLL
	20	0.55996	0.45872	0.33891	0.41865	0.42858	0.44980	0.11861	0.23587	BLL
	30	0.55784	0.45256	0.33875	0.41557	0.42817	0.44875	0.11665	0.23147	BLL
	50	0.55875	0.45158	0.33887	0.41987	0.42789	0.44489	0.11584	0.23178	BLL
	100	0.55389	0.45145	0.33881	0.41148	0.42675	0.44451	0.10457	0.22154	BLL
0.75	10	0.42125	0.43994	0.31665	0.50520	0.48665	0.51896	0.20689	0.30884	BLL
	20	0.42147	0.43897	0.31564	0.50456	0.48663	0.51887	0.20654	0.30845	BLL
	30	0.42410	0.43911	0.31478	0.50412	0.48538	0.51874	0.20568	0.30745	BLL
	50	0.42243	0.43742	0.31458	0.50408	0.48416	0.51754	0.20548	0.30581	BLL
	100	0.42145	0.43117	0.31354	0.50388	0.48314	0.51521	0.20487	0.30511	BLL
1.5	10	0.40789	0.45554	0.31897	0.50589	0.48897	0.51296	0.21884	0.30412	BLL
	20	0.40445	0.45078	0.31489	0.50568	0.48889	0.51285	0.20954	0.30989	BLL
	30	0.40325	0.45088	0.31258	0.50498	0.48875	0.51654	0.20852	0.30895	BLL
	50	0.40158	0.45245	0.31018	0.50477	0.48787	0.51491	0.20754	0.30845	BLL
	100	0.40125	0.45014	0.31011	0.50244	0.48584	0.51284	0.20451	0.30821	BLL

10. Conclusion and recommendation

- Table 1 shows that when (β=0.5), the output of Bayes estimator with many (loss functions) of β and Θ is better than the MLE.

TABLE 2: MLE and Bayes estimation for β *and* Θ by using Lindly when $\beta = 0.75$; $\Theta = 0.5, 0.75, 1.5$

Θ	n	MLE		Bayes under Square Error Loss		Bayes under Quadratic Loss		*Bayes under Linex Loss*		Best
		$\hat{\beta}_{MLE}$	$\hat{\Theta}_{MLE}$	$\hat{\beta}_{BSEL}$	$\hat{\Theta}_{BSEL}$	$\hat{\beta}_{BQL}$	$\hat{\Theta}_{BQL}$	$\hat{\beta}_{BLL}$	$\hat{\Theta}_{BLL}$	
0.5	10	0.21989	0.22911	0.30104	032817	0.10169	0.11199	0.01888	0.02220	BLL
	20	0.21852	0.22921	0.30101	032816	0.10155	0.11182	0.01824	0.02123	BLL
	30	0.21842	0.22846	0.30092	032801	0.10114	0.11145	0.01812	0.02112	BLL
	50	0.21753	0.22834	0.30088	032781	0.10111	0.11124	0.01801	0.02012	BLL
	100	0.21701	0.22784	0.30014	032711	0.10089	0.11119	0.01789	0.02010	BLL
0.75	10	0.21424	0.22412	0.30545	032894	0.10074	0.11985	0.01987	0.02625	BLL
	20	0.21421	0.22410	0.30254	032892	0.10064	0.11978	0.01963	0.02600	BLL
	30	0.21434	0.22385	0.30224	032887	0.10832	0.11894	0.01898	0.02875	BLL
	50	0.21415	0.22289	0.30214	032784	0.10823	0.11852	0.01874	0.02852	BLL
	100	0.21401	0.22265	0.30112	032745	0.10811	0.11724	0.01785	0.02812	BLL
1.5	10	0.24211	0.22321	0.30214	032721	0.10728	0.11700	0.01545	0.02942	BLL
	20	0.21987	0.22301	0.30005	032657	0.10721	0.11664	0.01527	0.02910	BLL
	30	0.21878	0.22102	0.30001	032584	0.10700	0.11657	0.01501	0.02828	BLL
	50	0.21258	0.22044	0.26541	032541	0.10681	0.11611	0.01456	0.02712	BLL
	100	0.21189	0.22001	0.25841	032441	0.10521	0.11421	0.01412	0.02621	BLL

TABLE 3: MLE and Bayes estimation for β *and* Θ by using Lindly when $\beta = 1.5$; $\Theta = 0.5, 0.75, 1.5$

Θ	n	MLE		Bayes under Square Error Loss		Bayes under Quadratic Loss		*Bayes under Linex Loss*		Best
		$\hat{\beta}_{MLE}$	$\hat{\Theta}_{MLE}$	$\hat{\beta}_{BSEL}$	$\hat{\Theta}_{BSEL}$	$\hat{\beta}_{BQL}$	$\hat{\Theta}_{BQL}$	$\hat{\beta}_{BLL}$	$\hat{\Theta}_{BLL}$	
0.5	10	0.15524	0.17878	0.29948	030753	0.12246	0.14019	0.00885	0.00727	BLL
	20	0.15515	0.17521	0.29936	030754	0.12237	0.14015	0.00881	0.00711	BLL
	30	0.15098	0.17510	0.29912	030742	0.12244	0.14014	0.00875	0.00621	BLL
	50	0.15085	0.17489	0.29901	030722	0.12213	0.14008	0.00871	0.00601	BLL
	100	0.15009	0.17467	0.29900	030712	0.12155	0.14004	0.00865	0.00561	BLL
0.75	10	0.15847	0.17457	0.29741	030697	0.12456	0.14028	0.00842	0.00552	BLL
	20	0.15879	0.17444	0.29714	030665	0.12424	0.14026	0.00833	0.00542	BLL
	30	0.15789	0.17432	0.29706	030642	0.12412	0.14014	0.00811	0.00532	BLL
	50	0.15741	0.17414	0.29712	030621	0.12368	0.14011	0.00784	0.00456	BLL
	100	0.15701	0.17211	0.29710	030612	0.12332	0.13001	0.00745	0.00444	BLL
1.5	10	0.15856	0.17212	0.29985	030822	0.12621	0.14901	0.00739	0.00424	BLL
	20	0.15846	0.17211	0.29963	030812	0.12611	0.14822	0.00718	0.00396	BLL
	30	0.15723	0.17200	0.29921	030811	0.12601	0.14802	0.00684	0.00352	BLL
	50	0.15710	0.17166	0.29901	030711	0.12564	0.14722	0.00665	0.00331	BLL
	100	0.15612	0.17122	0.29861	030701	0.12552	0.14692	0.00656	0.00153	BLL

- According to tables (2,3), when β=0.75,1.5 the performance of (MLE) of β and Θ is better than the estimator of Bayes under (SELF) for sample sizes.

- Tables (1,2,3) display that the Bayes estimator with both the Linex loss function and the Lindley approximation is the better estimator when compared to the other estimators and for all sample sizes.

- From both tables (2 and 3), as $\beta = 0.75, 1.5$, the performance of Bayes estimator over quadratic Loss function is better than MLE and the Bayes estimator with (SEL) function of all samples.

- The estimation values of (MLE) and Bayes estimator under different loss functions are very close to each other, as shown in the tables for estimate values of α and β.

- It should be noted that increasing the values of β due to decreases the values of (MSE) in both the different values of Θ and for all sample sizes.

- The study was carried out to determine the best estimator for the parameters of the Lomax distribution. We recommend using the Bayes estimator with Lindley approximation under the Linex loss function. Experiment with new loss functions as well.

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